Introduction to Fukaya Categories

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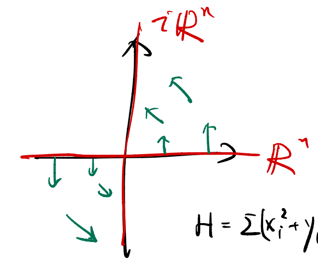
[Auroux] A beginner's introduction to Fukaya categories.

[Smith] A sympl. prolegomenon

- (M^{2n}, ω) is a symplectic manifold if ω = closed non-degenerate 2-form.
- $L^n \subset (M^{2n}, \omega)$ is a Lagrangian Submanifold if $\omega|_L = 0$, $\dim L = \frac{1}{2} \dim M$
- $H : M \to \mathbb{R}$ Hamiltonian $\Rightarrow dH = -\iota_{X_H} \omega \Rightarrow \varphi^t$ flow of X_H Hamiltonian isotopy



$H = \sum (x_i^2 + y_i^2)$

$\Rightarrow X_H = \sum (-y_i \partial_{x_i} + x_i \partial_{y_i})$

(1) $(\mathbb{R}^{2n}, \ \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$ $\mathbb{R}^n$, $i\mathbb{R}^n$ are Lagrangians.

(2) Σ = oriented surface, ω = volume form on Σ

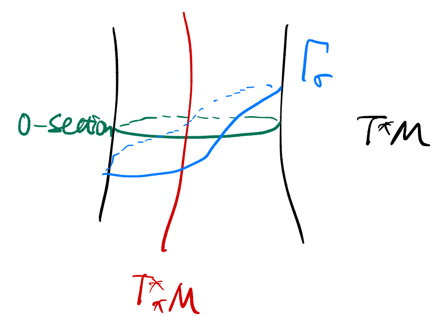
C = curve in Σ is a Lagrangian.

(3) $(T^*M, \ \omega = \sum dp_i \wedge dq_i)$,

$\forall$ closed 1-form σ on M, the graph

$\Gamma_\sigma = \{ (x, \sigma(x)) \mid \sigma(x) \in T_x^*M \}$ is a Lagrangian

Cotangent fiber is Lagrangian.



$H_t : M \to \mathbb{R}$

__Arnold conjecture__ : if $H: S^1 \times M \to \mathbb{R}$ and ϕ_H^1 has isolated fixed points (generic)

then $\# \mathrm{Fix}(\phi_H^1) \geq \sum \mathrm{rk}\, H^i(M)$

__"Everything is a Lagrangian"__ (Weinstein)

Given a symplectomorphism $\phi: M^{2n} \to M^{2n}$, $\underset{\{(x,\phi(x))\}}{\Gamma_\phi^{2n}} \subset \left((M \times M)^{4n}, (-\omega) \oplus \omega\right)$ Lagrangian

$\Delta \cap \Gamma_\phi \longleftrightarrow \mathrm{Fix}(\phi)$

$\rightsquigarrow$ Arnold conjecture $\Longleftrightarrow$ $\# \Delta \cap \Gamma_{\phi_H^1} \geq \mathrm{rk}\, H^*(\Delta)$ for generic H.

$\rightsquigarrow$ understand the intersection of two Lagrangians.

$$\Downarrow$$

Lagrangian Floer Cohomology. $HF^*(L,K)$

(under nice condition)

- $\chi(HF^*(L,K)) = L \cdot K$
- $HF^*(\phi_H^1(L), K) \simeq HF^*(L,K)$
- $\boxed{HF^*(L,L) \simeq H^*(L)}$
- $CF^*(L,K)$ gen by $L \cap K$

[Floer] $\# \phi_H L \cap L \geq \mathrm{rk}\, H^*(L)$ $\Leftarrow$

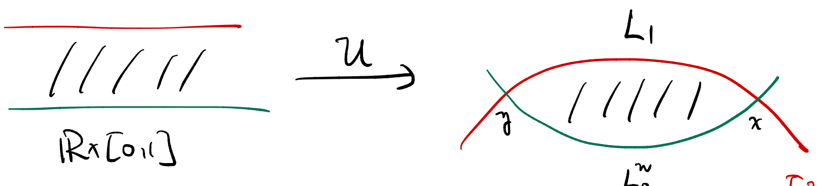
Definition of Lagrangian Floer cohomology :

L_0^w is $\phi_H^1(L_0)$ for some Hamiltonian H.

$\mathbb{Z}/2, \mathbb{Z}, \dots$

$$CF(L_0, L_1) = \sum_{x \in L_0^w \cap L_1} \mathbb{K}\cdot x \qquad \partial x = \sum_y n(x,y)\, y ,$$

$$n(x,y) = \# \left\{ u : \mathbb{R}\times[0,1] \longrightarrow M \;\middle|\; \begin{array}{l} \bar\partial_J u = \frac{1}{2}(du + J\circ du \circ j) = 0 \\ u(s,0) \in L_0^w, \quad u(s,1) \in L_1 \\ \lim_{s\to+\infty} u(s,t) = x, \quad \lim_{s\to-\infty} u(s,t) = y \\ E(u) = \int u^*\omega < \infty \end{array} \right\} \Big/ \mathbb{R}$$



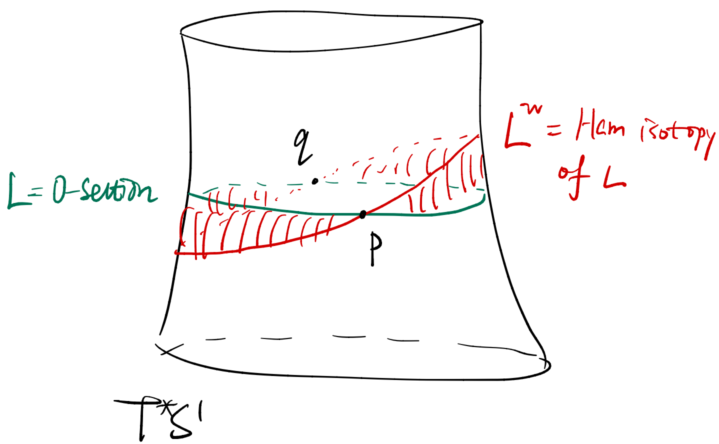
$[u : D \to M \mid u(\partial D) \subset L_0]$ $\qquad \int_D u^*\omega$

* under nice conditions (eg. $[\omega]\cdot\pi_2(M, L_0) = [\omega]\cdot\pi_2(M, L_1) = 0$ or ω exact)

$\partial^2 = 0 \quad \Rightarrow \quad HF(L_0, L_1)$, invariant of choices (H, J)

invariant of Hamiltonian isotopy

Example:



$$CF(L,L) = \mathbb{Z}/2 \cdot \{p,q\}$$

$$\partial p = \#\{u : D \to T^*S^1 \mid \dots\}\, q$$

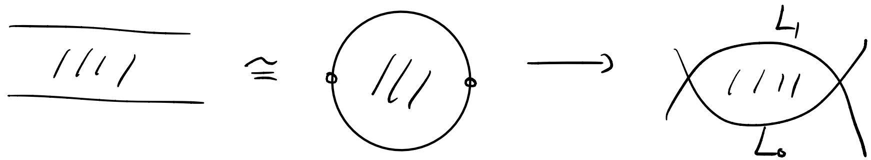
$$= 0$$

(the count $\# = 2$)

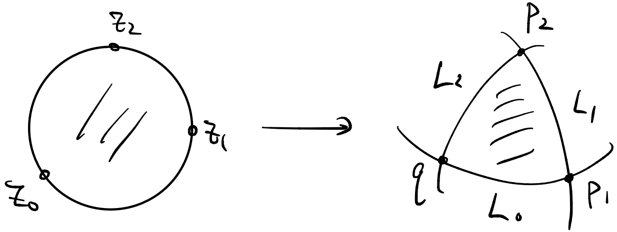
$$\partial q = 0$$

$$HF(L,L) = \mathbb{Z} \oplus \mathbb{Z} \cong H^*(L = S^1)$$

More Lagrangians and more operations $\Rightarrow$ Fukaya category



$\Rightarrow \mu_1 : CF(L_0, L_1) \longrightarrow CF(L_0, L_1)$

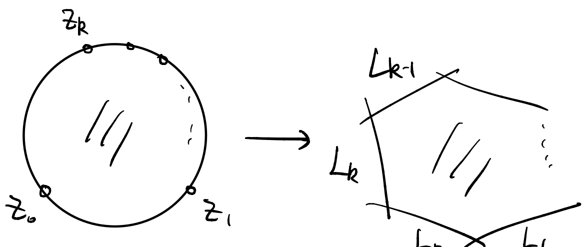


$\Rightarrow \mu_2 : CF(L_1, L_2) \otimes CF(L_0, L_1) \longrightarrow CF(L_0, L_2)$

$$(p_2, p_1) \longmapsto \sum_{q \in L_0 \cap L_2} n(p_2, p_1; q)\, q.$$

$\Downarrow$

$[\mu_2] : HF(L_1, L_2) \otimes HF(L_0, L_1) \longrightarrow HF(L_0, L_2)$ associative



$\Rightarrow$ higher operations: $\mu_k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L_1) \longrightarrow CF(L_0, L_k)$

$\Rightarrow$ invariant of symplectic manifold $Fuk(M, \omega)$ (defined in nice conditions)

Obj = Compact oriented Lagrangians

$hom(L, L') = CF(L, L')$

It is an A_∞-category as the higher operations μ_k satisfy A_∞-relations.

$0 = \mu_1^2$

$\pm \partial(p_2 \cdot p_1) \pm (\partial p_2) \cdot p_1 \pm p_2 \cdot (\partial p_1) = 0$

$0 = \pm \underline{\mu_1}(\underline{\mu_2}(p_2, p_1)) \pm \mu_2(\mu_1(p_2), p_1) \pm \mu_2(p_2, \mu_1(p_1))$

$0 = \pm \mu_2(\mu_2(p_3, p_2), p_1) \pm \mu_2(p_3, \mu_2(p_2, p_1)) \pm \mu_1(\mu_3(p_3, p_2, p_1))$

$\pm \mu_3(\mu_1(p_3), p_2, p_1) \pm \mu_3(p_3, \mu_1(p_2), p_1) \pm \mu_3(p_3, p_2, \mu_1(p_1))$

$\vdots$

Homological Mirror Symmetry Conjecture

[Kontsevich] Given (X, J, ω) Calabi-Yau manifold, $\exists$ mirror CY $(\check{X}, \check{J}, \check{\omega})$ s.t.

$$D\,\mathrm{Fuk}(X, \omega) \cong D\,\mathrm{Coh}(\check{X}, \check{J}) \quad , \quad D\,\mathrm{Coh}(X, J) \cong D\,\mathrm{Fuk}(\check{X}, \check{\omega})$$

(study Lagrangian Submanifolds of X $\Longleftrightarrow$ study holomorphic vector bundles of $\check{X}$)

Enlargement of $\mathrm{Fuk}(X, \omega)$ by adding exact triangles, direct sum, direct summand, degree shifts

$\underline{\mathrm{Coh}(\check{X}, \check{J})}$:

Obj = coherent sheaves

$\hom(A, B) = H^0(A^* \otimes B)$

proved for elliptic curves [Polishchuk-Zaslow], quartic surface [Seidel], etc.

Non-Calabi-Yau / Non-compact HMS:

Landau-Ginzburg model

* when X is Fano with anti-canonical divisor D, it is mirror to $(\check{X}, W: \check{X} \to \mathbb{C})$

wrapped

$$D\mathcal{W}(X \setminus D) \simeq MF(\check{X}, W)$$

$$D\,Coh(X) \simeq D\mathcal{FS}(\check{X}, W)$$

Fukaya-Seidel

($X \setminus D$ and $\check{X}$ are Liouville manifolds)

W is a way to incoporate the information of the divisor D.

proved for del Pezzo surfaces, weighted proj planes, [Auroux-Kartzarkov-Orlov], etc

* There is also HMS for varieties of general type.

  proved for genus ≥ 2 curves [Seidel, Efimov]

Geometric setting:

* **Liouville domain**: (M, θ) — 1-form

M = compact manifold w/ boundary

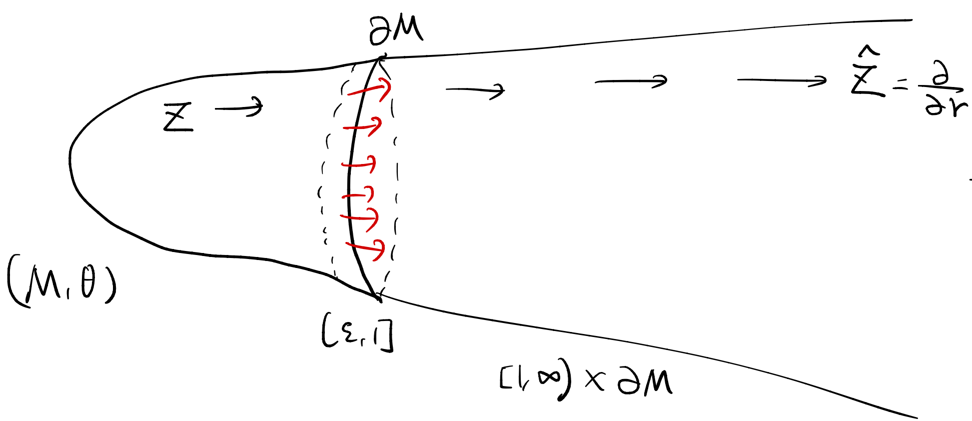
$\omega = d\theta$ is exact sympl. form.

with Liouville vector field Z pointing outwards along ∂M.

(defined by $\iota_Z \omega = \theta$) $\Rightarrow$ collar $N(\partial M) \cong (\partial M \times (\varepsilon, 1], \ r\alpha)$, $\alpha = \theta|_{\partial M}$.

* **Completion / Liouville manifold** $(\hat{M}, \hat{\theta}) = (M, \theta) \cup (\partial M \times [1, \infty), \ r\alpha)$

$\iota_{\hat{Z}} \hat{\omega} = \hat{\theta}$



* Examples: cotangent bundles, complex affine variety

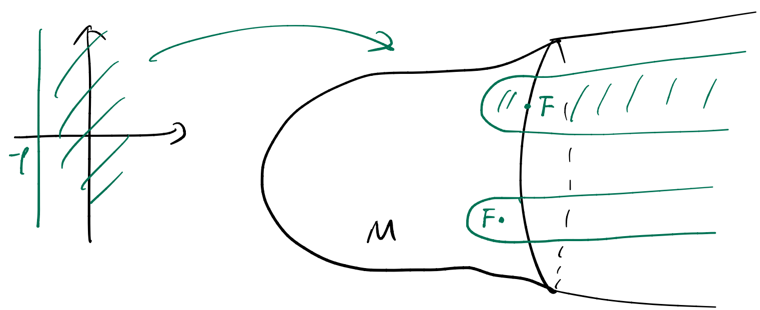
* Given (M^{2n}, θ_M), (F^{2n-2}, θ_F) Liouville domains.

Let $\mathbb{H}_\rho = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq -\rho\}$ with $\theta_{\mathbb{H}_\rho} = \frac{1}{2}(x\,dy - y\,dx)$

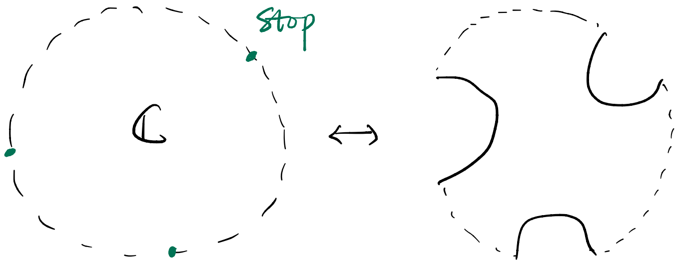
A stop in M with fiber F is a proper embedding

$$\sigma : \hat{F} \times \mathbb{H}_\rho \to \hat{M} \quad \text{s.t. } \sigma^* \hat{\theta}_M = \hat{\theta}_F + \theta_{\mathbb{H}_\rho} + df, \text{ for some cpt supp } f$$

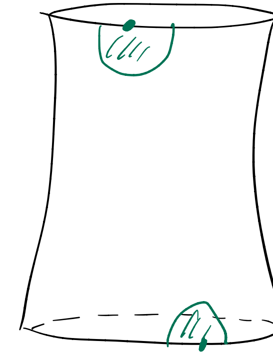
A Liouville sector is $\hat{M} \setminus \mathrm{Im}(\sigma)$



Examples

(1) 

Liouville sector

(2) T^*S^1 

(3) LG-model $W: M \longrightarrow \mathbb{C}$ gives a natural stop/sector

where stop = nbhd of $W^{-1}(\gamma)$, $\gamma: \mathbb{R}_{\geq 0} \hookrightarrow \mathbb{C}$, $F = W^{-1}(\gamma(0))$

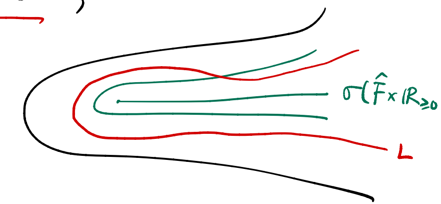
(4) F = page of open book decomposition of contact manifold ∂M.

* Let (M,θ,σ) be a Liouville domain with disjoint stops.

We consider Lagrangian $L \subset \hat{M}$, that is exact ($\theta|_L = df$, $f: L \to \mathbb{R}$),

properly embedded, parallel to $\hat{Z}$ outside a cpt set,

disjoint from $\sigma(\hat{F} \times \mathbb{R}_{\geq 0})$.



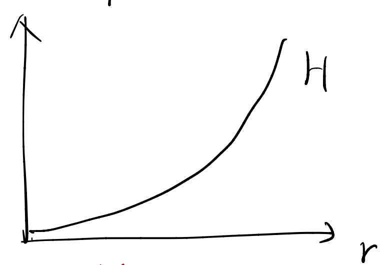
* Wrapped Floer cochain complex / cohomology [Sylvan, Ganatra-Pardon-Shende]

$$CW^*_\sigma(L_0, L_1) = \int^{\circ} CF^*(L_0^w, L_1)$$

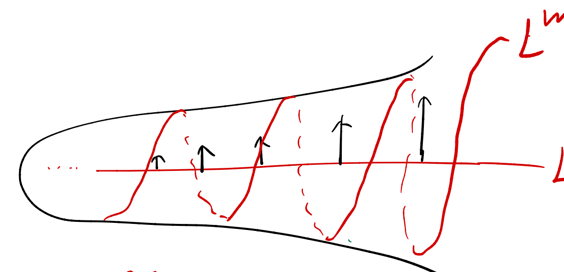
$L_0^w = \phi_H^1(L_0)$ for a quadratic positive H s.t. X_H is positively transverse to $\sigma(\hat{F} \times \mathbb{R}_+)$

$H(r) = r^2 + C$

$X_H = 2r R$



$\Rightarrow$



$CW^*(L_0, L_1) = CF^*(L_0^w, L_1)$

$\underset{r}{[1,\infty)} \times \partial M \cong$

* Filtration on $CF^*(L_0^w, L_1)$: each $x \in L_0^w \cap L_1$ corresponds to a integral curve γ_x of X_H Hamiltonian vector field, from L_0 to L_1

$$\mathcal{F}^i CF^*(L_0^w, L_1) = \{ x \mid \gamma \cdot \sigma(\hat{F} \times \mathbb{R}_+) \leq i \}$$

$\Rightarrow$ (partially) wrapped Floer cochain complex $CW_\sigma^*(L_0, L_1) := \mathcal{F}^0 CF^*(L_0^w, L_1)$

differential ∂ preserves filtration

gen. by $L_0' \pitchfork L_1$, where L_0' is wrapping of L_0 stopped by the stop.

$\Rightarrow HW_\sigma^*(L_0, L_1)$, indep of H, J.

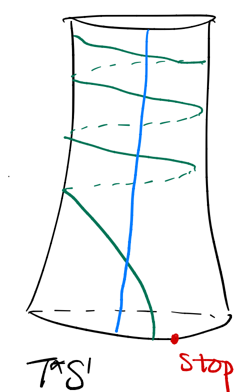
$\Rightarrow \mathcal{W}(M, \theta)$ wrapped Fukaya category

A_∞-cat

Obj = exact, properly embedded, cylindrical Lag

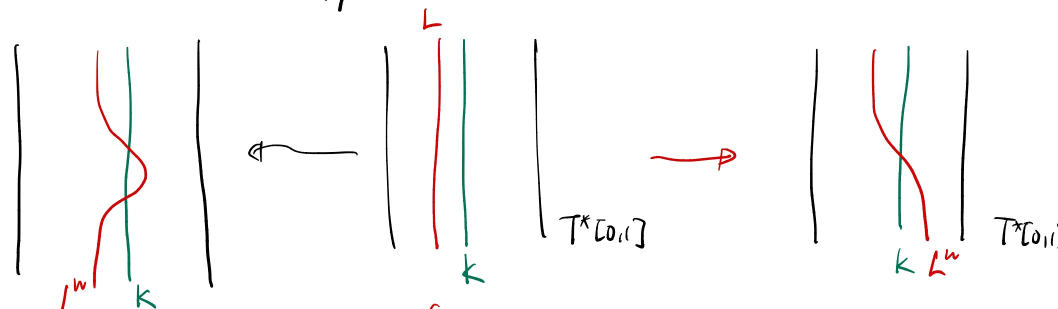
hom$(L_0, L_1) = CW^*(L_0, L_1)$

higher operations and A_∞-str.



Why wrapping?

* <u>M is non-compact</u>: $HF(L_0, L_1)$ is only invariant under compactly supp. Hamiltonian isotopy.



(not invariant under all Ham isotopies)

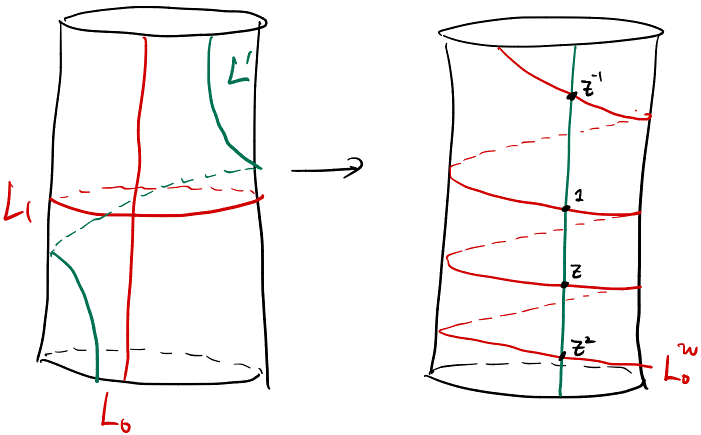
* <u>HMS requires something infinitely</u>:

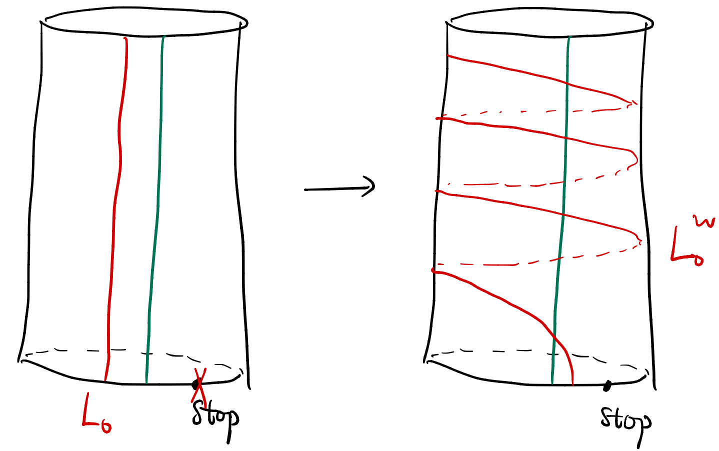
For example, $T^*S^1 \xleftrightarrow{\text{mirror}} \mathbb{C}^*$

Cotangent fiber F $\longleftrightarrow$ $\mathcal{O}_{\mathbb{C}^*}$

$$\hom(F,F) \cong \operatorname{Ext}(\mathcal{O}_{\mathbb{C}^*}, \mathcal{O}_{\mathbb{C}^*}) \cong \mathbb{C}[x, x^{-1}]$$

Example: T^*S^1 : $L' = L_1 \oplus L_0$





$HW(L_0, L_0) \cong \mathbb{K}[z, z^{-1}]$

Mirror $= \mathbb{C}^*$

$W(T^*S^1) = \langle L_0 \rangle$

$HW_\sigma(L_0, L_0) \cong \mathbb{K}[z]$

Mirror $= \mathbb{A}^1$

$W_\sigma(T^*S^1) = \langle L_0 \rangle$