Localization in homotopy theory
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Abstract. I will attempt to describe some of the ideas of chromatic homotopy theory, focusing on the role of periodicity. I will then describe "non-chromatic" periodicity operators that occur in motivic homotopy theory. All that is new here is joint work with Michael Andrews or work of his alone.

Thanks for this invitation: it's great to be in Bonn again.

Peter explained to me that this talk was an MPI Colloquium, that happened to land in the middle of this workshop on homotopy theory. I have taken this seriously. So please bear with me, all you experts! I'll begin by giving one view of the standard picture of "chromatic homotopy theory," and then go on to describe some non-chromatic phenomena in motivic homotopy theory.

I have in mind a naive meaning for "localization," namely, the effect of inverting an operator. I want to talk about several local computations in homotopy theory. Some are quite old, and some are newer – motivic – and represent work of Michael Andrews and some joint work I've done with him.

At heart, homotopy theory is the study of "meso-scale" phenomena in geometry: perhaps high dimensional but certainly not infinite dimensional. For example we want to know $\pi_k(S^n)$. Homotopy theory is much more than merely a febrile attempt to compute these abelian groups, which is just as well because, for given $n > 1$, we never will know than a modest finite number of them in terms of generators and relations. But "chromatic homotopy theory" provides certain structural features of the homotopy theory of finite complexes that shed light on regularities among these groups, and it leads us to consider certain natural localizations.

Ideas of localization have pervaded homotopy theory for a long time. There are two preliminary localizations:

Stabilization. There is an operation on pointed spaces $(X, *)$ which preserves homology: Form the reduced cone on X and then divide by X. This is the "suspension" of X, and

$$\overline{H}_n(X) \xrightarrow{\cong} \overline{H}_{n+1}(\Sigma X)\,.$$

Suspension kills the cup product in cohomology; but otherwise the spaces X and ΣX look quite similar. There is a rather simple process by which we may invert the operator Σ acting on pointed spaces. We receive thereby the "Stable Homotopy Category" $\mathcal{S}$, along with a functor

$$\Sigma^\infty : \mathbf{Top}_* \to \mathcal{S}$$

The objects in $\mathcal{S}$ are (unfortunately) called "spectra." $\mathcal{S}$ is an additive category, and the mapping cone construction provides it with a triangulation. It's a kind of non-abelian derived category. I might use curly brackets to denote Hom in this category.

Since homology, or generalized homology, commutes with suspension, we may define the homology of a spectrum. For example $\pi_n(X) = \{S^n, X\}$, "stable homotopy," is a homology theory, with coefficient ring $\pi_* = \pi_*(S^0)$. Similarly with cohomology. "Brown representability" asserts that cohomology – in fact any generalized cohomology theory – is representable in this category. Also, the smash product (product modulo axes) descends to a symmetric monoidal structure – it's a tensor triangulated category, with unit given by S^0, i.e.$(S^0, 0)$.

Arithmetic localization. Once we've eliminated the fundamental group, the homotopy type of a space can be analyzed one prime at a time. Serre described a way to localize at a prime p on the algebra side, and later Sullivan, Quillen, and Bousfield pushed this construction into topology. Stably, this is very easy: form the sequence

$$X \xrightarrow{n_1} X \xrightarrow{n_2} \cdots$$

where the numbers $n_1, n_2, \ldots$ run through all the positive integers prime to p. Then we take the direct limit. (Actually, we want to form a homotopy theoretic version of the direct limit, making the construction homotopy invariant. Appropriately for this occasion, it's known as a *telescope*.) We are inverting all primes other than p, and we'll write $X_{(p)}$ for the result.

This turns out to be an example of "Bousfield localization." For any homology theory E_*, there is a functor L_{E_*} from spaces to spaces – or spectra to spectra – and a natural transformation $X \to L_{E_*}X$ that is in the homotopy category the terminal $E_*(-)$-isomorphism. In fact

$$X_{(p)} = L_{\pi_*(-)\otimes\mathbb{Z}_{(p)}} X\,.$$

We will consider p-local spectra only; S^0 is shorthand for $S^0_{(p)}$, for example. Here are two avatars of the kind of localization theorem I have in mind.

[Serre] $p^{-1}X \xrightarrow{\simeq} L_{H\mathbb{Q}_*}(X)$.

Inverting p is the only interesting thing to do, by virtue of:

[**Nishida**] The nonzero elements of $\pi_0(S^0)$ exhaust the non-nilpotent elements in the graded ring $\pi_*(S^0)$.

Chromatic homotopy theory provides generalizations of these two facts.

Localization of the Moore spectrum. Of course $\pi_0(S^0) = \mathbb{Z}$, given by degree. Next, the Hopf map $\eta : S^3 \to S^2$ stabilizes to a generator of $\pi_1(S^0) = \mathbb{Z}/2\mathbb{Z}$. This famous element has a cousin at every prime p, less famous but no less important: α_1, a generator of the lowest-dimensional p-torsion. It is of order p:

$$\pi_{2p-3}(S^0) \cong \mathbb{Z}/p\mathbb{Z}\,.$$

For large p this group would be very complicated if we had not tensored with $\mathbb{Z}_{(p)}$!

Suspend α_1 once: $\alpha_1 : S^{2p-2} \to S^1$. Saying that $p\alpha_1 = 0$ is the same as saying that the diagonal factorization occurs in

$$\begin{array}{ccc} S^{2p-2}/p & \overset{v_1}{\dashrightarrow} & S^0/p \\ \uparrow & \searrow & \downarrow \\ S^{2p-2} & \xrightarrow{\alpha_1} & S^1 \end{array}$$

where S^{2p-2}/p denotes the effect of "coning off" the map $p : S^{2p-2} \to S^{2p-2}$:

$$S^{2p-2}/p = S^{2p-2} \cup_p e^{2p-1}\,.$$

For p odd this new map also has order p, and we receive a further factorization through a self-map of S^0/p, of degree $2p-2$, written v_1.

The map v_1 is certainly essential, since α_1 is. The interesting thing is that it is *non-nilpotent*: All of the composites

$$S^{2p-2}/p \longrightarrow S^0/p \longrightarrow S^{-(2p-2)}/p \longrightarrow S^{-2(2p-2)}/p \longrightarrow \cdots$$

are essential. You can't see this using homology! But there are other homology theories, and in fact v_1 induces an isomorphism in complex K-theory.

A principle interest in non-nilpotent self-maps is that they give rise to infinite families of elements in $\pi_*(S^0)$:

$$\begin{array}{ccc} S^{(2p-2)k}/p & \xrightarrow{v_1^k} & S^0/p \\ \uparrow & & \downarrow \\ S^{(2p-2)k} & \xrightarrow{\alpha_k} & S^1 \end{array}$$

each of which which turns out to be essential.

I can form the "mapping telescope" of this sequence to obtain a spectrum

$$v_1^{-1}S^0/p\,.$$

This is a *periodic* spectrum: it is isomorphic to its own $(2p-2)$-fold suspension. It has trivial homology but the map from S^0/p to it is a K-theory isomorphism so it's certainly nontrivial. Here are a couple of homotopy classes in $\pi_*(S^0/p)$:

$$\begin{array}{ccc} S^0 & \xrightarrow{\iota} & S^0/p \\ {\scriptstyle \alpha_1}\uparrow & \nearrow {\scriptstyle \iota\alpha_1} & \\ S^{2p-3} & & \end{array}$$

Computation. $v_1^{-1}\pi_*(S^0/p) = \pi_*(v_1^{-1}S^0/p) = \mathbb{F}_p[v_1^{\pm 1}]\langle \iota, \iota\alpha_1\rangle$.

Since $S^0/p \to v_1^{-1}S^0/p$ is a K-theory isomorphism, we get a canonical map, which turns out to be an equivalence:

Theorem. *$v_1^{-1}S^0/p \xrightarrow{\simeq} L_K S^0/p$.*

Question. Is there a more general expression, analogous to our expression of Serre's theorem, valid for all spectra? We shall see.

Adams, Smith, Toda. Something similar happens at $p = 2$; it's just a little more complicated. Adams constructed a K-theory isomorphism

$$v_1^4 : S^8/2 \to S^0/2\,,$$

and Mahowald computed $v_1^{-1}\pi_*(S^0/2)$. Adams also considered Moore spectra with larger cyclic groups, using the J-homomorphism.

Now let's see: $p : S^0 \to S^0$ and then (if $p > 2$) $v_1 : S^{2p-2}/p \to S^0/p$. This suggests that the the mapping cone

$$S^0/p, v_1$$

might have an interesting self-map of it. Larry Smith found one, for $p > 3$:

$$v_2 : S^{2p^2-2}/p, v_1 \to S^0/p, v_1 \, .$$

Both sides have trivial K-theory now, so he used MU to detect non-nilpotence; v_2 acts non-nilpotently in MU_*. Today we could equally well say that it is an isomorphism in a Morava K-theory. Morava K-theories form a family of ring spectra associated with complex bordism and hence controlled by the theory of one-dimensional formal groups. There's one for each n, with

$$\pi_*(K(n)) = \mathbb{F}_p[v_n^{\pm 1}] \, , \quad |v_n| = 2p^n - 2 \, .$$

This was hard work. It was continued by Toda and has been taken up by others to the present day – Behrens, Hopkins, Mahowald, Hill, ... It seemed that these finite complexes were very special; they enjoyed an extra symmetry in the form of an interesting non-nilpotent self-map.

Ravenel's conjectures; Hopkins-Smith. In the late 1970s Doug Ravenel formulated a series of conjectures, most of which were soon verified in work by Mike Hopkins and Jeff [unrelated to Larry] Smith. First of all,

Theorem. If X is finite and $f : \Sigma^? X \to X$ is nilpotent in MU_*, then f is nilpotent.

This generalizes Nishida's theorem. But there's more. Say that a finite spectrum X has "type n" if

$$K(i)_*(X) = 0 \quad \text{for} \quad i < n \, .$$

Theorem. (1) Let X be of type n. For some $k \geq 0$ there is a map

$$\phi : \Sigma^? X \to X$$

that induces multiplication by $v_n^{p^k}$ on $K(n)_*(X)$: a "v_n-self-map."
(2) v_n-self-maps are essentially canonical: Let $f : X \to Y$ be any map of type n spectra, let ϕ_X be any v_n-self-map of X and ϕ_Y any v_n-self-map of Y.

Then for some l and m, the following diagram commutes (in which I omit suspensions):

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \phi_X^{p^l}} & & \downarrow{\scriptstyle \phi_Y^{p^m}} \\ X & \xrightarrow{f} & Y \end{array}$$

(3) For any n there are finite spectra of type n and not of type $n+1$, and if X is of type n for every n then it's contractible.

So there's nothing special about these "generalized Moore spaces"! There's a *universal* "almost symmetry" on finite complexes of type n.

Among other things, this work shows that the telescopes described above are canonically associated to the finite spectrum you started with. In fact they are cases of a general construction: Start with any spectrum X; cone off all maps from finite $K(n)$-acyclic spectra to get $X \to X_1$; and continue. In the direct limit you get the "finite localization"

$$X \to L_n^f X$$

with respect to $K(n)$. Now if X is of type n and ϕ is a v_n-self map, then

$$\phi^{-1} X = L_n^f X \,.$$

Since $X \to L_n^f X$ is a $K(n)$-isomorphism, we get a map

$$L_n^f X \to L_{K(n)}(X)$$

This won't be an isomorphism in general, because on finite complexes the Morava K-theories are not independent of one another. Ravenel proved that if X is finite then $K(n)_*(X) = 0 \Longrightarrow K(n-1)_*(X) = 0$. So a $K(n)$ acyclic finite spectrum is actually acyclic with respect to

$$K(\leq n) = K(0) \vee \cdots \vee K(n)$$

as well. Write L_n for $L_{K(\leq n)}$. Now

$$X \to L_n^f X$$

is a $K(\leq n)$-equivalence, so we get a natural map

$$L_n^f X \to L_n X \,.$$

The right hand side here is amenable to attack by high powered machinery. In the case $n = 1$ for example it's explictly constructible by means of Adams operations. The Computation above (and an analogue at $p = 2$ due to Mark Mahowald) leads to the

Telescope Theorem. $L_1^f X \xrightarrow{\cong} L_1 X$.

This is an analogue of Serre's theorem.

For $n > 1$ the whole of the theory of formal groups can be brought to bear on it. So the following question due to Doug Ravenel is very important!

Telescope Question: Is this map an equivalence?

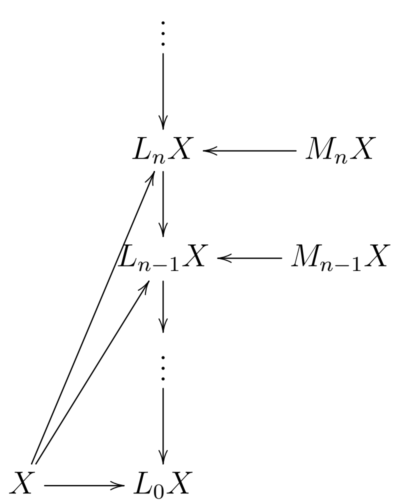
In my view this is the major outstanding conjecture in chromatic homotopy theory. A single finite complex of type n but not $n+1$ for which the conjecture is true would verify itfor all such. But Ravenel himself, and others, have spent a great deal of effort on it, and have presented evidence that it is probably false for $n > 1$.

And the corresponding localization questions are largely unanswered.

Example: What is $v_2^{-1}\pi_*(S^0/p, v_1) = \pi_*(L_2^f S^0/p, v_1)$?

We do not know. The computation of $\pi_*(L_2 S^0/p, v_1)$ is known and quite interesting – it's 12 dimensional over $\mathbb{F}_p[v_2^{\pm 1}]$ – and a great deal of work has gone in to understanding $\pi_*(L_2 S^0/p)$ and $\pi_*(L_2 S^0)$.

Because they localize at larger and larger spectra, the L_n's fit into a tower, the "chromatic tower":



This is like a prism, breaking the white light of X into its constituent colors.

Novikov. There is a systematic way to use operations in a cohomology theory to get more information than merely evaluating it on a map: the Adams spectral sequence. Since MU detects non-nilpotence, the Adams spectral sequence based on MU is particularly adapted to picking out chromatic phenomena. It takes the form

$$E_2^s(X; MU) = \mathrm{Ext}^s_{MU_*MU}(MU_*, MU_*(X)) \Longrightarrow \pi_*(X)\,.$$

Long ago, Doug Ravenel, Steve Wilson and I used this spectral sequence to study chromatic phenomena.

While every non-nilpotent self-map of X (for X finite) is detected on the 0-line, there are lots of non-nilpotent elements in E_2. For example, when $p = 2$ the element detecting η is non-nilpotent! One would like to understand $\eta^{-1}E_2(S^0; MU)$. I will describe it.

The MU "Hopf invariant" is a homomorphism

$$\pi_n(S^0) \to E_2^{1,n+1}\,.$$

These groups are trivial for n odd, and in his initiating work on this spectral sequence, Novikov proved that the other groups are cyclic and computed their order. Doug, Wilson, and I showed that with the exception of the one in $E_2^{1,4}$ all the generators of these cyclic groups support η towers.

A new result:

Theorem (with Michael Andrews) These generate the η-free part of E_2.

What happens for p odd, by the way? $\alpha_1^2 = 0$ since it's odd dimensional. On the other hand homotopy theory offers us a variant of α_1^2, namely the p-fold Massey product of α_1 with itself. This is β_1, and that element is non-nilpotent in E_2. By Nishida's theorem it has to be nilpotent in π_*, but it tries hard not to be: $\beta_1^{p^2-2p+1} \neq 0$. It would be great to compute the localization $\beta_1^{-1}E_2$, but I have no idea how to do that.

Motivic homotopy theory. We now know that very nearby there are worldsheets containing alternate realities, homotopy theories very much like the one we inhabit but different. They are controlled by a choice of base field k. These "motivic" homotopy theories, denoted $\mathcal{S}(k)$, were defined by Morel and Voevodsky, and have been much studied in the past decade or so. I am a novice at this, but Voevodsky and others have done enough to let a simple homotopy theorist such as myself have some fun here.

The first thing to know is that homotopy groups are naturally *bi*-graded, and the Hopf map lies in $\pi_{1,1}(S^0)$. The second thing to realize is:

η is non-nilpotent motivically.

This explains why MU thought that η was non-nilpotent; MU knew about motivic homotopy theory long before we did.

Let's just stick to the simplest case, $k = \mathbb{C}$. In that case there is a realization functor $\mathcal{S}(\mathbb{C}) \to \mathcal{S}$, called "taking complex points."

It turns out there is a motivic analogue of the Novikov spectral sequence, analyzed by Hu, Kriz, and Ormsby and by Dugger and Isaksen. Its E_2 term is the classical one extended by a polynomial generator; so the computation we did above gives a computation of the η localization of the motivice Novikov E_2 term. There is a d_3, and we end up with the following computation, verifying a conjecture of Guillou and Isaksen:

Theorem (with Andrews). There are elements

$$\sigma \in \pi_{7,4}(S^0) \quad , \quad \mu \in \pi_{9,5}(S^0)$$

such that

$$\eta^{-1}\pi_{*,*}(S^0) = \pi_{*,*}(\eta^{-1}S^0) = \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu]/\sigma^2 \,.$$

So there are non-chromatic periodic operators in motivic homotopy. A related study of thick subcategories of the motivic homotopy category has been made by Ruth Joachimi in her thesis.

The chromatic experience leads one to wonder whether now S^0/η has an interesting non-nilpotent self-map.

Theorem (Andrews). There is a non-nilpotent self-map

$$w_1 : S^{20,12}/\eta \to S^{0,0}/\eta \,.$$

A periodicity operator of this type was studied many years ago, by Harvey Margolis and Mark Mahowald. Apparently they also knew about the existence of motivic homotopy theory long before the rest of us did.

In fact Andrews has a sketch of a construction of a spectrum X_n in $\mathcal{S}(\mathbb{R})$ with a self-map whose real points give a v_n self map of a type n spectrum, and which base-changes to $\mathcal{S}(\mathbb{C})$ to an operator with properties extending those of w_1. The detection scheme he used for powers of w_1 doesn't work anymore, though, so we don't yet know whether w_n is non-nilpotent.

My guess is that there is a second series of non-nilpotent operators in motivic homotopy over $\mathbb{C}$, reflecting the existence of the real subfield. v_n corresponds to Milnor's ξ_{n+1}. The element η could be written w_0. It corresponds to ξ_1^2, and the higher w_n's should correspond to the squares of the other ξ_n's. Since it is beyond the standard color range of chromatic homotopy theory, I propose to call it "technicolor." Technicolor was an MIT high tech spin-off, introduced one century ago, in 1916. The current MIT campus opened in Cambridge opened the same year. This cinematic technique did not use colored pigments. Instead, each frame had two pictures. A red beam was projected through one and a green beam through the other. Just so; we have the chromatic family, in green, but now also a parallel family, in red.

Thank you very much!