REPORT ON "DYNAMICS IN QUASICRYSTALS"

JENS MARKLOF

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1. The setting

The Lorentz gas is defined as an ensemble of non-interacting point particles moving in an array of spherical scatterers placed at the elements of a given point set $\mathcal{P} \subset \mathbb{R}^d$ ($d \geq 2$, and we assume that the scatterers do not overlap). Each particle travels with constant velocity along straight lines until it collides with a scatterer, and is then reflected elastically. We denote by $\boldsymbol{q}(t), \boldsymbol{v}(t)$ the position and velocity of a particle at time t. Since the reflection is elastic, speed is a constant of motion; we may assume without loss of generality that $\|\boldsymbol{v}(t)\| = 1$. The "phase space" is then the unit tangent bundle $\mathrm{T}^1(\mathcal{K}_\rho)$ where $\mathcal{K}_\rho \subset \mathbb{R}^d$ is the complement of the set $\mathcal{B}_\rho^d + \mathcal{P}$ (the "billiard domain"), and $\mathcal{B}_\rho^d$ denotes the open ball of radius ρ, centered at the origin. We parametrize $\mathrm{T}^1(\mathcal{K}_\rho)$ by $(\boldsymbol{q}, \boldsymbol{v}) \in \mathcal{K}_\rho \times \mathrm{S}_1^{d-1}$, where we use the convention that for $\boldsymbol{q} \in \partial\mathcal{K}_\rho$ the vector $\boldsymbol{v}$ points away from the scatterer (so that $\boldsymbol{v}$ describes the velocity *after* the collision). The Liouville measure on $\mathrm{T}^1(\mathcal{K}_\rho)$ is

$$d\nu(\boldsymbol{q}, \boldsymbol{v}) = d\mathrm{vol}(\boldsymbol{q})\, d\omega(\boldsymbol{v}) \tag{1.1}$$

where vol and ω refer to the Lebesgue measures on $\mathbb{R}^d$ and S_1^{d-1}, respectively.

The first collision time corresponding to the initial condition $(\boldsymbol{q}, \boldsymbol{v}) \in \mathrm{T}^1(\mathcal{K}_\rho)$ is

$$\tau_1(\boldsymbol{q}, \boldsymbol{v}; \rho) = \inf\{t > 0 : \boldsymbol{q} + t\boldsymbol{v} \notin \mathcal{K}_\rho\}. \tag{1.2}$$

Since all particles are moving with unit speed, we may also refer to $\tau_1(\boldsymbol{q}, \boldsymbol{v}; \rho)$ as the free path length. The distribution of free path lengths in the limit of small scatterer density (Boltzmann-Grad limit) has been studied extensively when $\mathcal{P}$ is a fixed realisation of a random point process (such as a spatial Poisson process) [5, 11, 23, 29] and when $\mathcal{P}$ is a Euclidean lattice [1, 2, 6, 7, 9, 10, 12, 16, 22, 23]. In the Boltzmann-Grad limit, the Lorentz process in fact converges to a random flight process, see [11, 29, 5] for the case of random $\mathcal{P}$ and [8, 17, 18, 19] for periodic $\mathcal{P}$.

2. Cut and project

In the present work, we consider the Lorentz gas for scatterer configurations $\mathcal{P}$ given by regular cut-and-project sets, cf. [14, 31]. Examples of such $\mathcal{P}$ include large classes of quasicrystals, for instance the vertex set of any of the classical Penrose tilings [26]. Further examples include all locally finite periodic point sets such as graphene's honeycomb lattice [3, 4].

To give a precise definition of cut-and-project sets in $\mathbb{R}^d$, denote by π and π_{int} the orthogonal projection of $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m$ onto the first d and last m coordinates, and refer to $\mathbb{R}^d$ and $\mathbb{R}^m$ as the *physical space* and *internal space*, respectively. Let $\mathcal{L} \subset \mathbb{R}^n$ be a lattice of full rank. Then the closure of the set $\pi_{\mathrm{int}}(\mathcal{L})$ is an abelian subgroup $\mathcal{A}$ of $\mathbb{R}^m$. We denote by $\mathcal{A}^\circ$ the connected subgroup of $\mathcal{A}$ containing $\mathbf{0}$; then $\mathcal{A}^\circ$ is a linear subspace of $\mathbb{R}^m$, say of dimension

m_1, and there exist $\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{m_2} \in \mathcal{L}$ ($m = m_1 + m_2$) such that $\pi_{\rm int}(\boldsymbol{b}_1),\ldots,\pi_{\rm int}(\boldsymbol{b}_{m_2})$ are linearly independent in $\mathbb{R}^m/\mathcal{A}^\circ$ and

$$\mathcal{A} = \mathcal{A}^\circ + \mathbb{Z}\pi_{\rm int}(\boldsymbol{b}_1) + \ldots + \mathbb{Z}\pi_{\rm int}(\boldsymbol{b}_{m_2}). \tag{2.1}$$

Given $\mathcal{L}$ and a bounded subset $\mathcal{W} \subset \mathcal{A}$ with non-empty interior, we define

$$\mathcal{P}(\mathcal{W},\mathcal{L}) = \{\pi(\boldsymbol{y}) : \boldsymbol{y} \in \mathcal{L},\ \pi_{\rm int}(\boldsymbol{y}) \in \mathcal{W}\} \subset \mathbb{R}^d. \tag{2.2}$$

We will call $\mathcal{P} = \mathcal{P}(\mathcal{W},\mathcal{L})$ a *cut-and-project set*, and $\mathcal{W}$ the *window*. We denote by $\mu_{\mathcal{A}}$ the Haar measure of $\mathcal{A}$, normalized so that its restriction to $\mathcal{A}^\circ$ is the standard m_1-dimensional Lebesgue measure. If $\mathcal{W}$ has boundary of measure zero with respect to $\mu_{\mathcal{A}}$, we will say $\mathcal{P}(\mathcal{W},\mathcal{L})$ is *regular*. Set $\mathcal{V} = \mathbb{R}^d \times \mathcal{A}^\circ$; then $\mathcal{L}_{\mathcal{V}} = \mathcal{L} \cap \mathcal{V}$ is a lattice of full rank in $\mathcal{V}$. Let $\mu_{\mathcal{V}} = \mathrm{vol}\times\mu_{\mathcal{A}}$ be the natural volume measure on $\mathbb{R}^d \times \mathcal{A}$ (this restricts to the standard $d + m_1$ dimensional Lebesgue measure on $\mathcal{V}$). It follows from Weyl equidistribution (see [13]) that for any regular cut-and-project set $\mathcal{P}$ and any bounded $\mathcal{D} \subset \mathbb{R}^d$ with boundary of measure zero with respect to Lebesgue measure,

$$\lim_{T\to\infty} \frac{\#\{\boldsymbol{b} \in \mathcal{L} \,:\, \pi(\boldsymbol{b}) \in \mathcal{P} \cap T\mathcal{D}\}}{T^d} = \delta_{d,m}(\mathcal{L})\,\mathrm{vol}(\mathcal{D})\mu_{\mathcal{A}}(\mathcal{W}) \tag{2.3}$$

where

$$\delta_{d,m}(\mathcal{L}) := \frac{1}{\mu_{\mathcal{V}}(\mathcal{V}/\mathcal{L}_{\mathcal{V}})}. \tag{2.4}$$

A further condition often imposed in the quasicrystal literature is that $\pi|_{\mathcal{L}}$ is injective (i.e., the map $\mathcal{L} \to \pi(\mathcal{L})$ is one-to-one); we will not require this here. To avoid coincidences in $\mathcal{P}$, we simply assume in the following that the window is appropriately chosen so that the map $\pi_{\mathcal{W}} : \{\boldsymbol{y} \in \mathcal{L} : \pi_{\rm int}(\boldsymbol{y}) \in \mathcal{W}\} \to \mathcal{P}$ is bijective. Then (2.3) implies

$$\lim_{T\to\infty} \frac{\#(\mathcal{P} \cap T\mathcal{D})}{T^d} = \delta_{d,m}(\mathcal{L})\,\mathrm{vol}(\mathcal{D})\mu_{\mathcal{A}}(\mathcal{W}). \tag{2.5}$$

Under the above assumptions $\mathcal{P}(\mathcal{W},\mathcal{L})$ is a Delone set, i.e., uniformly discrete and relatively dense in $\mathbb{R}^d$.

We may obviously extend the definition of cut-and-project sets $\mathcal{P}(\mathcal{W},\widetilde{\mathcal{L}})$ to affine lattices $\widetilde{\mathcal{L}} = \mathcal{L} + \boldsymbol{x}$, for any $\boldsymbol{x} \in \mathbb{R}^n$; note that $\mathcal{P}(\mathcal{W},\mathcal{L}+\boldsymbol{x}) = \mathcal{P}(\mathcal{W} - \pi_{\rm int}(\boldsymbol{x}),\mathcal{L}) + \pi(\boldsymbol{x})$.

3. The distribution of free path lengths in the Boltzmann-Grad limit

In order to study the distribution of the free path length for random initial data $(\boldsymbol{q},\boldsymbol{v})$ we need to specify a probability measure on $\mathrm{T}^1(\mathcal{K}_\rho)$. A natural choice is of course any Borel probability measure which is absolutely continuous with respect to the Liouville measure ν. Given $s > 0$ and a Borel probability measure Λ on $\mathrm{T}^1(\mathbb{R}^d)$, we define the family of Borel probability measures $\Lambda^{(s)}$ on $\mathrm{T}^1(\mathbb{R}^d)$ by

$$\Lambda^{(s)}(E) = \Lambda\big(\{(s^{-1}\boldsymbol{q},\boldsymbol{v}) \,:\, (\boldsymbol{q},\boldsymbol{v}) \in E\}\big). \tag{3.1}$$

Theorem 3.1. *Given any regular cut-and-project set $\mathcal{P}$ there is a non-increasing continuous function $F_{\mathcal{P}} : [0,\infty] \to [0,1]$ with $F_{\mathcal{P}}(0) = 1$, $F_{\mathcal{P}}(\infty) = 0$, such that for any Borel probability measure Λ on $\mathrm{T}^1(\mathbb{R}^d)$ which is absolutely continuous with respect to Liouville measure, and any $s_0 > 0$, $\xi > 0$, we have*

$$\Lambda^{(s)}(\{(\boldsymbol{q},\boldsymbol{v}) \in \mathrm{T}^1(\mathcal{K}_\rho) : \rho^{d-1}\tau_1(\boldsymbol{q},\boldsymbol{v};\rho) \geq \xi\}) \to F_{\mathcal{P}}(\xi), \tag{3.2}$$

as $\rho \to 0$, uniformly over all $s \geq s_0$.

We highlight the fact that the limit distribution is independent of Λ. Our techniques will allow us to prove limit theorems for more singular measures. A natural example is to fix a generic point $\boldsymbol{q} \notin \mathcal{P}$ and take $\boldsymbol{v}$ random:

Theorem 3.2. *Given any regular cut-and-project set $\mathcal{P}$ there is a subset $\mathfrak{S} \subset \mathbb{R}^d$ of Lebesgue measure zero such that for any $\boldsymbol{q} \in \mathbb{R}^d \setminus \mathfrak{S}$, any $\xi > 0$ and any Borel probability measure λ on S_1^{d-1} which is absolutely continuous with respect to Lebesgue measure, we have*

$$\lim_{\rho\to 0} \lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \rho^{d-1}\tau_1(\boldsymbol{q},\boldsymbol{v};\rho) \geq \xi\}) = F_{\mathcal{P}}(\xi), \tag{3.3}$$

with $F_{\mathcal{P}}(\xi)$ as in Theorem 3.1.

In fact our proof shows that the limit in (3.3) exists for *every* $\boldsymbol{q} \in \mathbb{R}^d$; however for $\boldsymbol{q} \in \mathfrak{S}$ the limit in general depends on $\boldsymbol{q}$.

Another possibility is to specify the location $\boldsymbol{q} \in \mathcal{P}$ of a scatterer and consider the initial data $(\boldsymbol{q}_{\rho,\boldsymbol{\beta}}(\boldsymbol{v}),\boldsymbol{v}) \in \mathrm{T}^1(\mathbb{R}^d)$ where $\boldsymbol{q}_{\rho,\boldsymbol{\beta}}(\boldsymbol{v}) := \boldsymbol{q}+\rho\boldsymbol{\beta}(\boldsymbol{v})$ is on (or near) the scatterer's boundary. Here $\boldsymbol{\beta} : \mathrm{S}_1^{d-1} \to \mathbb{R}^d$ is some fixed continuous function and $\boldsymbol{v}$ is again chosen at random on S_1^{d-1}. To avoid pathologies, we assume that $(\boldsymbol{\beta}(\boldsymbol{v}) + \mathbb{R}_{>0}\boldsymbol{v}) \cap \mathcal{B}_1^d = \emptyset$ for all $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$. Let us also write $\beta_\perp(\boldsymbol{v}) = \sqrt{\|\boldsymbol{\beta}(\boldsymbol{v})\|^2 - (\boldsymbol{\beta}(\boldsymbol{v})\cdot\boldsymbol{v})^2}$ for the length of the orthogonal projection of $\boldsymbol{\beta}(\boldsymbol{v})$ onto the orthogonal complement of $\boldsymbol{v}$ in $\mathbb{R}^n$.

Theorem 3.3. *Given any regular cut-and-project set $\mathcal{P}$ and $\boldsymbol{q} \in \mathcal{P}$, there is a continuous function $F_{\mathcal{P},\boldsymbol{q}} : [0,\infty] \times \mathbb{R}_{\geq 0} \to [0,1]$ with $F_{\mathcal{P},\boldsymbol{q}}(\,\cdot\,,r)$ non-increasing, $F_{\mathcal{P},\boldsymbol{q}}(0,r) = 1$, $F_{\mathcal{P},\boldsymbol{q}}(\infty,r) = 0$ for all $r \in \mathbb{R}_{\geq 0}$, such that for any $\xi > 0$ and any Borel probability measure λ on S_1^{d-1} which is absolutely continuous with respect to Lebesgue measure, we have*

$$\lim_{\rho\to 0} \lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \rho^{d-1}\tau_1(\boldsymbol{q}_{\rho,\boldsymbol{\beta}}(\boldsymbol{v}),\boldsymbol{v};\rho) \geq \xi\}) = \int_{\mathrm{S}_1^{d-1}} F_{\mathcal{P},\boldsymbol{q}}(\xi,\beta_\perp(\boldsymbol{v}))\,d\lambda(\boldsymbol{v}). \tag{3.4}$$

The convergence in (3.4) is uniform over all $\boldsymbol{q} \in \mathcal{P}$.

We remark that the proof actually shows that (3.4) holds for any fixed $\boldsymbol{q} \in \pi(\mathcal{L})$, and uniformly over all $\boldsymbol{q}$ in any set of the form $\pi(\mathcal{L} \cap \pi_{\mathrm{int}}^{-1}(B))$ with B a bounded subset of $\mathcal{A}$.

4. Spaces of quasicrystals

We will now characterise the limit distributions in Theorems 3.2 and 3.3 in terms of a certain homogeneous space $(\Gamma \cap H_g)\backslash H_g$ equipped with a translation-invariant probability measure μ_g. In analogy with the space of Euclidean lattices of covolume one, $\mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})$, we will call such a space a *space of quasicrystals*.

Set $G = \mathrm{ASL}(n,\mathbb{R}) = \mathrm{SL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$, $\Gamma = \mathrm{ASL}(n,\mathbb{Z})$. The multiplication law in G is defined by

$$(M,\boldsymbol{\xi})(M',\boldsymbol{\xi}') = (MM', \boldsymbol{\xi}M' + \boldsymbol{\xi}'). \tag{4.1}$$

For $g \in G$ we define an embedding of $\mathrm{ASL}(d,\mathbb{R})$ in G by

$$\varphi_g : \mathrm{ASL}(d,\mathbb{R}) \to G, \quad (A,\boldsymbol{x}) \mapsto g\left(\begin{pmatrix} A & 0 \\ 0 & 1_m \end{pmatrix}, (\boldsymbol{x},\boldsymbol{0})\right) g^{-1}. \tag{4.2}$$

We also set $G^1 = \mathrm{SL}(n,\mathbb{R})$ and $\Gamma^1 = \mathrm{SL}(n,\mathbb{Z})$, and identify G^1 with a subgroup of G in the standard way; similarly we identify $\mathrm{SL}(d,\mathbb{R})$ with a subgroup of $\mathrm{ASL}(d,\mathbb{R})$. It follows from Ratner's work [24], [25] that there exists a unique closed connected subgroup H_g of G such that $\Gamma \cap H_g$ is a lattice in H_g, $\varphi_g(\mathrm{SL}(d,\mathbb{R})) \subset H_g$, and the closure of $\Gamma\backslash\Gamma\varphi_g(\mathrm{SL}(d,\mathbb{R}))$ in $\Gamma\backslash G$ is given by $\Gamma\backslash\Gamma H_g$ (cf. in particular [25, p. 237 (lines 1–2 and Cor. B)], and note that $\varphi_g(\mathrm{SL}(d,\mathbb{R}))$ is connected and generated by Ad-unipotent one-parameter subgroups of G). Note that $\Gamma\backslash\Gamma H_g$ can be naturally identified with the homogeneous space $(\Gamma \cap H_g)\backslash H_g$. We denote the unique right-H_g invariant probability measure on either of these spaces by μ_g; sometimes we will also let μ_g denote the corresponding Haar measure on H_g.

Similarly, there exists a unique closed connected subgroup $\widetilde{H}_g$ of G such that $\Gamma \cap \widetilde{H}_g$ is a lattice in $\widetilde{H}_g$, $\varphi_g(\mathrm{ASL}(d,\mathbb{R})) \subset \widetilde{H}_g$, and the closure of $\Gamma\backslash\Gamma\varphi_g(\mathrm{ASL}(d,\mathbb{R}))$ in $\Gamma\backslash G$ is given by $\Gamma\backslash\Gamma\widetilde{H}_g$. Note that $\Gamma\backslash\Gamma\widetilde{H}_g$ can be naturally identified with the homogeneous space $(\Gamma\cap\widetilde{H}_g)\backslash\widetilde{H}_g$.

We denote the unique right-$\widetilde{H}_g$ invariant probability measure on either of these spaces by $\mu_{\widetilde{H}_g}$; sometimes we will also use $\mu_{\widetilde{H}_g}$ to denote the corresponding Haar measure on $\widetilde{H}_g$. Of course, $H_g \subset \widetilde{H}_g$, and $\widetilde{H}_g = \widetilde{H}_{g(1_n,\boldsymbol{x})}$ for any $\boldsymbol{x} \in \mathbb{R}^d \times \{\mathbf{0}\}$.

We will refer to H_g and $\widetilde{H}_g$ as Ratner subgroups. Note that if $g \in G^1$ then $H_g \subset G^1$; in fact in this case H_g is the unique closed connected subgroup of G^1 such that $\Gamma^1 \cap H_g$ is a lattice in H_g, $\varphi_g(\mathrm{SL}(d,\mathbb{R})) \subset H_g$, and the closure of $\Gamma^1\backslash\Gamma^1\varphi_g(\mathrm{SL}(d,\mathbb{R}))$ in $\Gamma^1\backslash G^1$ is given by $\Gamma^1\backslash\Gamma^1 H_g$.

Given $g \in G$ and $\delta > 0$ we set $\mathcal{L} = \delta^{1/n}(\mathbb{Z}^n g)$ and let $\mathcal{A} = \overline{\pi_{\mathrm{int}}(\mathcal{L})}$ as before. Then $\overline{\pi_{\mathrm{int}}(\delta^{1/n}(\mathbb{Z}^n hg))} \subset \mathcal{A}$ for all $h \in \widetilde{H}_g$ and $\overline{\pi_{\mathrm{int}}(\delta^{1/n}(\mathbb{Z}^n hg))} = \mathcal{A}$ for $\mu_{\widetilde{H}_g}$-almost all $h \in \widetilde{H}_g$ and also for μ_g-almost all $h \in H_g$. We fix $\delta > 0$ and a window $\mathcal{W} \subset \mathcal{A}$, and consider the map from $\Gamma\backslash\Gamma\widetilde{H}_g$ to the set of point sets in $\mathbb{R}^d$,

$$\Gamma\backslash\Gamma h \mapsto \mathcal{P}(\mathcal{W}, \delta^{1/n}(\mathbb{Z}^n hg)). \tag{4.3}$$

We denote the image of this map by $\widetilde{\mathfrak{Q}}_g = \widetilde{\mathfrak{Q}}_g(\mathcal{W},\delta)$, and define a probability measure on $\widetilde{\mathfrak{Q}}_g$ as the push-forward of $\mu_{\widetilde{H}_g}$ (for which we will use the same symbol). This defines a random point process in $\mathbb{R}^d$ which is invariant under the natural action of $\mathrm{ASL}(d,\mathbb{R})$ on $\mathbb{R}^d$. Similarly we denote by $\mathfrak{Q}_g = \mathfrak{Q}_g(\mathcal{W},\delta)$ the image of $\Gamma\backslash\Gamma H_g$ under the map (4.3), and define a probability measure on $\mathfrak{Q}_g$ as the push-forward of μ_g; this again defines a random point process in $\mathbb{R}^d$, invariant under the natural action of $\mathrm{SL}(d,\mathbb{R})$ on $\mathbb{R}^d$.

We let $\mathfrak{Z}_\xi$ be the cylinder in $\mathbb{R}^d$ defined by

$$\mathfrak{Z}_\xi = \big\{(x_1,\ldots,x_d) \in \mathbb{R}^d : 0 < x_1 < \xi,\ x_2^2 + \ldots + x_d^2 < 1\big\}. \tag{4.4}$$

The following theorem provides formulas for the limit distributions in Theorems 3.1, 3.2 and 3.3 in terms of $\widetilde{H}_g$ and H_g.

Theorem 4.1. *Let $\mathcal{P} = \mathcal{P}(\mathcal{L},\mathcal{W})$ be a regular cut-and-project set, and $\boldsymbol{q} \in \mathbb{R}^d$. Choose $g \in G$ and $\delta > 0$ so that $\mathcal{L} - (\boldsymbol{q},\mathbf{0}) = \delta^{1/n}(\mathbb{Z}^n g)$. Then the function $F_{\mathcal{P}}(\xi)$ in Theorems 3.1 and 3.2 is given by*

$$F_{\mathcal{P}}(\xi) = \mu_{\widetilde{H}_g}(\{\mathcal{P}' \in \widetilde{\mathfrak{Q}}_g \;:\; \mathfrak{Z}_\xi \cap \mathcal{P}' = \emptyset\}). \tag{4.5}$$

In fact if $\boldsymbol{q} \in \mathbb{R}^d \setminus \mathfrak{S}$ (as in Theorem 3.2), then $H_g = \widetilde{H}_g$ and this group is independent of the choice of $\boldsymbol{q}$. On the other hand if $\boldsymbol{q} \in \mathcal{P}$, then the function $F_{\mathcal{P},\boldsymbol{q}}(\xi, r)$ in Theorem 3.3 is given by

$$F_{\mathcal{P},\boldsymbol{q}}(\xi, r) = \mu_g(\{\mathcal{P}' \in \mathfrak{Q}_g \;:\; (\mathfrak{Z}_\xi + r\boldsymbol{e}_d) \cap \mathcal{P}' = \emptyset\}) \tag{4.6}$$

with $\boldsymbol{e}_d = (0,\ldots,0,1)$.

5. The Siegel integral formula for quasicrystals

The Siegel integral formula is a fundamental identity in the geometry of numbers [27, 28]. We will prove an analogue for the space of quasicrystals, which in fact is a special case of the Siegel-Veech formula [30, Thm. 0.12]. Let $f \in \mathrm{L}^1(\mathbb{R}^d)$. Define for every $\mathcal{P} \in \mathfrak{Q}_g$ the Siegel transform

$$\widehat{f}(\mathcal{P}) = \sum_{\boldsymbol{q} \in \mathcal{P}\setminus\{\mathbf{0}\}} f(\boldsymbol{q}). \tag{5.1}$$

Recall the definition of $\delta_{d,m}(\mathcal{L})$ in (2.4); for $\mathcal{L}$ an affine lattice we extend the definition by setting $\delta_{d,m}(\mathcal{L}) := \delta_{d,m}(\mathcal{L} - \mathcal{L})$; note that $\mathcal{L} - \mathcal{L}$ is the lattice in $\mathbb{R}^n$ of which $\mathcal{L}$ is a translate.

Theorem 5.1. *Let $\mathcal{L} = \delta^{1/n}(\mathbb{Z}^n g)$ and $\mathfrak{Q}_g = \mathfrak{Q}_g(\mathcal{W},\delta)$ as above, and assume that $\mathcal{P} = \mathcal{P}(\mathcal{W},\mathcal{L})$ is regular and the map $\pi_{\mathcal{W}} : \{\boldsymbol{y} \in \mathcal{L} : \pi_{\mathrm{int}}(\boldsymbol{y}) \in \mathcal{W}\} \to \mathcal{P}$ is bijective. Then for any $f \in \mathrm{L}^1(\mathbb{R}^d)$ we have*

$$\int_{\mathfrak{Q}_g} \widehat{f}(\mathcal{P})\, d\mu_g(\mathcal{P}) = \delta_{d,m}(\mathcal{L})\mu_{\mathcal{A}}(\mathcal{W}) \int_{\mathbb{R}^d} f(\boldsymbol{x})\, d\mathrm{vol}(\boldsymbol{x}). \tag{5.2}$$

The continuity for $\xi < \infty$ of the limit distributions $F_{\mathcal{P}}$ and $F_{\mathcal{P},\boldsymbol{q}}$ in Theorems 3.1, 3.2 and 3.3 is an immediate consequence of Theorem 5.1 and the formulas in Theorem 4.1; for $F_{\mathcal{P}}$ one uses also the fact that each $\widetilde{\mathfrak{Q}}_g$ can be obtained as $\mathfrak{Q}_{g'}$ for an appropriate g'.

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School of Mathematics, University of Bristol, Bristol BS8 1TW, U.K.
j.marklof@bristol.ac.uk