ON THE ENUMERATION OF SURFACE COVERINGS

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We look at branched covers of surfaces $f : Y \to X$ ramified over branch points $x_1, \ldots, x_n \in X$ with ramification profiles $\mu_1, \ldots, \mu_n,\ \mu_i \vdash d = \deg f$. As usual, we count covers with weights reciprocal to $|\mathrm{Aut} f|$.

Problem: How to enumerate topological types of branched covers with a given ramification profile $\mu_1, \ldots, \mu_n$?

Examples:

(1) *Simple Hurwitz numbers:* $X = \mathbb{C}P^1,\ \mu_1 = \ldots = \mu_{n-1} = [1^{d-2}2]$, and $\mu_n = \mu$ is arbitrary ($x_n = \infty$).
(2) *Belyi maps:* $X = \mathbb{C}P^1,\ n = 3,\ (x_1, x_2, x_3) = (0, 1, \infty)$, and μ_1, μ_2, μ_3 are arbitrary (in this case Y is an algebraic curve defined over $\overline{\mathbb{Q}}$).
(3) *Square-tiled surfaces:* $X = E$ (elliptic curve), $n = 1$ and $\mu_1 = \mu$ is arbitrary ($x_1 = 0$).
(4) *Pillowcase surfaces:* $X = \mathbb{C}P^1,\ n = 4,\ d = \deg f$ is even, $\mu_1 = \mu_2 = \mu_3 = [2^{d/2}]$, and $\mu_4 = \mu$ is arbitrary ($x_4 = \infty$).

The latter two examples are relevant to dynamics of flat billiards.

Coverings of type (2)–(4) admit an interpretation in terms of properly colored 3-valent ribbon graphs whose vertices can be colored in either black or white, and edges can be colored in either red (R), green (G) or blue (B).

Theorem 1. *(i) Belyi maps are in one-to-one correspondence with 3-valent bipartite 3-edge colored graphs with the ribbon graph structure given by the cyclic order of edges R-G-B at black vertices and R-B-G at white vertices;*

(ii) Square-tiled covers are in one-to-one correspondence with 3-valent bipartite 3-edge colored graphs with the ribbon graph structure given by the order R-G-B at all vertices;

(iii) Pillowcase covers are in one-to-one correspondence with 3-valent 3-edge colored graphs with the ribbon graph structure given by the order R-G-B at all vertices.

(Colorings are considered up to permutations of colors.)

Proof. Case (i) of this theorem is proven in [2], Lemma 2. To prove (ii) we notice that the graph in Fig. 1 (a) is a spine of a once punctured torus and its preimage on a square-tiled cover is a bipartite 3-edge colored 3-valent ribbon graph. Vice versa, every such graph uniquely covers the graph in Fig. 1 (a) respecting the colors, and this covering extends to a branched cover of a pointed torus. To prove (iii), we assume that the branch points $x_1, x_2, x_3 \in \mathbb{C}P^1$ are the cubic roots of unity. Then the preimage of the 3-prong star, cf. Fig. 1 (b), on a pillowcase cover is a 3-edge colored 3-valent ribbon graph, establishing the bijection claimed in (iii). □

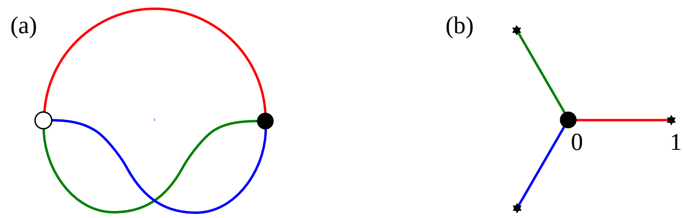

FIGURE 1

This theorem allows to enumerate branched covers (2)–(4) (at least, for small d) using well-developed routines for generating 3-valent graphs (both bipartite and not) and for counting 3-edge colorings in 3-valent graphs.

However, there are better ways to enumerate Belyi maps. Denote by $N_{k,l}(\mu)$ the number of (isomorphism classes of) Belyi maps with $k = |f^{-1}(0)|$, $l = |f^{-1}(1)|$ and ramification profile $\mu = (d_1, \ldots, d_m)$ over ∞ (the poles of f are labeled by integers from 1 to m and have orders $d_1, \ldots, d_m$). In many respects the numbers $N_{k,l}(\mu)$ behave similar to the simple Hurwitz numbers that are quite well studied. More specifically, consider the generating function

$$F(t, u, v, p_1, p_2, \ldots) = \sum_{k,l,m \geq 1} \frac{1}{m!} \sum_{\mu \in \mathbb{Z}_+^m} N_{k,l}(\mu) t^d u^k v^l \, p_{d_1} \ldots p_{d_m} \, .$$

In analogy with the generating function of simple Hurwitz numbers, F satisfies

- Virasoro constraints,
- Evolution equation of "cut-and-join" type,
- Kadomtsev-Petviashvili (KP) hierarchy,
- Topological recursion in the sense of Eynard-Orantin.

Further details one can find in [1].

Currently it is not known whether any of these (or similar) properties hold in the cases of square-tiled or pillowcase covers.

REFERENCES

[1] Kazarian, M., Zograf, P.: Virasoro constraints and topological recursion for Grothendieck's dessin counting. arXiv:1406.5976 (2014).
[2] Zograf, P.: Enumeration of Grothendieck's dessins and KP hirerarchy. arXiv:1312.2538 (2013).

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