NO SEMICONJUGACY TO A MAP OF CONSTANT SLOPE

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There are some special classes of interval maps. One important class consists of piecewise linear maps of constant slope (we will call them simply *maps of constant slope*). Here by "slope" we mean what is sometimes called the absolute value of slope. When we say "piecewise", at this moment we mean that there are finitely many pieces.

Maps of constant slope are easier to investigate than other maps. They also have some special properties. For instance:

Lemma 1. *Assume that a transitive piecewise monotone interval map has constant slope. Then it has a unique measure with maximal entropy, and this measure is equivalent to the Lebesgue measure.*

The proof is an application of various results: [M. and Szlenk 1980], [Blokh 1987], [Hofbauer 1981], [Denker, Keller and Urbański 1990], and [Li and Yorke 1978].

The question of which maps are conjugate or semiconjugate to maps of constant slope was being asked for a long time. The first answer was given by Parry in:

W. Parry, *Symbolic dynamics and transformations of the unit interval*, Trans. Amer. Math. Soc. **122** (1966), 368-378.

He proved:

Theorem 2. *A continuous transitive piecewise monotone interval map of positive entropy is conjugate to a map of constant slope, where the slope is the exponential of the entropy.*

The next answer was given by Milnor and Thurston in:

J. Milnor and W. Thurston, *On iterated maps of the interval*, Dynamical systems, 465-563, Lecture Notes in Math. **1342**, Springer, Berlin, 1988

(the preprint of this paper circulated much earlier).

They proved:

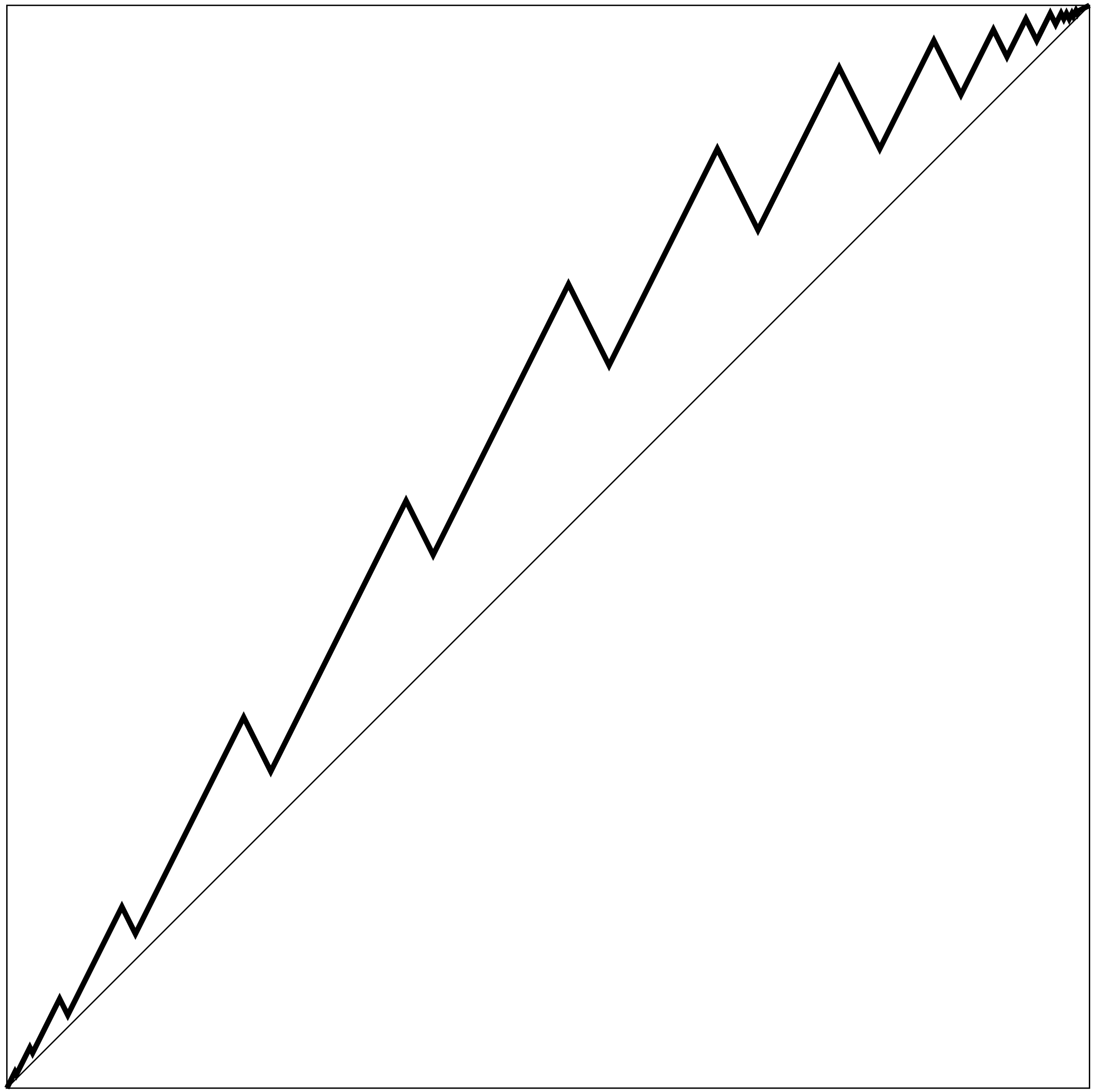
Theorem 3. *A continuous piecewise strictly monotone interval map of positive entropy is semiconjugate, via a nondecreasing map, to a map of constant slope, where the slope is the exponential of the entropy.*

A very natural way of generalizing the problem of (semi)conjugacy to a map of constant slope is to consider piecewise monotone interval maps with *countably* many pieces.

The first problem is a definition of such maps. What should we assume about the set of turning points (local extrema)? For instance, if we allow the closure of this set to be a Cantor set, there may be a substantial dynamics on it, not captured by our considerations. Thus, it is reasonable to assume that the closure of the set of turning points is countable.

The second problem is, what should the slope be? For countably piecewise monotone maps of constant slope it is no longer true that the entropy is the logarithm of the slope. There are obvious counterexamples, where all points of the interval, except the endpoints, are moved to the right, so the entropy is zero, but the slope is larger than 1. Thus, there is no natural choice of the slope of the map to which our map should be (semi)conjugate.

Here the slope is 2 but the entropy is 0:



Recently, Bobok considered the case of continuous, Markov, countably piecewise monotone interval maps in:

J. Bobok, *Semiconjugacy to a map of a constant slope*, Studia Math. **208** (2012), 213–228.

He found a necessary and sufficient condition for the existence of a nondecreasing semiconjugacy to a map of constant slope in terms of the existence of an eigenvector for a certain operator. The operator is given by a countably infinite 0-1 matrix representing the transitions in the Markov system, and the criterion asks for a nonnegative eigenvector in the space of sequences ℓ^1. Bobok described a rich class of examples satisfying this criterion and proved that for many of these examples the constant slope so obtained is the exponential of the topological entropy of the original interval map. However, he did not give any examples that violate the criterion.

Samuel Roth and I considered general countably piecewise continuous piecewise monotone interval maps in:

M. M. and S. Roth, *No semiconjugacy to a map of constant slope*, preprint (2014).

As the title suggests, we concentrate on counterexamples.

We define a class $\mathcal{C}$ of maps f for which there exists a closed, countable set $P \subset [0,1]$, $f : [0,1] \setminus P \to [0,1]$, and f is continuous and strictly monotone on each P-basic interval (a component of $[0,1] \setminus P$). Note that we assume strict monotonicity; while it is possible to do everything that we do assuming only monotonicity, the technical details would be much more involved and they would obscure the ideas.

Similarly as in measure theory, where two functions are considered equal if they differ only on a set of measure zero, we will consider two elements of $\mathcal{C}$ equal if they are equal on a complement of a closed countable set. This gives us a possibility of using different sets P for a given map $f \in \mathcal{C}$. Each such set for which f is continuous and strictly monotone on P-basic intervals will be called *f-admissible*.

Lemma 4. *A composition of two maps from $\mathcal{C}$ belongs to $\mathcal{C}$.*

By induction it follows that if $f \in \mathcal{C}$ then $f^n \in \mathcal{C}$ for every natural n.

We do not want to abandon continuous maps. Therefore we consider the class $\mathcal{CC}$ of continuous maps $f : [0,1] \to [0,1]$ for which there exists a countable closed set $P \subset [0,1]$ such that $f|_{[0,1] \setminus P} \in \mathcal{C}$. Results for maps from this class will follow easily from the results for maps from $\mathcal{C}$.

We will say that a map $f \in \mathcal{C}$ (or $f \in \mathcal{CC}$) has *constant slope* λ, if for some f-admissible set P, f restricted to each P-basic interval is affine with slope of absolute value λ. Clearly, this property depends only on the map f, and not on the choice of an f-admissible set P.

We will say that a nonatomic measure defined on the Borel σ-algebra on the interval $[0, 1]$ is *strongly σ-finite* if there is a closed countable set $P \subset [0, 1]$ such that each P-basic interval has finite measure. We denote by $\mathcal{M}$ the set of all such measures. Observe that $\mathcal{M}$ is closed under addition and under multiplication by positive real scalars.

Each map $f \in \mathcal{C}$ induces an operator $T_f : \mathcal{M} \to \mathcal{M}$ that acts on a measure $\mu \in \mathcal{M}$ as follows. Choose an f-admissible set P. For each P-basic interval I, consider the homeomorphism $f|_I : I \to f(I)$. Pull back the measure $\mu|_{f(I)}$ by $f|_I$ (that is, push it forward by $(f|_I)^{-1}$) to a measure on I. This defines $T_f\mu$ on the interval I:

$$(T_f\mu)|_I = (f|_I)^*(\mu|_{f(I)}), \quad I \in \mathcal{B}(P).$$

More explicitly,

$$(T_f\mu)(A) = \sum_{I \in \mathcal{B}(P)} \mu(f(I \cap A))$$

for all Borel sets A.

Note the linearity properties $T_f(\mu + \nu) = T_f\mu + T_f\nu$ and $T_f(\alpha\mu) = \alpha T_f\mu$ for $\mu, \nu \in \mathcal{M}$ and $\alpha \geq 0$.

Although $\mathcal{M}$ is not a true linear space (multiplication by negative scalars is not permitted), it is nevertheless quite fruitful to consider eigenvectors for positive eigenvalues. Let us denote the Lebesgue measure by m.

Lemma 5. *The map $f \in \mathcal{C}$ has constant slope λ if and only if $T_f m = \lambda m$.*

Theorem 6. *Let $f \in \mathcal{C}$ and let $\lambda > 0$. Then f is semiconjugate via a nondecreasing map φ to some map $g \in \mathcal{C}$ of constant slope λ if and only if there exists a probability measure $\mu \in \mathcal{M}$ such that $T_f \mu = \lambda \mu$.*

This theorem holds also with $\mathcal{C}$ replaced by $\mathcal{CC}$, for both f and g. Moreover, we can do the same for circle maps.

Now we want to find conditions that prevent semiconjugacy to a map of constant slope. We start with a technical lemma.

Lemma 7. *Let $f \in \mathcal{C}$. Suppose that there exist $\lambda > 2$, $\delta > 0$, $\mu \in \mathcal{M}$ and an f-admissible set P such that $T_f\mu = \lambda\mu$ and the measure of every P-basic interval I satisfies $\delta \leq \mu(I) < \infty$. Then for μ almost every x in $[0, 1]$ there exist infinitely many times $n_1 < n_2 < \dots$ such that x belongs to an interval which is mapped monotonically by f^{n_k} to an interval of μ-measure at least δ.*

Unlike in the case of piecewise monotone maps with finitely many pieces, we cannot get rid of the assumption $\lambda > 2$ by switching to an iterate of f, because it may happen that the iterates of f do not satisfy our assumptions.

The following lemma is an analog of Lebesgue's density theorem.

Lemma 8. *Let $\mu \in \mathcal{M}$ and let A be a Borel set. Then for μ almost every $x \in A$ the measures of all one-sided neighborhoods of x are positive, and*

$$\lim_{\delta \searrow 0} \frac{\mu(A \cap [x, x+\delta))}{\mu([x, x+\delta))} = \lim_{\delta \searrow 0} \frac{\mu(A \cap (x-\delta, x])}{\mu((x-\delta, x])} = 1.$$

In the next theorem we need an assumption stronger than transitivity. Since the term "strong transitivity" is sometimes used for the property that the union of the images of every nonempty open set is the whole space, we have to use another term. We will say that a map $f \in \mathcal{C}$ is *substantially transitive* if for every nonempty open set $U \in [0,1]$ the set $[0,1] \setminus \bigcup_{n=0}^{\infty} f^n(U)$ is countable.

If $f \in \mathcal{CC}$ is transitive, we get substantial transitivity automatically. We will state this fact for graph maps.

Lemma 9. *Let X be a graph and let f be a topologically transitive continuous map of X to X. Let U be a nonempty open subset of X. Then the set $X \setminus \bigcup_{n=0}^{\infty} f^n(U)$ is finite.*

The following theorem gives sufficient conditions for nonexistence of semiconjugacy to a map of constant slope with a given slope.

Theorem 10. *Let $f \in \mathcal{C}$ be a substantially transitive map and let $\lambda > 2$. Assume that there exist $\delta > 0$, an infinite measure $\mu \in \mathcal{M}$, and an f-admissible set P, such that $T_f \mu = \lambda\mu$ and the measure of every P-basic interval I satisfies $\delta \leq \mu(I) < \infty$. Then there is no probability measure $\nu \in \mathcal{M}$ such that $T_f \nu = \lambda\nu$.*

Idea of the proof of Theorem 10

Assume that such ν exists. We can replace μ by $\mu + \nu$ and then ν is absolutely continuous with respect to μ. This allows us to find a fully invariant set E of positive but finite measure μ. Then we use Lemma 8 to get high density of E in a small interval, then Lemma 7 to transport it to a long interval, and then substantial transitivity to transport it to the whole space. By the invariance of E this construction gives us infinite measure of E, which is impossible.

Even with Theorem 10, it is not immediately obvious how to construct concrete examples. In particular, how to consider all λs?

Theorem 11. *Assume that* $f : \mathbb{S}^1 \to \mathbb{S}^1$ *is a continuous degree one map that is piecewise monotone with finitely many pieces and has constant slope* $\lambda > 1$*. Assume also that* f *has a lifting* $F : \mathbb{R} \to \mathbb{R}$ *that is topologically transitive. Take any continuous interval map* $g : [0,1] \to [0,1]$ *such that* $g|_{(0,1)}$ *is topologically conjugate to* F*. Then there does not exist any nondecreasing semiconjugacy of* g *to an interval map of constant slope.*

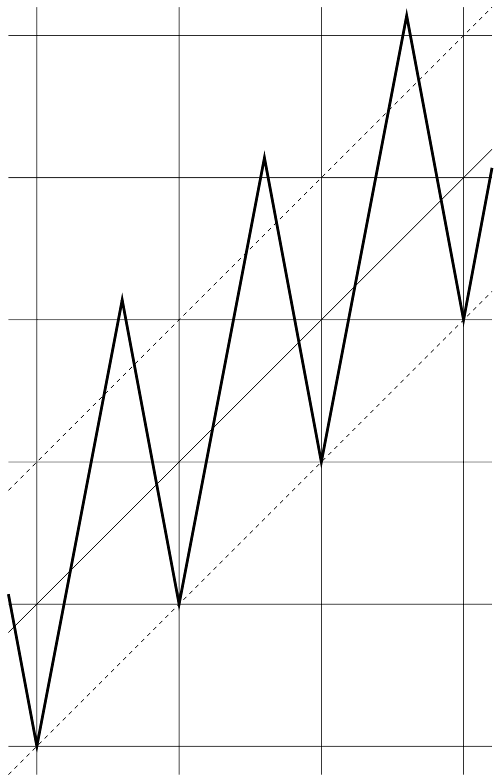
Here λs other than $e^{h(f)}$ are excluded, because semiconjugacy of g to a map of constant slope λ implies semiconjugacy of f to a map of constant slope λ. Since f is piecewise monotone with finitely many pieces, the only possibility is $\lambda = e^{h(f)}$.

In Theorem 11 there is no difficulty in finding a continuous interval map $g : [0,1] \to [0,1]$ such that $g|_{(0,1)}$ is topologically conjugate to F. Let h be any homeomorphism of $(0,1)$ with $\mathbb{R}$ and define $g = h^{-1} \circ F \circ h$ with additional fixed points at 0 and 1. We obtain continuity of g at the points $0, 1$, because F was assumed to be the lifting of a degree one circle map.

If we wish to construct explicit examples that satisfy the hypotheses of Theorem 11, the only possible difficulty is in verifying the transitivity of the lifting F. Fortunately, there is a simple condition, broadly applicable and easy to verify, that guarantees transitivity.

Theorem 12. *Assume that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a continuous degree one map that is piecewise monotone with finitely many pieces and has constant slope. Assume also that $F : \mathbb{R} \to \mathbb{R}$ is a lifting of f. Let P denote the set of turning points of F. If for each P-basic interval I there are points x_L, x_R in the closure of I such that $F(x_L) = x_L - 1$ and $F(x_R) = x_R + 1$, then F is topologically transitive.*

Let the lifting F_λ (with slope $\lambda \geq 2+\sqrt{5}$) of a circle map looks like this:



Now if we choose any homeomorphism $h : (0,1) \to \mathbb{R}$, we will get a map $g_\lambda = h^{-1} \circ F_\lambda \circ h$ (with additional fixed points at 0 and 1), which belongs to $\mathcal{CC}$, but is not semiconjugate by a nondecreasing map to a map of constant slope.

Using symbolic dynamics, we can prove:

Theorem 13. *Fix an integer* $n \geq 2$*. Then there are uncountable sets* $\Lambda_M \subset [2n+1, 2n+3]$ *and* $\Lambda_{nM} \subset [2n+1, 2n+3]$ *such that for every* $\lambda \in \Lambda_M$ *the lifting* F_λ *is Markov and for every* $\lambda \in \Lambda_{nM}$ *the lifting* F_λ *is not Markov.*