Higher enveloping algebras and configuration spaces of manifolds

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13 February 2017

Configuration spaces

If M is an n-manifold, the *configuration space* of k unordered points in M is

$$B_k(M) = \{(x_1, \ldots, x_k) \in M^k \mid x_i \neq x_j \text{ if } i \neq j\}/\Sigma_k.$$

The homotopy type of this space can be complicated, but maybe we can understand its rational homology.

Plan

Set $B(M) = \coprod_k B_k(M)$.

$$\begin{array}{ccccc}
\text{Global data} & \quad & C_*(B(M)) & \xrightarrow{\sim} & !!! \\
\uparrow{\scriptstyle \text{hocolim}} & & \uparrow & & \uparrow \\
\text{Local data} & & C_*(B(\mathbb{R}^n)) & \xrightarrow{\sim} & ???
\end{array}$$

Locality

$$\mathcal{D}(M) \subset \mathrm{Op}(M) \xrightarrow{A := C_*(B(-))} \mathcal{C}\mathrm{h}_{\mathbb{Q}}$$

Objects $U \cong \amalg_k \mathbb{R}^n$, $k > 0$

Arrows π_0-surjections

A sends π_0-bijections to quasi-isomorphisms and disjoint union to tensor product.

Jargon A is a (nonunital, locally-constant) *factorization algebra.*

Key fact $\mathrm{hocolim}_{\mathcal{D}(M)} A \simeq C_*(B(M))$.

Local structure

For $U \cong \mathbb{R}^n$, $A(U)$ has a "Hopf" structure:

- To multiply, stack configurations:

$$\begin{array}{ccc} A(U) \otimes A(U) & \dashrightarrow & A(U) \\ \wr\uparrow & & \uparrow \\ A(V) \otimes A(W) & \xrightarrow{\sim} & A(V \amalg W) \end{array}$$

- To comultiply, split configurations:

$$\{x_1, \ldots, x_k\} \mapsto \sum_{i+j=k,\, \mathrm{Sh}(i,j)} \{x_{\sigma(1)}, \ldots, x_{\sigma(i)}\} \otimes \{x_{\sigma(i+1)}, \ldots, x_{\sigma(k)}\}$$

Over $\mathbb{Q}$, cocommutative Hopf algebras are enveloping algebras...

Higher enveloping algebras

Theorem (K)

There are

(1) *an adjunction of ∞-categories*

$$\mathrm{Shv}^{\mathrm{loc}}(M, \mathrm{Alg}_{\mathcal{L}}(\mathcal{C}\mathrm{h}_{\mathbb{Q}})) \underset{\longleftarrow}{\overset{U_M}{\longrightarrow}} \mathrm{Fact}^{\mathrm{loc}}_{\mathrm{nu}}(M, \mathcal{C}\mathrm{h}_{\mathbb{Q}});$$

(2) *a natural equivalence*

$$U_M(\mathfrak{g}) \simeq C^{\mathcal{L}}(\Gamma_c(\mathfrak{g})),$$

where $C^{\mathcal{L}}$ is the Chevalley-Eilenberg complex

$$C^{\mathcal{L}}(\mathfrak{h}) = \big(\mathrm{Sym}(\mathfrak{h}[1]), \pm[-,-]\big).$$

Plan revisited

$$
\begin{array}{ccccc}
\text{Global data} & & C_*(B(M)) & \xrightarrow{\ \sim\ } & C^{\mathcal{L}}(\Gamma_c(M;\mathcal{L}(\mathbb{Q}^w[n-1]))) \\
\uparrow{\scriptstyle \text{hocolim}} & & \uparrow & & \uparrow{\scriptstyle (2)+\epsilon} \\
\text{Local data} & & C_*(B(\mathbb{R}^n)) & \xrightarrow{\ \sim\,(1)\ } & U_{\mathbb{R}^n}(\mathcal{L}(\mathbb{Q}[n-1]))
\end{array}
$$

Ignoring some subtleties...

Theorem (K)

There is a bigraded isomorphism

$$H_*(B(M);\mathbb{Q}) \cong H^{\mathcal{L}}(\mathfrak{g}_M),$$

where

$$\mathfrak{g}_M = \begin{cases} H_c^{-*}(M;\mathbb{Q}^w) \otimes v & n \text{ odd} \\ H_c^{-*}(M;\mathbb{Q}^w) \otimes v \oplus H_c^{-*}(M;\mathbb{Q}) \otimes [v,v] & n \text{ even} \end{cases}$$

and $|v| = (n-1, 1)$.

This generalizes results of Bödigheimer-Cohen-Taylor, Bödigheimer-Cohen, and Félix-Thomas.

The Chevalley-Eilenberg complex allows for many computations.

Theorem (Drummond-Cole-K)

The unstable Betti number $\dim H_i(B_{i-1}(\Sigma_g))$ *is equal to*

$$\sum_{j=0}^{g-1}\sum_{m=0}^{j}(-1)^{g+j+1}\frac{2j-2m+2}{2j-m+2}\binom{\frac{6j+2i+2g-2m-5-3(-1)^{i+j+g+m}}{4}}{m,2j-m+1}$$

for $i \geq 5$*, with special cases*

$$\dim H_i(B_{i-1}(\Sigma_g)) = \begin{cases} 0 & i=1 \\ 1 & i=2 \\ 0 & i=3 \\ 2g & i=4. \end{cases}$$

Models

$$\mathrm{Shv}^{\mathrm{loc}}(M, \mathrm{Alg}_{\mathcal{L}}(\mathcal{C}\mathrm{h}_{\mathbb{Q}})) \xleftarrow{\quad} \mathrm{Fact}^{\mathrm{loc}}_{\mathrm{nu}}(M, \mathcal{C}\mathrm{h}_{\mathbb{Q}}), \qquad U_M : \mathrm{Shv}^{\mathrm{loc}}(M, \mathrm{Alg}_{\mathcal{L}}(\mathcal{C}\mathrm{h}_{\mathbb{Q}})) \to \mathrm{Fact}^{\mathrm{loc}}_{\mathrm{nu}}(M, \mathcal{C}\mathrm{h}_{\mathbb{Q}})$$

Idea Factorization algebras *are* Lie algebras (somewhere).

$$\left\{ \begin{matrix} \text{monoidal} \\ \text{model} \end{matrix} \right\} \xrightarrow{\text{convolution}} \left\{ \begin{matrix} \text{coalgebra} \\ \text{model} \end{matrix} \right\} \xrightarrow{\text{duality}} \left\{ \begin{matrix} \text{Lie} \\ \text{model} \end{matrix} \right\}$$

Puzzle How is a symmetric monoidal functor like a cocommutative coalgebra?

(Right) Day convolution

Given symmetric monoidal $F, G : \mathcal{V} \to \mathcal{W}$, define $F \otimes^{\mathrm{RD}} G$ by right Kan extension:

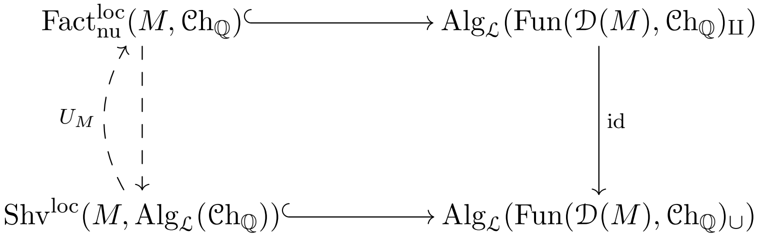
$$\begin{array}{ccccc} \mathcal{V}\times\mathcal{V} & \xrightarrow{F\times G} & \mathcal{W}\times\mathcal{W} & \xrightarrow{\otimes} & \mathcal{W} \\ {\scriptstyle\otimes}\downarrow & & & \nearrow{\scriptstyle F\otimes^{\mathrm{RD}}G} & \\ \mathcal{V} & & & & \end{array}$$

This operation is usually not associative, but when it is, we have

$$\mathrm{Fun}^{\otimes}(\mathcal{V}, \mathcal{W}) \subseteq \mathrm{Fun}^{\mathrm{oplax}}(\mathcal{V}, \mathcal{W}) \simeq \mathrm{Coalg}_{\mathrm{Com}}(\mathrm{Fun}(\mathcal{V}, \mathcal{W}))$$

Adjunction

$$(F \otimes^{\amalg} G)(I \times \mathbb{R}^n) \simeq \bigoplus_{I = I_1 \amalg I_2} F(I_1 \times \mathbb{R}^n) \otimes G(I_2 \times \mathbb{R}^n)$$



$$(F \otimes^{\cup} G)(I \times \mathbb{R}^n) \simeq \bigoplus_{I = I_1 \cup I_2} F(I_1 \times \mathbb{R}^n) \otimes G(I_2 \times \mathbb{R}^n)$$