This note is a summery of the talk I gave at the Dynamical Numbers conference on July 2014 at MPI Bonn based on a joint work with Barak Weiss bearing the same title. My aim in this note is to describe what I perceive as the main result in [SW14] and along the way illustrate some arguments dealing with simplifications.

Let \( X_n \) denote the space of \( n \)-dimensional unimodular lattices in \( \mathbb{R}^n \) where \( G \) is the group of \( n \times n \) matrices with determinant 1, \( \Gamma \) is the group of \( n \times n \) matrices with positive determinant, \( \kappa(x) \) is a function defined as

\[
\kappa(x) = \sup \left\{ 2^{-n} \text{vol}(B) : B \text{ is a symmetric } x \text{-admissible box with faces parallel to the hyperplanes of the axis} \right\}.
\]

Here a set \( B \) is \( x \)-admissible if \( x \cap B = \{0\} \). We refer to \( \kappa(x) \) as the Mordell constant of \( x \) and to the image of \( \kappa \) as the Mordell-Gruber spectrum. Note that the upper bound \( \kappa(x) \leq 1 \) is a consequence of Minkowski’s convex body theorem and that indeed there are lattices (such as \( \mathbb{Z}^n \)) with Mordell constant 1.

The dynamical interpretation of the Mordell constant is as follows. Let \( X_n \) denote the space of \( n \)-dimensional unimodular lattices in \( \mathbb{R}^n \) where \( G \) is the group of \( n \times n \) matrices with determinant 1, \( \Gamma \) is the group of \( n \times n \) matrices with positive determinant, \( \kappa(x) \) is a function defined as

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The dynamical interpretation of the Mordell constant is as follows. Let us denote by \( \| \cdot \| \) the \( \infty \)-norm on \( \mathbb{R}^n \) and for \( \epsilon > 0 \) let \( X_n(\epsilon) \) be the set of \( x \) where \( \epsilon \) is a constant of \( x \) and to the image of \( \kappa \) as the Mordell-Gruber spectrum. Note that the upper bound \( \kappa(x) \leq 1 \) is a consequence of Minkowski’s convex body theorem and that indeed there are lattices (such as \( \mathbb{Z}^n \)) with Mordell constant 1.

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The following measure theoretical analogue of the above topological statement will be more suitable for our discussion: Given an $A$-invariant and ergodic Radon measure $\mu$ on $X_n$, it is natural to define $\kappa_\mu$ as the $\mu$-almost sure value of the $A$-invariant function $\kappa$. By (0.2)

$$\kappa_\mu = \max \{ \kappa(x) : x \in \text{supp}(\mu) \}.$$ 

It is clear that when supp $\mu_1 \subset$ supp $\mu_2$ then $\kappa_{\mu_1} \leq \kappa_{\mu_2}$. Our discussion deals with trying to understand when does a strict inequality $\kappa_{\mu_1} < \kappa_{\mu_2}$ holds. Note that if $\mu = m_{X_n}$ is the $G$-invariant probability measure on $X_n$ then $\kappa_\mu = 1$. The following observation was the starting point of our study.

**Proposition 0.1.** Let $\mu$ be an $A$-invariant measure supported on a compact set. Then $\kappa_\mu < 1$.

**Proof.** The following short argument relies on a rather heavy tool, namely Hajós Theorem [Haj49]. In our context this theorem (which settled a conjecture of Minkowski) asserts the equality

$$X_n(1) = \sqcup \sigma UZ^n,$$

where $U < G$ is the subgroup of upper triangular unipotent matrices and $\sigma$ is a permutation matrix. A straightforward check shows the inclusion $\supset$ in the above equation and the content of Hajós' theorem is the inclusion $\subset$. What we need to take out of this theorem is the fact that any lattice in $X_n(1)$ contains a non-trivial vector on one of the axis. In particular, by (0.2), if $\kappa_\mu = 1$ then in supp $\mu$ there exists a lattice having a non-trivial vector on one of the axis. This vector could then be made as short as we wish by acting upon with a suitable element of $A$ in contradiction to the compactness of supp $\mu$. \hfill $\Box$

The following is a first approximation of the main result I wish to describe here. It generalizes the above proposition when $\mu$ is assumed to be homogeneous.

**Theorem 0.2.** Let $H_1 < H_2$ be a strict containment between two connected closed subgroups of $G$ containing $A$. Let $\mu_i$ be an $H_i$-invariant Radon measure supported on an $H_i$-orbit (i.e. a homogeneous measure). Suppose that supp $\mu_1 \subset$ supp $\mu_2$ and that $\mu_1(X_n) < \infty$. Then

$$\kappa_{\mu_1} < \kappa_{\mu_2}.$$ 

Note that the assumption $\mu_1(X_n) < \infty$ is needed as is shown by considering the orbit containment $AZ^n \subset GZ^n$ each of which supports an $A$-invariant and ergodic Radon measures having generic Mordell constants equal to 1.

\footnote{One can show that in this case $\mu_i$ are $A$-ergodic.}
The main theorem that we prove in [SW14] is stronger than Theorem 0.2 but is more elaborate to state. We attach to each $A$-invariant and ergodic homogeneous Radon measure $\mu$ an algebraic invariant; namely a certain finite dimensional $\mathbb{Q}$-algebra $A_\mu$ in such a way that if $\mu_i$ are such measures and $\text{supp} \mu_1 \subset \text{supp} \mu_2$ then there is a reversed inclusion $A_{\mu_2} \hookrightarrow A_{\mu_1}$. The associated algebras $A_\mu$ that arise in this way are of the form $\bigoplus_i \mathbb{F}_i$ where $\mathbb{F}_i$ are number fields. We say that an inclusion $A_{\mu_2} \hookrightarrow A_{\mu_1}$ is essential if it is onto when post-composing with the projections onto the number field components of $A_{\mu_1}$. Otherwise this inclusion is said to be non-essential.

**Theorem 0.3.** Let $\mu_i$ be two measures as in Theorem 0.2 but without the finiteness assumption $\mu_1 (X_n) < \infty$. Then, if the containment of the associated algebras $A_{\mu_2} \hookrightarrow A_{\mu_1}$ is non-essential, then there is a strict inequality $\kappa_{\mu_1} < \kappa_{\mu_2}$.

We end noting two things:

1. In the notation of the Theorem, if $\mu_1 (X_n) < \infty$ then the containment $A_{\mu_2} \hookrightarrow A_{\mu_1}$ is automatically non-essential and so Theorem 0.3 indeed implies Theorem 0.2.

2. In the example of orbit-inclusion $A\mathbb{Z}^n \subset G\mathbb{Z}^n$ discussed above, giving rise to an equality between the generic Mordell constants, the associated algebras turn to be $A_{\mu_1} = \bigoplus_i \mathbb{Q}$, and $A_{\mu_2} = \mathbb{Q}$. So, the (diagonal) inclusion $\mathbb{Q} \hookrightarrow \bigoplus_i \mathbb{Q}$ is essential and so this fits with Theorem 0.3.

**References**
