Adding machine factorizations and iterations of quadratic polynomials

V. Nekrashevych

The talk is based on a joint work with D. Dudko and L. Bartholdi.

1 Iterated monodromy groups

1.1 Algebraic definition

The following definition is due to R. Pink.

Let $f(x) \in \mathbb{C}$ be a rational function. We denote by $f^n(x)$ its nth iteration. Let $\overline{\Omega}$ be an algebraic closure of the field of functions $\mathbb{C}(t)$. Let $\Omega_n \subset \overline{\Omega}$ be the field obtained by adjoining all solutions of the equation $f^n(x) = t$ to $\mathbb{C}(t)$. It is easy to see that $\Omega_n \subset \Omega_{n+1}$. Let Ω_∞ be the union of the fields Ω_n.

Galois iterated monodromy group is the Galois group of the extension $\Omega_\infty/\mathbb{C}(t)$.

1.2 Topological version

Suppose that $f(x)$ is post-critically finite, i.e., that the union P_f of the forward orbits of critical points of f is finite. Let $\hat{\mathbb{C}}$ be the Riemann sphere. Let $\mathcal{M} = \hat{\mathbb{C}} \setminus P_f$ and $\mathcal{M}_1 = f^{-1}(\mathcal{M})$. Then $\mathcal{M}$ and $\mathcal{M}_1$ are punctured spheres, $\mathcal{M}_1 \subseteq \mathcal{M}$, and

$$f : \mathcal{M}_1 \longrightarrow \mathcal{M}$$

is a covering map. Similarly, $f^n : \mathcal{M}_n \longrightarrow \mathcal{M}$ for $\mathcal{M}_n = f^{-n}(\mathcal{M})$ is a covering map. Consider a basepoint $t \in \mathcal{M}$, and the *tree of preimages*

$$T = \bigsqcup_{n \geq 0} f^{-n}(t),$$

where $f^{-0}(t) = \{t\}$. The fundamental group $\pi_1(\mathcal{M}, t)$ acts on each level of the tree T by the classical monodromy action. Taking these actions together, we get an action of $\pi_1(\mathcal{M}, t)$ on the tree T by automorphisms. The action is not faithful in general, and the quotient of $\pi_1(\mathcal{M}, t)$ by the kernel of the action is called the *iterated monodromy group* $\mathrm{IMG}(f)$ of the map f.

2 Self-similar groups

2.1 Definition

The tree T can be identified with Cayley graph of the free monoid X^*, where $|X| = \deg f$, in a nice (though not canonical way). The iterated monodromy group (as a group acting on X^*) becomes *self-similar* in the following sense.

A group G acting faithfully on X^* is said to be *self-similar* if for every $x \in X$ and $g \in G$ there exist $y \in X$ and $h \in G$ such that

$$(vx)^g = v^h y$$

for all $v \in X^*$.

For example, $\mathrm{IMG}(z^2)$ is generated by one element a satisfying

$$(v0)^a = v1, \qquad (v1)^a = v^a 0,$$

for all $v \in X^*$. Here $X = \{0, 1\}$. We see that a acts precisely by the rule of adding 1 to a binary integer.

For $X = \{0, 1\}$ we write

$$g = (g_0, g_1)$$

if $(v0)^g = v^{g_0}0$, $(v1)^g = v^{g_1}1$, and we write

$$g = (g_0, g_1)\sigma$$

if $(v0)^g = v^{g_0}1$, $(v1)^g = v^{g_1}0$ for all v.

3 Some examples

As it was mentioned above, $\mathrm{IMG}(z^2)$ is generated by a single automorphism of the tree, defined by

$$a = (1, a)\sigma,$$

where 1 is the identity automorphism, and hence is isomorphic to the infinite cyclic group $\mathbb{Z}$.

The iterated monodromy group of $z^2 - 2$ is generated by

$$a = (1, 1)\sigma, b = (a, b).$$

It is isomorphic to the infinite dihedral group.

These two groups are the only examples of finitely presented iterated monodromy groups of post-critically finite quadratic polynomials.

The iterated monodromy group of $z^2 - 1$ is generated by two elements a and b satisfying

$$a = (1, b)\sigma, b = (1, a).$$

This group is infinitely presented. It is amenable (by a result of B. Virag and L. Bartholdi) but can not be constructed from groups of sub-exponential growth by the usual group-theoretic constructions preserving amenability (this was shown by R. Grigorchuk and A. Żuk).

The iterated monodromy group of $z^2 + i$ is generated by three elements

$$a = (1, 1)\sigma, b = (a, c), c = (b, 1).$$

It has intermediate growth (by a result of K.-U. Bux and R. Perez).

4 Adding machine factorizations

4.1 Periodic case

Let $f(z) = z^2 + c$ be a post-critically finite quadratic polynomial. Its action near infinity topologically (i.e., up to homotopy) is the same as the action of z^2. It follows that $\mathrm{IMG}(f)$ contains $\mathrm{IMG}(z^2)$ generated by the loop a around infinity. The iterated monodromy group of $f(z)$ is generated by loops around the finite post-critical points of f. We can choose these generators $a_1, a_2, \ldots, a_n$ in such a way that $a = a_1 a_2 \cdots a_n$, and thus arrive at *adding machine factorizations* in the automorphism group of the tree X^*.

Consider at first the case when the critical point 0 of $f(z) = z^2 + c$ belongs to a cycle of iterations of f. One can show, after choosing special generating set a_i and special encoding of the preimage tree by X^* that $\mathrm{IMG}\, f$ can be defined in the following way.

Let $w = x_1 x_2 \ldots \in X^\infty$ be a periodic sequence (here $X = \{0, 1\}$, as before). Denote by $s : X^\infty \longrightarrow X^\infty$ the shift map $s(x_1 x_2 \ldots) = x_2 x_3 \ldots$. The iterated monodromy group is generated by automorphisms τ_u indexed by infinite sequences $u \in \{s^n(w) \ : \ s \in \mathbb{N}\}$, and defined by the recursions:

$$\tau_u = (\tau_{0u}, \tau_{1u}),$$

if $u \neq w$, and

$$\tau_w = (\tau^{\infty}_{(1w,0w)}, \tau_{[1w,0w]})\sigma,$$

where τ_I denotes the product of all generators τ_u belonging to the interval I in the inverse lexicographic order. If u is not a shift of w, then $\tau_u = 1$.

We show that for every quadratic polynomial f with periodic critical point there exists a periodic sequence w such that the group generated by τ_u is $\mathrm{IMG}(f)$, and conversely, for every periodic sequence w the group G_w generated by τ_u is the iterated monodromy group of a quadratic polynomial with periodic critical point.

4.2 General case

In fact, one can use above recurrent relations to define automorphisms τ_u for *any* sequence w. Note that then $\tau_{(1w,0w)}$ and $\tau_{[1w,0w]}$ are products of infinite number of elements τ_u, but it is still well defined, as a product in a profinite group of a linearly ordered countable set converging to identity.

For a sequence $w \in X^\infty$, denote by G_w the group generated by products τ_I, where I is an interval whose endpoints are eventually periodic sequences.

5 Mandelbrot set

The Mandelbrot set $\mathcal{M}$ is the set of complex numbers c such that the orbit of 0 under iterations of $z^2 + c$ is bounded. It is known that the complement of $\mathcal{M}$ in $\mathbb{C}$ is bi-holomorphically isomorphic to the complement of the closed unit disc $\mathbb{D}$. Let $\Phi : \mathbb{C} \setminus \mathbb{D} \longrightarrow \mathbb{C} \setminus \mathcal{M}$ be the isomorphism tangent to identity at infinity. An *external ray* R_θ is the image of the ray $\{re^{2\pi i\theta} \;:\; r > 1\}$ under Φ. It is not known if all rays *land*, i.e., if the limit $\lim_{r\to 1+} \Phi(re^{2\pi i\theta})$ exists for every θ. This depends on the famous question of local connectivity of $\mathcal{M}$. It is known, however, that all rays with rational θ land.

Theorem. *Let $\theta_1, \theta_2 \in [0, 1]$ be rational numbers, and let $w_1, w_2 \in X^\infty$ be their binary expansions. Then R_{θ_1}, R_{θ_2} land on the same point of $\mathcal{M}$ if and only if $G_{w_1} = G_{w_2}$. If $\mathcal{M}$ is locally connected, then the same statement is true for all angles θ_1, θ_2.*