Segal Approach for Algebraic Structures

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Segal objects

Γ-spaces (Segal, 1974)

Denote by **Fin** the category of finite sets and by $\mathbf{Fin}_+$ the category with the same objects as in **Fin**, and with morphisms given by partially defined maps, $S \supset U \to T$.

Denote 1 a one-element set. Given a finite set S, its elements $\{s\} \subset S$ induce the morphisms $\rho_s : S \hookleftarrow 1 \to 1$ in $\mathbf{Fin}_+$. A Γ-space is then a functor

$$A : \mathbf{Fin}_+ \longrightarrow \mathbf{Top},$$

such that the induced morphism (*the Segal map*)

$$\prod A(\rho_S) : A(S) \longrightarrow A(1)^S$$

is a (weak) homotopy equivalence for each $S \in \mathbf{Fin}_+$.

Homotopy coherent multiplication

Given $S \in \mathbf{Fin}_+$, we have the usual map $\pi_S : S \overset{=}{\hookleftarrow} S \to 1$ defined everywhere on S. Consider the span

$$\begin{array}{ccc} & A(S) & \\ {\scriptstyle \prod A(\rho_S)} \swarrow {\scriptstyle \sim} & & \searrow {\scriptstyle A(\pi_S)} \\ A(1)^S & & A(1). \end{array}$$

which left arrow is a homotopy equivalence. Inverting it, we obtain multiplication operations $m_S : A(1)^S \to A(1)$.

In $\mathrm{Ho}\,\mathbf{Top}$, the homotopy type of $A(1)$ is a commutative monoid. But a Γ-space contains much more information than an H-space ("Segal delooping machinery").

Operator categories (Barwick 2013)

One can replace **Fin** by **Ord**, the category of finite ordered sets. Somewhat equivalently, one can work with Δ^{op} instead of $\mathbf{Fin}_+$. A way to systematise:

Definition. An *operator category* C is a small category with a terminal object 1, such that the hom-sets $C(1, x)$ are finite for each $x \in C$ and that pullbacks exist along any map $1 \to x$ of C.

Examples. Already mentioned **Fin**, **Ord**. Consider also the category **B**:

1. Its objects are injections $f : S \hookrightarrow D$ with domain $S \in \mathbf{Fin}$, into the 2-disc D same thing as $|S|$ distinct points in D.
2. A map between $f : S \hookrightarrow D$ and $f' : S' \hookrightarrow D$ is given by a set map $\alpha : S \to S'$ and by a path f to $f' \circ \alpha$ in the stratified groupoid $\Pi_1^{EP}(Cf(S, D))$ of the configuration space $Cf(S, D) = \{S \to D\}$.

Intuition: **B** is like **Fin**, but with braid-automorphisms.

Algebra Classifiers

Definition. Let C be an operator category. Its *algebra classifier* is the category C_+ such that

1. $\mathrm{Ob}\, C_+ = \mathrm{Ob}\, C$,
2. the hom-sets $C_+(x, y)$ are given by equivalence classes of span diagrams $x \hookleftarrow z \to y$, where $z \to y$ is in C and $z \hookrightarrow x$ is an *admissible monomorphism*: a composition of pullbacks of elementary (admissible) monos $1 \to t$.

For example, $\mathbf{Fin}_+$ is the same category as before. For **Ord**, the admissible monos are interval inclusions. All monos are admissible in **B**.

A morphism in C_+ is *inert* if it can be presented as $z \hookleftarrow x \xrightarrow{=} x$, and *active* if it can be presented as $y \overset{=}{\hookleftarrow} y \to t$. Inert and active maps form a factorisation system (In_C, Act_C) on C_+.

Homotopical algebra with Segal objects

Given an operator category C, we can now define a Segal C-spaces as a functor $A : \mathrm{C}_+ \to \mathbf{Top}$ with Segal conditions: that

$$\prod A(\rho_x) : A(x) \longrightarrow A(1)^{|x|}$$

is a homotopy equivalence for $x \in \mathrm{C}_+$, $|x| = \mathrm{C}(1, x)$.

For example, Segal **Ord**-spaces are homotopy associative monoids.

Because of the correspondence between constructible sheaves on $Cf(S, D)$ and functors $\Pi_1^{EP}(Cf(S, D)) \to \mathbf{Top}$, Segal **B**-spaces are the same data as constructible factorisation spaces over a 2-disc, hence E_2-algebras in **Top**.

Barwick shows how to construct operator categories related to E_n-spaces. There are also other approaches (Batanin-Markl, Berger).

Resolutions

Resolutions (of operator categories)

Definition. A functor $F : \mathcal{D} \to \mathcal{C}$ is a *resolution* if for each $\mathbf{c}_{[n]} = c_0 \to \ldots \to c_n$ de $\mathbb{C}$, the category $\mathcal{D}(\mathbf{c}_{[n]})$ of strings $d_0 \to \ldots \to d_n$ in $\mathsf{Fun}([n], \mathcal{D})$ with isomorphisms $(Fd_0 \to \ldots \to Fd_n) \cong \mathbf{c}_{[n]}$ is contractible.

Formal examples: $\int N\mathcal{C} \to \mathcal{C}^{\mathsf{op}}$, $\int N\mathcal{C} \to \mathcal{C}$, equivalences of categories, (op)fibrations with contractible fibres.

Definition. A functor $F : \mathrm{D} \to \mathrm{C}$ between operator categories is a resolution if:

1. The functor F preserves limits and $\mathrm{D}(1, x) \cong \mathrm{C}(1, F(x))$,
2. The functor F is a resolution.

NB: No requirement for $\mathrm{D}_+ \to \mathrm{C}_+$ to be a resolution!

Example: classifying spaces and representations

Let G and denote BG its classifying space. Let I be a regular cell decomposition of BG, viewed as a poset. There is a natural functor

$$F : I \longrightarrow \Pi_1(BG) \cong G$$

given by choosing a point in each cell. The functor F is a resolution.

Denoting $\mathcal{D}(\mathcal{C}, k)$ the derived category of functors $\mathcal{C} \to \mathbf{DVect}_k$, we have

$$F^* : \mathsf{DRep}(G, k) \cong \mathcal{D}(\Pi_1(BG), k) \to \mathcal{D}(I, k) \cong \mathcal{D}(A(I)\text{-}\mathbf{Mod}).$$

One can prove that F^* is full and faithful, and that its image consists of $X : I \to \mathbf{DVect}_k$ which are *locally constant*: for each $f : i \to i'$ de I, $X(f)$ is a quasi-isomorphism.

Does this example arise in operator categories?

Planar trees

A *planar tree* T is

1. a connected unoriented graph without cycles possessing a distinguished vertex of valency 1 called the root,
2. both the set of vertices $V(T)$ and the set of edges $E(T)$ are finite,
3. for each $v \in V(T)$ there is a datum of cyclic order on the set of edges attached to v. This makes T into an oriented graph.

A morphism $f : T \to T'$ is an oriented cellular map $|f| : |T| \to |T'|$ between the geometric realisations, such that

1. $|f|$ preserves the roots,
2. for each $a, b \in V(T)$, the image by $|f|$ of a geodesic between a and b in $|T|$ is a geodesic between $|f|(a)$ et $|f|(b)$.

Planar trees form a category $\mathbf{T}_0$. It is an operator category with zero object.

Stable planar trees

A *marked* planar tree is a pair (T, S), where $T \in \mathbf{T}_0$ and $S \subset V(T)$ is a subset not containing the root. A marked planar tree is *stable* if each non-marked vertex (but the root) has valency at least three 3.

Definition. An object of the category $\mathbf{T}$ is a marked stable planar tree (T, S). A morphism $(T, S) \to (T', S')$ is given by a map $f : T \to T'$ in $\mathbf{T}_0$ such that f sends S to S'.

The category $\mathbf{T}$ is an operator category.

There is another category $\tilde{\mathbf{T}}$ with the objects given by those of $\mathbf{T}$ plus an immersion into a 2-disc which sends all roots to one fixed point on the boundary. The forgetful functor $\tilde{\mathbf{T}} \to \mathbf{T}$ is an equivalence of categories.

Forgetting everything but the marked vertices induces another functor $\tilde{\mathbf{T}} \to \mathbf{B}$. Inverting the equivalence $\tilde{\mathbf{T}} \xrightarrow{\sim} \mathbf{T}$, we get $F : \mathbf{T} \to \mathbf{B}$.

Resolution of $\mathbf{B}$ by planar trees

Theorem PT (partially observed by Kontsevich-Soibelman, Kaledin). *The functor $F : \mathrm{T} \to \mathrm{B}$ is a resolution of operator categories.*

Corollary. *The inverse image functor between categories of Segal objects*

$$F^* : \mathrm{Ho\ Fun}_{Seg}(\mathbf{B}_+, \mathbf{Top}) \to \mathrm{Ho\ Fun}_{Seg}(\mathbf{T}_+, \mathbf{Top})$$

is full and faithful, and its essential image consists of $A : \mathbf{T}_+ \to \mathbf{Top}$ such that whenever $F(f)$ is an iso in $\mathbf{B}_+$, the image $A(f)$ is a weak homotopy equivalence.

Thus to construct E_2-algebras in **Top**, do it first over **T** in a good manner, then descend. Examples might involve another proof of

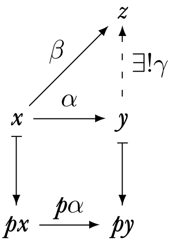
Deligne Conjecture. *For a dg-algebra A, there is an E_2-algebra structure on $CH^\bullet(A, A)$ (which realises the well-known operations on $HH^\bullet(A, A)$).*

...except how to do Segal objects in non-cartesian monoidal setting?

Grothendieck (op)fibrations

Opcartesian morphisms (à l'ancienne)

Let $p : \mathcal{E} \to \mathcal{C}$ be a functor. A morphism $\alpha : x \to y$ of $\mathcal{E}$ is *p-opcartesian* if for each $\beta : x \to z$ such that $p\beta = p\alpha$ there exists unique factorisation $\beta = \gamma\alpha$, where $p(\gamma) = id_{p(y)}$:

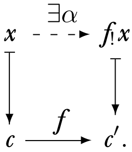


This definition is from SGA1; modern references call this notion locally opcartesian.

Opfibrations

A functor $p : \mathcal{E} \to \mathcal{C}$ is a Grothendieck opfibration if:

1. For each $f : c \to c'$ of $\mathcal{C}$ and $x \in \mathcal{E}$ with $px = c$ there exists an opcartesian lifting $\alpha : x \to f_! x$, $p\alpha = f$:



2. The composition of opcartesian morphisms is opcartesian.

For $c \in \mathcal{C}$, denote by $\mathcal{E}(c) = p^{-1}c$ the fibre of p over c. A choice of opcartesian liftings along $f : c \to c'$ defines a functor $f_! : \mathcal{E}(c) \to \mathcal{E}(c')$.

Dual notions: cartesian maps, Grothendieck fibrations.

Example: symmetric monoidal categories (Lurie, Segal)

Given a symmetric monoidal category $\mathcal{M}$ with $\otimes$, construct an opfibration $\mathcal{M}^{\otimes} \to \mathbf{Fin}_+$ as follows.

1. An object of $\mathcal{M}^{\otimes}$ is a pair consisting of $S \in \mathbf{Fin}_+$ and a S-indexed family $\{X_s\}_{s \in S}$ of objects of $\mathcal{M}$.
2. A morphism in $\mathcal{M}^{\otimes}$, $(S, \{X_s\}_{s \in S}) \to (T, \{Y_t\}_{t \in T})$, consists of a map $f : S \to T$ in $\mathbf{Fin}_+$ together with maps $\otimes_{s \in f^{-1}(t)} X_s \to Y_t$ for each $t \in T$.
3. The projection $(S, \{X_s\}_{s \in S}) \to S$ gives a functor $\mathcal{M}^{\otimes} \to \mathbf{Fin}_+$.

We see that $\mathcal{M}^{\otimes}(S) \cong \mathcal{M}^S$. The map $S \mapsto \mathcal{M}^{\otimes}(S)$ can be made into a (pseudo-)functor which satisfies Segal conditions in $\mathbf{Cat}$. For other monoidal structures (associative, braided) one can make similar considerations.

Algebras as sections

Let $p : \mathcal{E} \to \mathcal{C}$ be an opfibration. A *section* of p is a functor $A : \mathcal{C} \to \mathcal{E}$ such that $pA = id$. The sections of p form a category $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ with fibrewise natural transformations.

In the example of $\mathcal{M}^{\otimes} \to \mathbf{Fin}_+$, consider a section $A : \mathbf{Fin}_+ \to \mathcal{M}^{\otimes}$ such that for each inert morphism

$$j : S \hookleftarrow T \xrightarrow{=} T,$$

the morphism $A(j)$ is opcartesian. Then we get $A(S) \cong (A(1), \dots, A(1))$ and we obtain maps $A(1)^{\otimes S} \to A(1)$ in $\mathcal{M}^{\otimes}(1) = \mathcal{M}$. This way, the object $A(1)$ becomes a commutative monoid in $\mathcal{M}$, and this construction can be reversed.

However, commutative algebras in $\mathcal{M} = \mathbf{DVect}_k$ do not form a good homotopical category.

Homotopy theory of sections?

One way to remedy the issue consists of doing $\mathcal{M}^{\otimes} \to \mathbf{Fin}_+$ and similarly for more general opfibrations $\mathcal{E} \to \mathcal{C}$ is to pass to higher-categorical context, taking a higher localisation of $\mathcal{E}$. Associated difficulties arise ("very fibrant replacement").

We still have no Segal description for monoids in $\mathcal{M}$. In general, an ordinary section A of an opfibration $\mathcal{E} \to \mathcal{C}$ produces, out of $f : c \to c'$, a map $f_! A(c) \to A(c')$. Can we get a "weak section" X, which, out of $f : c \to c'$, would produce a diagram

$$\begin{array}{ccc} & X(f) & \\ \swarrow^{\sim} & & \searrow \\ f_! X(c) & & X(c') \end{array}$$

with left arrow a weak equivalence?

The approach proposed below addresses the latter point, and works in categorical or higher-categorical setting.

Derived, or Segal, sections

Simplicial replacements

For a small category $\mathcal{C}$, its *simplicial replacement* is the category $\mathbb{C}$ such that

1. An object $\mathbf{c}_{[n]} \in \mathbb{C}$ is a sequence of composable arrows of $\mathcal{C}$:

$$\mathbf{c}_{[n]} = c_0 \to c_1 \to \dots \to c_n.$$

2. A morphism $\alpha : \mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ is given by $a : [m] \to [n]$ in Δ such that $c_{a(i)} = c'_i$ for each $i \in [m]$.

Defined this way, $\mathbb{C} = (\int N\mathcal{C})^{\mathsf{op}}$. The maps $\mathbf{c}_{[n]} \mapsto c_0$, $\mathbf{c}_{[n]} \mapsto c_n$ yield functors $\mathbb{C} \xrightarrow{h} \mathcal{C}$ and $\mathbb{C} \xrightarrow{t} \mathcal{C}^{\mathsf{op}}$.

Extension of $\mathcal{E} \to \mathcal{C}$ to $\mathbb{C}$

Recall the final object map $t : \mathbb{C} \to \mathcal{C}^{\mathsf{op}}$, $\mathbf{c}_{[n]} \mapsto c_n$. We want to use it to lift the opfibration (covariant family) $\mathcal{E} \to \mathcal{C}$ to $\mathbb{C}$. However, for this we have to replace $\mathcal{E} \to \mathcal{C}$ by its *transposed fibration* (contravariant family) $\mathcal{E}^{\top} \to \mathcal{C}^{\mathsf{op}}$.

This family is characterised by the facts that $\mathcal{E}^{\top}(c) \cong \mathcal{E}(c)$ and that for each morphism $f : c' \leftarrow c$ of $\mathcal{C}^{\mathsf{op}}$, the transition functor $\mathcal{E}^{\top}(c) \to \mathcal{E}^{\top}(c')$ is isomorphic to $f_! : \mathcal{E}(c) \to \mathcal{E}(c')$. This is, however, a fibration, so a normed section of $(\mathcal{M}^{\otimes})^{\top} \to \mathbf{Fin}_{+}^{\mathsf{op}}$ would correspond to a *coalgebra* object in $\mathcal{M}$.

This is natural for Segal formalism, which treats algebraic objects as "bigger" coalgebraic objects with special conditions.

We can now consider $t^{*}\mathcal{E}^{\top} \to \mathbb{C}$, the inverse image of $\mathcal{E}^{\top} \to \mathcal{C}^{\mathsf{op}}$ along $t : \mathbb{C} \to \mathcal{C}^{\mathsf{op}}$.

Sections of $t^*\mathcal{E}^\top \to \mathbb{C}$

Let us consider a section $X : \mathbb{C} \to t^*\mathcal{E}^\top$ of the fibration $t^*\mathcal{E}^\top \to \mathbb{C}$. For $f : c \to c'$ (associated functor $f_! : \mathcal{E}(c) \to \mathcal{E}(c')$), we get a span in $\mathcal{E}(c')$ of the desired form:

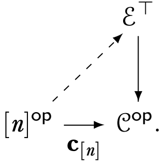
$$\begin{array}{ccccccc} & c \xrightarrow{f} c' & & & & X(c \to c') & \\ \swarrow & & \searrow & \mapsto & \swarrow & & \searrow \\ c & & c' & & f_! X(c) & & X(c'). \end{array}$$

If $\mathcal{E} \to \mathcal{C}$ is equipped with a homotopical structure (e.g. weak equivalences in each $\mathcal{E}(c)$ preserved by $f_!$), we can ask for the left arrow to be a weak equivalence.

This is a naïve picture to keep, and works well in higher-categorical setting. However, in model-categorical setting, such objects are insufficiently homotopy coherent.

Simplicial extension of an opfibration

Given $\mathcal{E} \to \mathcal{C}$, its *simplicial extension* $\mathbf{E} \to \mathbb{C}$ is a family with fibres given by $\mathbf{E}(\mathbf{c}_{[n]}) = \mathsf{Sect}([n]^{\mathsf{op}}, \mathbf{c}_{[n]}^{*}\mathcal{E}^{\top})$, where we consider $\mathbf{c}_{[n]}$ as a functor:



If the opfibration $\mathcal{E} \to \mathcal{C}$ is fibrewise (finitely) complete, then the family $\mathbf{E} \to \mathbb{C}$ is a bifibration (trivally an opfibration, but also a fibration).

Definition. The category of *presections* is the category $\mathsf{PSect}(\mathcal{C}, \mathcal{E}) := \mathsf{Sect}(\mathbb{C}, \mathbf{E})$. A presection $X : \mathbb{C} \to \mathbf{E}$ is derived (or Segal) if the image $X(\alpha)$ of any left interval inclusion $\alpha : \mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ factors as a weak equivalence followed by cartesian. We thus have

$$\mathsf{DSect}(\mathcal{C}, \mathcal{E}) \subset \mathsf{PSect}(\mathcal{C}, \mathcal{E}) = \mathsf{Sect}(\mathbb{C}, \mathbf{E}).$$

The model structure

Model category PSect

Let $\mathcal{E} \to \mathcal{C}$ be a *model opfibration*, that is, each $\mathcal{E}(x)$ is a model category, and the transition functors $\mathcal{E}(x) \to \mathcal{E}(y)$ preserve weak equivalences and fibrations. (Think $\mathbf{DVect}_k^{\otimes} \to \mathbf{Fin}_+$)

Theorem. *In this case, the presections category* $\mathsf{PSect}(\mathcal{C}, \mathcal{E}) = \mathsf{Sect}(\mathbb{C}, \mathbf{E})$ *possesses a model structure, with weak equivalences fibrewise.*

Implication: the category $\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ is realised as a full homotopical subcategory of a model category $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$. Denote by $\operatorname{Ho}\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ and $\operatorname{Ho}\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ the corresponding localisations.

This result is a consequence of a more general theorem for families of model categories over Reedy categories.

Semifibrations

A *semifibration* over a factorisation category $(\mathcal{C}, \mathcal{L}, \mathcal{R})$ is a functor $p : \mathcal{S} \to \mathcal{C}$ such that

1. for each morphism $l : x \to y$ of $\mathcal{L}$ and $Y \in \mathcal{S}(y)$ there is a cartesian lift $l^* Y \to Y$ of l,
2. for each morphism $r : x \to y$ of $\mathcal{R}$ and $X \in \mathcal{S}(x)$ there is an opcartesian lift $X \to r_! X$ of r,
3. given a morphism $\alpha : X \to Y$ of $\mathcal{S}$ and a factorisation of $p(\alpha)$ as $x \xrightarrow{r} z \xrightarrow{l} y$ (wrong arrow order), there is a decomposition of α as

$$X \xrightarrow{\rho} Z \xrightarrow{\omega} Z' \xrightarrow{\lambda} Y,$$

such that $p(\rho) = r$, $p(\lambda) = l$ and $p(\omega) = id_z$.

Theorem MS

Let $\mathcal{R}$ be a Reedy category. A *model semifibration* over $\mathcal{R}$ is a semifibration $\mathcal{S} \to \mathcal{R}$ for the Reedy factorisation system $(\mathcal{R}, \mathcal{R}_-, \mathcal{R}_+)$ such that

1. each fibre $\mathcal{S}(x)$ is a model category,

$2L$. for each $l : x \to y$ de $\mathcal{R}_-$, the transition functor $l^* : \mathcal{S}(y) \to \mathcal{S}(x)$ preserves fibrations and trivial fibrations,

$3L$. for each x in $\mathcal{R}$, either

the matching category $Mat(x)$ is a disjoint union of categories with initial objects, or

the functor $\mathsf{Sect}(Mat(x), \mathcal{S}) \to Fun(Mat(x), \mathcal{S}(x))$ preserves limits,

and dually, $2R, 3R$.

Theorem MS. *The category* $\mathsf{Sect}(\mathcal{R}, \mathcal{S})$ *of sections of a model semifibration* $\mathcal{S} \to \mathcal{R}$ *has a model structure, in which weak equivalences are fibrewise, and the fibrations and cofibrations are Reedy.*

Theorem MS: discussion

When $\mathcal{S} \to \mathcal{R}$ is a bifibration, the result reduces to that of Hirschowitz-Simpson (theory of Quillen presheaves).

Corollary. Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration, then the simplicial extension $\mathbf{E} \to \mathbb{C}$ is a model semifibration.

Proof. Each fibre $\mathbf{E}(\mathbf{c}_{[n]}) = \mathsf{Sect}([n]^{\mathsf{op}}, \mathcal{E}^{\top})$ is a model category by Theorem MS, and then we apply Theorem MS (or H.-S.) again, globally to $\mathbf{E} \to \mathbb{C}$.

Contrary to H.-S., we have a case in which nothing is assumed on transition functors (adjoints, exactness...). This allows us to consider n-fold tensor products.

For a fibrewise-presentable, accessible higher opfibration $\mathcal{E} \to \mathcal{C}$ over a 1-category, the presentability of $\mathsf{PSect}(\mathcal{C}, \mathcal{E}) = \mathsf{Sect}(\mathbb{C}, \mathbf{E})$ is almost readily apparent.

Resolutions and Segal sections

Locally constant derived sections

Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration and $Iso(\mathcal{C}) \subset \mathcal{S} \subset \mathcal{C}$ a subcategory.

Définition. A derived section $X \in \mathsf{DSect}(\mathcal{C}, \mathcal{E})$ is $\mathcal{S}$-*locally constant* if X sends to weakly cartesian arrows those maps $\mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ which verify the following

1. the induced morphism $[m] \to [n]$ in Δ is a right interval inclusion,
2. the maps $c_{i-1} \to c_i$, $1 \leq i \leq n-1$, belong to $\mathcal{S}$.

Example. Each algebra $A : \mathbf{Fin}_+ \to \mathcal{M}^{\otimes}$ gives a derived section locally constant along the inert morphisms $In_{\mathbf{Fin}}$.

Denote by $\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E}) \subset \mathsf{DSect}(\mathcal{C}, \mathcal{E})$ the subcategory of derived $\mathcal{S}$-locally constant sections.

Theorem RES

Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration, $\mathit{Iso}(\mathcal{C}) \subset \mathcal{S} \subset \mathcal{C}$ a subcategory and $F : \mathcal{D} \to \mathcal{C}$ a functor. Then the functor F induces

$$F^* : \mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E}) \longrightarrow \mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D}, F^*\mathcal{E}),$$

where $F^*\mathcal{E} \to \mathcal{D}$ is the pullback of $\mathcal{E} \to \mathcal{C}$, and $\subset F^*\mathcal{S} \subset \mathcal{D}$ is a subcategory given by those f of $\mathcal{D}$ such that $F(f) \in \mathcal{S}$.

Theorem RES. *If moreover $F : \mathcal{D} \to \mathcal{C}$ is a resolution, then*

$$\mathsf{h}F^* : \mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E}) \longrightarrow \mathrm{Ho}\,\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D}, F^*\mathcal{E})$$

is an equivalence of categories.

If $\mathcal{S} = \mathit{Iso}(\mathcal{C})$, then $F^*\mathit{Iso}(\mathcal{C})$ is a subcategory of morphisms of $\mathcal{D}$ which become isomorphisms in $\mathcal{C}$.

Some comments on the proof

To prove Theorem RES, we construct a functor $\mathsf{h}F_!$ inverse to $\mathsf{h}F^*$, as

$$\mathsf{h}F_! \;:=\; \mathbb{L}p_{F,!} \circ \mathsf{h}\mu_F^* \circ \mathbb{R}\delta_{\mathcal{D},*}, \text{ where:}$$

1. The functor $\delta_{\mathcal{D},*} : \mathsf{PSect}(\mathcal{D}, \mathcal{E}) = \mathsf{Sect}(\mathbb{D}, \mathbf{E}) \to \mathsf{Sect}(\mathbb{D}_\Pi, \mathbf{E}_\Pi)$ is a right Kan extension along $\delta_{\mathcal{D}} : \mathbb{D} \to \mathbb{D}_\Pi$. The category $\mathbb{D}_\Pi$ is the Π-*replacement* of $\mathcal{D}$, its objects are $\mathbf{c}_P : P \to \mathcal{D}$, with $P \in \Pi$ a finite poset with initial and final objects.
2. The functor $\mu_F^* : \mathsf{Sect}(\mathbb{D}_\Pi, \mathbf{E}_\Pi) \to \mathsf{Sect}(\mathbb{T}(F), \mu^*\mathbf{E}_\Pi)$ is the inverse image along $\mu : \mathbb{T}(F) \to \mathbb{D}_\Pi$. Here, we note by $p_F : \mathbb{T}(F) \to \mathbb{C}$ *the tower of* F, an opfibration which fibres are simplicial replacements of $\mathcal{D}(\mathbf{c}_{[n]})$.
3. The functor

$$p_{F,!} : \mathsf{Sect}(\mathbb{T}(F), \mu^*\mathbf{E}_\Pi) \to \mathsf{Sect}(\mathbb{T}(F), p_F^*\mathbf{E}) \to \mathsf{Sect}(\mathbb{C}, \mathbf{E}) = \mathsf{PSect}(\mathcal{C}, \mathcal{E})$$

is obtained from a left Kan extension along the opfibration $p_F : \mathbb{T}(F) \to \mathbb{C}$.

Derived algebras

Recall: For an operator category C, the algebra classifier C_+ consists of partially defined maps with admissible domain.

Definition. Let C be an operator category. An C-*monoidal category* is a Grothendieck opfibration $\mathcal{M}^{\otimes} \to C_+$ such that for each $x \in C_+$, the induced functor

$$\mathcal{M}^{\otimes}(x) \longrightarrow \prod_{(x \to 1) \in In_C} \mathcal{M}^{\otimes}(1)$$

is an equivalence of categories.

A C-*monoidal model category* is a C-monoidal C category $\mathcal{M}^{\otimes} \to C_+$ which is also a model opfibration. (use suitable presentability/accessibility for highercat setting)

Definition. Given an C-monoidal model category, its *category of derived algebras* is $\mathsf{DAlg}(C, \mathcal{M}) := \mathsf{DSect}_{In_C}(C_+, \mathcal{M}^{\otimes})$, that is the category of derived sections of $\mathcal{M}^{\otimes} \to C_+$ which are In_C-locally constant.

Theorem RES-ALG

Definition (reminder). A functor $F : \mathrm{D} \to \mathrm{C}$ between operator categories is a resolution if:

1. The functor F preserves limits and $\mathrm{D}(1, x) \cong \mathrm{C}(1, F(x))$,
2. The functor F is a resolution.

Theorem RES-ALG. *Given a* C*-monoidal model category* $\mathcal{M}^{\otimes} \to \mathrm{C}_{+}$ *and a resolution of operator categories* $F : \mathrm{D} \to \mathrm{C}$*, the induced functor*

$$\mathsf{h}F^{*} : \mathrm{Ho}\,\mathsf{DAlg}(\mathrm{C}, \mathcal{M}) \longrightarrow \mathrm{Ho}\,\mathsf{DAlg}_{F^{*}\mathit{Iso}(\mathrm{C})}(\mathrm{D}, F^{*}\mathcal{M})$$

is an equivalence of categories, where $\mathsf{DAlg}_{F^{*}\mathit{Iso}(\mathrm{C})}(\mathrm{D}, F^{*}\mathcal{M})$ *is the category of derived algebras locally constant along* $F^{*}\mathit{Iso}(\mathrm{C}) \subset \mathrm{D} \subset \mathrm{D}_{+}$.

Preuve. Repeated “black box” application of Theorem RES.

Resolution of $\mathbf{B}$ and Segal algebras

Theorem PT (reminder). *There is a functor $F : \mathrm{T} \to \mathrm{B}$ which is a resolution of operator categories.*

Theorems PT and RES-ALG imply that the inverse image functor

$$\mathsf{h}F^* : \mathrm{Ho}\,\mathsf{DAlg}(\mathbf{B}, \mathcal{M}) \to \mathrm{Ho}\,\mathsf{DAlg}_{F^*\mathit{Iso}(\mathbf{B})}(\mathbf{T}, \mathcal{M})$$

is an equivalence of categories.

This can be used to prove the Deligne conjecture outside of the operad formalism. For $\mathbf{DVect}_k^{\otimes} \to \Gamma_+$ and a *dg*-algebra A over k, there is a combinatorial way to construct a derived algebra $CH^{\bullet}_{\mathbf{T}}(A) \in \mathsf{DAlg}(\mathbf{T}, \mathbf{DVect}_k)$ whose value at $1 \in \mathbf{T}$ is $CH^{\bullet}(A, A)$ and which is locally constant.

Sketch of construction

Over $\mathbf{T}_+$, there is an opfibration $\mathbf{Bimod}_A^{\mathbf{T}} \to \mathbf{T}_+$ with fibres over (T, S) equivalent to $\prod_{v \in S}(A^{\otimes out(v)} \otimes A\text{-}\mathbf{Bimod})$ (bimodules viewed as functors of many arguments).

This opfibration has two distinguished sections $L(A)$, $R(A)$, induced by the bimodules $A^{\otimes out(v)} \otimes A$ and $\mathrm{Hom}_k(A^{\otimes out(v)}, A)$ in each fibre, respectively.

Taking a hom-pairing between the corresponding derived sections (amounts to projectively deriving $L(A)$) produces $CH^\bullet_{\mathbf{T}}(A) \in \mathsf{DAlg}(\mathbf{T}, \mathbf{DVect}_k)$.

Descending the obtained derived section to $\mathbf{B}$ gives us

$$CH^\bullet_{\mathbf{B}}(A) \in \mathsf{DAlg}(\mathbf{B}, \mathbf{DVect}_k),$$

a presentation of $CH^\bullet(A, A)$ as an E_2-algebra.

Thank you.