BILLIARD FLOWS, LAPLACE EIGENFUNCTIONS, AND SELBERG ZETA FUNCTIONS

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Let $\mathbb{H}$ denote the hyperbolic plane and let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a discrete subgroup of Riemannian isometries of $\mathbb{H}$, not necessarily a lattice. The relation between the geometric and the spectral properties of the hyperbolic Riemannian orbifold $X := \Gamma\backslash\mathbb{H}$ is of huge interest in various areas, including dynamical systems, spectral theory, harmonic analysis, representation theory, number theory and quantum chaos.

We present transfer operator techniques which show an intimate relation between periodic geodesics and the L^2-eigenfunctions, in particular Maass cusp forms, and resonances of the Laplace–Beltrami operator Δ on X. It is well-known that the spectral parameters of eigenfunctions and the resonances are encoded in the zeros of the meromorphic continuation of the Selberg zeta function Z_X of X, which is a generating function for the primitive geodesic length spectrum L_X:

$$Z_X(s) = \prod_{\ell \in L_X} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right).$$

The transfer operator techniques complement and enhance this result. In this talk we restrict the exposition to the Hecke triangle groups

$$\Gamma_\lambda = \left\langle S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T_\lambda := \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \right\rangle \leq \mathrm{PSL}_2(\mathbb{R}),$$

where (cofinite case) $\lambda = \lambda(q) = 2\cos\frac{\pi}{q}$, $q \in \mathbb{N}_{\geq 3}$, or (infinite co-area case) $\lambda > 2$.

First let $\lambda = \lambda(q)$ for some $q \in \mathbb{N}_{\geq 3}$. The discretization for the geodesic flow on $\Gamma_\lambda\backslash\mathbb{H}$ from [Poh14b] leads to the transfer operator family ($s \in \mathbb{C}$)

$$\mathcal{L}_{s,q} = \sum_{k=1}^{q-1} \tau_s(g_k)$$

acting on functions $f\colon \mathbb{R}_{>0} \to \mathbb{C}$, where $g_k := \left((T_\lambda S)^k S\right)^{-1}$ and

$$\tau_s\big(g^{-1}\big) f(x) := \big(|\det g| \cdot g'(x)\big)^s f(g.x)$$

for $g \in \mathrm{PGL}_2(\mathbb{R})$.

Theorem A ([MP13]). *Let $s \in \mathbb{C}$, $1 > \mathrm{Re}\, s > 0$. Then the space of Maass cusp forms for Γ_λ with eigenvalue $s(1-s)$ is isomorphic to the space of eigenfunctions $f \in C^\omega(\mathbb{R}_{>0};\mathbb{C})$ with eigenvalue 1 of $\mathcal{L}_{s,q}$ for which the function*

$$x \mapsto \begin{cases} f(x) & \text{for } x > 0 \\ |x|^{-2s} f\big(-\frac{1}{x}\big) & \text{for } x < 0 \end{cases}$$

extends smoothly to all of $\mathbb{R}$.

The proof of this theorem uses the characterization of Maass cusp forms in parabolic 1-cohomology as provided by Bruggeman–Lewis–Zagier [BLZ13]. The isomorphism from Maass cusp forms to eigenfunctions of $\mathcal{L}_{s,q}$ is effective and given by an integral transform. The converse isomorphism is an averaging process.

The Hecke triangle groups commute with the element $Q := \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathrm{PGL}_2(\mathbb{R})$, which corresponds to the Riemannian isometry $z \mapsto \overline{z}^{-1}$. Let $\widetilde\Gamma_\lambda := \langle \Gamma_\lambda, Q\rangle \leq \mathrm{PGL}_2(\mathbb{R})$ be the triangle group underlying Γ_λ. A discretization of the billiard flow (with Dirichlet resp. Neumann boundary value conditions) on $\widetilde\Gamma_\lambda\backslash\mathbb{H}$ allows us to improve Theorem A.

Theorem B ([Poh14a])**.** *Let $s \in \mathbb{C}$, $1 > \operatorname{Re} s > 0$. Then the space of odd resp. even Maass cusp forms for Γ_λ with eigenvalue $s(1-s)$ is isomorphic to the space of 1-eigenfunctions of $\mathcal{L}^-_{s,q}$ resp. $\mathcal{L}^+_{s,q}$ of the same regularity as in Theorem A, where*

$$\mathcal{L}^\pm_{s,q} = \sum_{k=\frac{q+1}{2}}^{q-1} \tau_s(g_k) \pm \tau_s(Qg_k) \qquad \textit{for } q \textit{ odd},$$

$$\mathcal{L}^\pm_{s,q} = \sum_{k=\frac{q}{2}+1}^{q-1} \tau_s(g_k) \pm \tau_s(Qg_k) + \frac{1}{2}\tau_s(g_{\frac{q}{2}}) \pm \frac{1}{2}\tau_s(Qg_{\frac{q}{2}}) \qquad \textit{for } q \textit{ even}.$$

Theorems A and B provide classical dynamical characterizations of Maass cusp forms, and the eigenfunctions of the tranfer operators serve as period functions. These new characterizations might allow novel approaches to Maass cusp forms. In particular, the Phillips–Sarnak conjecture on nonexistence of even Maass cusp forms (for large q) translates to nonexistence of highly regular 1-eigenfunctions of $\mathcal{L}^+_{s,q}$.

By inducing the discretizations on the involved parabolic elements, the associated discrete dynamical systems become uniformly expanding and hence the associated transfer operators become nuclear operators of order zero on certain Banach spaces of holomorphic functions. For the geodesic flow, this family of nuclear transfer operators is

$$\widetilde{\mathcal{L}}_{s,q} = \begin{pmatrix} 0 & \sum\limits_{k=2}^{q-2}\tau_s(g_k) & \sum\limits_{n\in\mathbb{N}}\tau_s(g^n_{q-1}) \\ \sum\limits_{n\in\mathbb{N}}\tau_s(g^n_1) & \sum\limits_{k=2}^{q-2}\tau_s(g_k) & \sum\limits_{n\in\mathbb{N}}\tau_s(g^n_{q-1}) \\ \sum\limits_{n\in\mathbb{N}}\tau_s(g^n_1) & \sum\limits_{k=2}^{q-2}\tau_s(g_k) & 0 \end{pmatrix}.$$

It allows us to represent the Selberg zeta function Z_q for $\Gamma_\lambda\backslash\mathbb{H}$ as a Fredholm determinant.

Theorem C ([MP13])**.** *We have*

$$Z_q(s) = \det\left(1 - \widetilde{\mathcal{L}}_{s,q}\right).$$

The expressions for the induced transfer operators $\widetilde{\mathcal{L}}^\pm_{s,q}$ for the billiard flow are also explicit. Their Fredholm determinants can be evaluated and provide dynamical zeta functions $Z^\pm_q(s) := \det\left(1 - \widetilde{\mathcal{L}}^\pm_{s,q}\right)$. We then have the following extension of results by Mayer [May91] and [Efr93].

Theorem D ([Poh14a]). *Let $s \in \mathbb{C}$, $\operatorname{Re} s = 1/2$. Then s is a zero of Z_q^+ resp. Z_q^- if and only if s is an even resp. odd spectral parameter. Moreover, $Z_q(s) = Z_q^+(s) Z_q^-(s)$ for $s \in \mathbb{C}$.*

By Theorem D, the Phillips–Sarnak conjecture is equivalent to the nonexistence of any 1-eigenfunction of $\widetilde{\mathcal{L}}_{s,q}^+$ for $\operatorname{Re} s = 1/2$.

The underlying geometry as well as Selberg theory leads to the following conjecture, which has already been established for $q = 3$ [CM99, LZ01].

Conjecture. The eigenspaces of $\mathcal{L}_{s,q}^\pm$ and $\widetilde{\mathcal{L}}_{s,q}^\pm$ are isomorphic.

Some parts of these results could already be extended to the infinite co-area Hecke triangle groups Γ_λ, $\lambda > 2$. As before, using tailor-made discretizations for the billiard flow on the underlying triangle group, we construct a pair of finite-term transfer operators $\mathcal{L}_{s,\lambda}^\pm$ and a pair of infinite-term nuclear transfer operators $\widetilde{\mathcal{L}}_{s,\lambda}^\pm$, all given by explicit formulas.

Theorem E ([Poh14c]). *We have*

$$Z_{\Gamma_\lambda \backslash \mathbb{H}}(s) = \det\left(1 - \widetilde{\mathcal{L}}_{s,\lambda}^+\right) \det\left(1 - \widetilde{\mathcal{L}}_{s,\lambda}^-\right).$$

Theorem E is the first example of a thermodynamic formalism approach to infinite co-area Fuchsian groups with a cusp. The previously known results for convex cocompact Fuchsian groups feature in the recent progress by Bourgain and Kontorovich towards Zaremba's conjecture [BK14].

Motivated by the finite co-area case, we make the following conjecture.

Conjecture. The 1-eigenfunctions of $\mathcal{L}_{s,\lambda}^\pm$ characterize the residue function at the resonance s.

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