Higher categories and the (∞,n)-category of bordisms

Last week: • categories & bicategories of cobordisms
• cobordism hypothesis: $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_2^{fr}, \mathcal{C}) \simeq (\mathcal{C}^{fd})^{\simeq}$

Today: • higher structures on these categories
• ∞-version of the cobordism hypothesis
• model of (∞,n)-categories used by Lurie to prove this result

I. Motivating examples of categories

Category $\mathcal{C}$: • objects $A, B, C, \ldots$
• morphisms $A \xrightarrow{f} B$
• composition $A \xrightarrow{f} B \xrightarrow{g} C = A \xrightarrow{g \circ f} C$

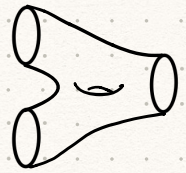
which is associative: $h \circ (g \circ f) = (h \circ g) \circ f$

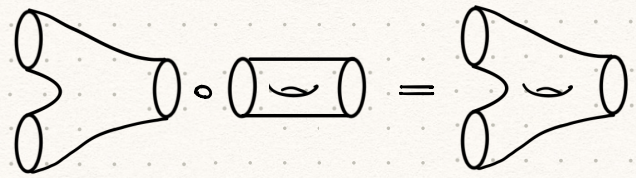
Examples of categories:

(Top) $X \in \mathrm{Top}$; $\Pi_1 X$: obj: vertices in X
mor: htpy classes of paths in X
comp: concatenation

(Cob) $n\mathrm{Cob}$: obj: closed $(n-1)$-manifolds
mor: diffeo classes of n-cobordisms
comp: gluing

2Cob: obj: $\leftrightsquigarrow$ $n \in \mathbb{N}$

mor:  $\leftrightsquigarrow$ $2 \longrightarrow 1$

comp: 

$2 \longrightarrow 1 \longrightarrow 1$ $\quad$ $2 \longrightarrow 1$

(BiMod) A, B-rings

$\mathrm{BiMod}_{(A,B)}$: obj: (A,B)-bimodules

mor: homomorphisms of bimodules

comp: usual

<u>Why are higher structures useful?</u>

To retain full information of:

- paths & homotopies in X
- n-cobordisms & diffeomorphisms

To extend from below:

- start with closed 0-manifolds as objects but still see the n-cobordisms
- see all bimodules for all possible rings at once.

II. One step at a time: one dimension up

Bicategory $\mathcal{A}$: objects $A, B, C, \dots$

morphisms $A \xrightarrow{f} B$

2-morphisms $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd} \rightsquigarrow \alpha : f \Rightarrow g$

horizontal comp: $A \xrightarrow{f} B \xrightarrow{g} C = A \xrightarrow{g \circ f} C$

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\alpha"] \arrow[r, bend left, "g"{name=U'}] \arrow[r, bend right, "g'"'{name=D'}] & C \arrow[Rightarrow, from=U', to=D', "\beta"] \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "g \circ f"{name=U}] \arrow[r, bend right, "g' \circ f'"'{name=D}] & C \arrow[Rightarrow, from=U, to=D, "\beta \circ_h \alpha"] \end{tikzcd}$$

vertical comp: $\begin{tikzcd} A \arrow[r, bend left=60, "f"{name=U}] \arrow[r, "g"{name=M}] \arrow[r, bend right=60, "h"'{name=D}] & B \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M, to=D, "\beta"] \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "h"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\beta \circ_v \alpha"] \end{tikzcd}$

$\rightsquigarrow f \overset{\alpha}{\Longrightarrow} g \overset{\beta}{\Longrightarrow} h = f \overset{\beta \circ_v \alpha}{\Longrightarrow} h$

such that comp. of morphisms is associative up to 2-isomorphisms: $h \circ (g \circ f) \overset{\cong}{\Longrightarrow} (h \circ g) \circ f$ (+ compatibility)

$\rightsquigarrow$ category $\mathcal{A}(A,B)$: obj: morphisms $A \xrightarrow{f} B$

mor: 2-morphisms $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$

comp: vertical comp.

Examples:

(Top) $X \in \mathrm{Top}$; $\Pi_2 X$: obj: vertices in X

mor: paths in X

2-mor: htpy classes of htpies btw paths

Concatenation of paths is only associative up to homotopy: $\gamma \star (\beta \star \alpha) \underset{\text{invertible 2-mor}}{\simeq} (\gamma \star \beta) \star \alpha$

($Cob^{\uparrow}$) $\underline{nCob}$: obj : closed $(n-1)$-manifolds

mor : n-cobordisms

2-mor : diffeomorphisms

Gluing of n-cobordisms is only associative up to diffeomorphism

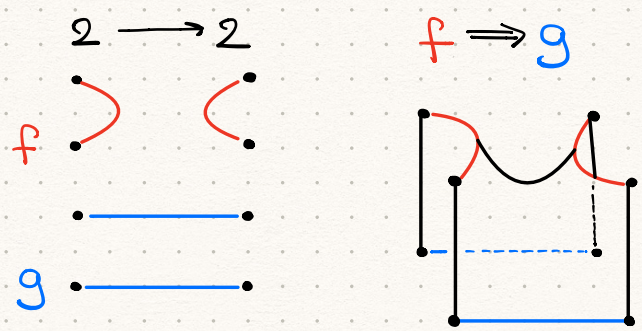
($Cob^{\downarrow}$) $hBord_2$: obj : closed 0-manifolds

mor : 1-cobordisms

2-mor : diffeo classes of 2-cobordisms with corners

Ex : object morphism 2-morphism

2



(BiMod) BiMod : obj : rings $A, B, C, \ldots$

mor $A \xrightarrow{M} B$: (A,B)-bimodules ${}_AM_B$

2-mor : homomorphisms

horizontal comp :

$$\begin{array}{ccccc} {}_AM_B & \circ & {}_BN_C & = & {}_AM \otimes_B N_C \\ \downarrow \varphi & & \downarrow \psi & & \downarrow \varphi \otimes_B \psi \\ {}_AM'_B & \circ & {}_BN'_C & = & {}_AM' \otimes_B N'_C \end{array}$$

vertical comp : usual comp. of homomorphisms

Tensors are only associative up to isomorphism.

III. All steps at once: going to infinity

(∞,n)-category: obj

1-mor

2-mor

⋮

n-mor

(n+1)-mor } invertible

⋮

} $(\infty,n-1)$-category of mor

⋮

All compositions are associative up to higher invertible morphisms.

(∞,n)-categories "=" enriched in $(\infty,n-1)$-categories

"=" internal to $(\infty,n-1)$-categories with discrete $(\infty,n-1)$-cat of objects.

Examples:

(Top) $X \in \mathrm{Top}$: obj: vertices in X

1-mor: paths in X

2-mor: htpies btw paths

3-mor: htpies btw htpies

⋮

} invertible

⋮

$\rightsquigarrow$ $(\infty,0)$-categories $= \infty$-groupoids

$(\mathrm{Cob}^{\updownarrow})$ Bord_n: obj: closed 0-manifold

1-mor: 1-cobordisms

2-mor: 2-cobordisms

$\vdots$

n-mor: n-cobordisms

$\left.\begin{array}{l}(n+1)\text{-mor: diffeomorphisms} \\ (n+2)\text{-mor: isotopies btw diffeo}\end{array}\right\}$ invertible

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

<u>Cobordism hypothesis</u> [Lurie]

Let $\mathcal{C}$ be a symmetric monoidal (∞,n)-category

There is an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \simeq (\mathcal{C}^{fd})^{\simeq}$$

where $(-)^{\simeq}$ takes the underlying ∞-groupoid.

<u>IV. Models of (∞,n)-categories</u>

<u>$n=0$</u>: ∞-groupoids = topological spaces

<u>Other model</u>: Kan complexes $\subseteq$ simplicial sets

$\Delta \underset{\text{full}}{\subseteq} \mathrm{Cat}$ on the objects $[n] = \{0 \to 1 \to 2 \to \dots \to n\}$

models composition

Simplicial set: $X : \Delta^{op} \to \mathrm{Set}$; $\mathrm{sSet} := \mathrm{Set}^{\Delta^{op}}$

X_0 = set of objects ; X_1 = set of morphisms , ...

Kan complex: conditions $\rightsquigarrow$ composition + invertibility

$\underline{n=1}$: Complete Segal spaces [Rezk]

Internal to ∞-groupoids with discrete ∞-gpd of obj

$X : \Delta^{op} \longrightarrow (sSet)_{Kan}$ such that

$X_0 = \infty$-gpd of objects; $X_1 = \infty$-gpd of morphisms

(i) <u>Segal</u>: $X_m \xrightarrow{\sim} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is an eq of ∞-gpd

$$\left.\begin{array}{ccccc} X_1 \times_{X_0} X_1 & \xleftarrow{\sim} & X_2 & \xrightarrow{d_1} & X_1 \\ \nearrow\searrow & \longmapsto & \underline{\nearrow\searrow} & \longmapsto & \longrightarrow \end{array}\right\} \text{composition}$$

+ discreteness: X_0 is a set $\rightsquigarrow$ Segal categories

(ii) <u>Completeness</u>: $X_0 \xrightarrow{\sim} X_1^{eq} \subseteq X_1$

sub-∞-gpd of invertible mor

$\underline{n=2}$: 2-fold complete Segal spaces [Barwick]

Internal to $(\infty,1)$-categories with discrete $(\infty,1)$-cat of obj

$X : \Delta^{op} \longrightarrow (sSet^{\Delta^{op}})_{CSS}$ such that:

$X_0 = (\infty,1)$-cat of objects; $X_1 = (\infty,1)$-cat of mor + 2-mor

(i) <u>Segal</u>: $X_m \xrightarrow{\sim} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is an eq of $(\infty,1)$-cat's

(ii) <u>Completeness</u>:

(a) $X_{0,0} \xrightarrow{\sim} X_{0,k}$ ($X_{0,-} : \Delta^{op} \to (sSet)_{Kan}$ is htpically constant $\rightsquigarrow$ X_0 is an ∞-gpd)

(b) $X_{0,0} \xrightarrow{\sim} X_{1,0}^{eq}$

$\underline{n>2}$: n-fold complete Segal spaces [Barwick]

$X : \Delta^{op} \to (sSet^{\Delta^{op,n-1}})_{(n-1)CSS}$ such that

(i) <u>Segal</u>: $X_m \xrightarrow{\sim} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is an eq of $(\infty, n-1)$-cat's

(ii) <u>Completeness</u>:

(a) $X_{0,-} : \Delta^{op,n-1} \to (sSet)_{KQ}$ is htpically constant

(b) $X_{0,0,\ldots,0} \xrightarrow{\sim} X^{eq}_{1,0,\ldots,0}$