Bounds in tropical geometry: between complex and real

We show that bounds in tropical geometry can have for some problems features of complex geometry and for other problems of real geometry.

We prove (jointly with V. Podolskii) a tropical dual effective Nullstellensatz which states that a system of tropical polynomials $f_1, \ldots, f_k$ in n variables of degrees $d_1 \geq \cdots \geq d_k$ has a solution iff a finite (truncated) submatrix of (infinite) Macauley matrix of the system has a tropical solution. The size of truncation depends on the sum of degrees $d_1 + \cdots + d_k$ in case of a tropical semi-ring without infinity. In case of a tropical semiring with infinity the size of truncation depends on the product of degrees $d_1 \cdots d_n$ (Bezout number). The bounds are close to sharp. This phenomenon resembles the classical effective Nullstellensatz in the complex geometry for which a similar bound being the sum of degrees holds in the projective case (Macauley-Lazard), while the bound being the product of degrees emerges in the affine case (Brownawell-Galligo-Giusti-Heintz-Kollar). We consider the dual Nullstellensatz since the classical Hilbert's one fails in the tropical world.

On the other hand, the sum of Betti numbers of a tropical prevariety given by $f_1, \ldots, f_k$ is less than by $\binom{k+n}{n} \cdot d_1 \cdots d_n$ (a joint result with A. Davydow), which resembles Oleinik-Petrovsky-Milnor-Thom bound on the sum of Betti numbers of a real semi-algebraic set. Again, this bound is close to sharp.