Geometric methods in representation theory of supergroups

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Support variety

Let $\mathcal{C}$ be some abelian (or triangulated) tensor category. For every object $M \in \mathcal{C}$ we want to associate a geometric object (projective variety) X_M satisfying

- $X_{M \oplus N} = X_M \cup X_N$;
- $X_{M \otimes N} = X_N \cap X_N$;
- $X_{M*} = X_M$;
- M is projective iff $X_M = \varnothing$.

Examples:

- Representations of finite groups in positive characteristic.
- Representations of restricted Lie algebras.
- Finite (super)group schemes.
- Balmer spectrum of triangulated tensor categories.

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Approaches.

1. Homology: $X_{\mathbf{k}} = \operatorname{Spec}\operatorname{Ext}^{\bullet}(\mathbf{k}, \mathbf{k})$, $X_M = \operatorname{Spec}\operatorname{Ext}^{\bullet}(M, M)$.
2. Rank variety: maps of elementary objects $\pi : \mathbf{k}[t]/(t^p) \to \mathbf{k}[G]$ for finite groups, p-nilpotent elements for restricted Lie algebras. Check when restriction is not projective.
3. Rank variety and homological support coincide.

Goal: Generalize to supergroups.

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Quasireductive algebraic supergroups

Now we switch to the ground field $\mathbb{C}$. The following three objects are the same (categories are equivalent):

- Affine algebraic supergroup G.
- Pair $(\mathfrak{g}, G_0)$ where $\mathfrak{g}$ is a Lie superalgebra, G_0 an algebraic group with $\operatorname{Lie} G_0 = \mathfrak{g}_{\bar{0}}$, representation of G_0 in $\mathfrak{g}$ whose differential is the adjoint representation.
- Commutative finitely generated Hopf superalgebra $\mathbb{C}[G]$.

By $\operatorname{Rep} G$ we denote the category of finite-dimensional representations of G and by $\overline{\operatorname{Rep}} G$ the category of all representations.

We call G reductive if $\operatorname{Rep} G$ is semisimple (any finite-dimensional representation is completely reducible).

Theorem

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We call G quasireductive if G_0 is reductive.

$\operatorname{Rep} G$ has many remarkable properties:

- Frobenius category. Enough projective and injective objects, every projective is injective and vice versa.
- $\operatorname{Ind}_{G_0}^{G} M$ is finite-dimensional if M is finite-dimensional.
- If $K \subset G$ is quasireductive then G/K is affine.
- If $K \subset G$ are both quasireductive then $\operatorname{Ind}_K^G : \overline{\operatorname{Rep}} K \to \overline{\operatorname{Rep}} G$ is exact maps projective modules to projective modules.

A lot of similarities with finite groups in positive characteristic.
Examples. $GL(m|n)$, $Q(n)$.
Homological approach.[Boe, Kujawa, Nakano.]

- $\operatorname{Ext}^\bullet(\mathbb{C}, \mathbb{C}) = \mathbb{C}[\mathfrak{g}_1^{G_0}]$ is a Noetherian (supercommutative) ring.
- Polynomial algebra in most classical examples.

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Rank variety

Let $\mathfrak{g} = \operatorname{Lie} G$. We call $x \in \mathfrak{g}$ semisimple if $[x, x] \in \mathfrak{g}_0$ is semisimple. By Q_x we denote the quasireductive subgroup of G generated by x. Denote by $\mathfrak{g}_1^{ss}$ the set of all odd semisimple elements of $\mathfrak{g}$.

A functor $S_x : \operatorname{Rep} G \to \operatorname{Svect}$ as the composition

$$\operatorname{Rep} G \xrightarrow{Res} \operatorname{Rep} Q_x \xrightarrow{S} \operatorname{Svect},$$

where S is the semisimplification functor. Equivalently,

$$S_x M := \operatorname{Ker} x_M / (xM \cap \operatorname{Ker} x_M).$$

S_x is a symmetric monoidal functor although it is not exact!

Let M be a G-module. We define the support $\mathcal{X}_M$ as

$$\mathcal{X}_M = \{x \in \mathfrak{g}_1^{ss} \mid S_x M \neq 0\}.$$

Immediate: $\mathcal{X}_{M \otimes N} = \mathcal{X}_M \cap \mathcal{X}_N$, $\mathcal{X}_{M \oplus N} = \mathcal{X}_M \cup \mathcal{X}_N$, $\mathcal{X}_{M^*} = \mathcal{X}_M$. Need: projectivity detection.
Although $\mathfrak{g}_1^{ss}$ and $\mathcal{X}_M$ are not closed in $\mathfrak{g}_1$, we should pass to the quotient $\mathfrak{g}_1 / G_0$ using G_0-equivariance.

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Detecting subgroups

Let $K \subset G$ be both quasireductive. Then $\mathcal{O}(G/K) = \operatorname{Ind}_K^G \mathbb{C}$ contains a trivial G-submodule $\mathbb{C}$. We call K detecting in G if $\mathbb{C} \to \mathcal{O}(G/K)$ splits.

Theorem

The following conditions on the subgroup K in G are equivalent:

1. *K is detecting;*
2. *Any G-module M splits as a direct summand in $\operatorname{Ind}_K^G M$.*
3. *For any pair of G-modules M, M', the restriction morphism*

$$\operatorname{Ext}^i_G(M, M') \to \operatorname{Ext}^i_K(M, M')$$

is injective for all i.

Example

Let G be a finite group but ground field has a positive characteristic p. Then K is detecting if it contains a p-Sylow subgroup of G.

If K is a detecting subgroup of G, then a G-module M is projective if and only if its restriction to K is projective.

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Transitivity. Given $K \subset H \subset G$.

1. If K is a detecting subgroup in G, then H is also a detecting subgroup in G.
2. If K is a detecting subgroup of H, and H is a detecting subgroup of G, then K is a detecting subgroup of G.

Theorem (I. Entova-Aizenbud-V.S.-A. Sherman)

If K is a detecting subgroup of G then the restriction functor maps negligible morphisms to negligible morphisms. The restriction functor descends to the functor between semisimplifications of $\operatorname{Rep} G$ and $\operatorname{Rep} K$.

A morphism (G-equivariant map) $V \xrightarrow{f} W$ is called negligible if for any $W \xrightarrow{g} V$ the supertrace of $f \circ g$ is zero.

Theorem (V.S.-A. Sherman)

Let K be a detecting subgroup of G. Then any G_0-orbit in $\mathfrak{g}_1^{ss}$ has non-empty intersection with $\mathfrak{k}_1$.

Idea of proof. If $G_0 x \cap \mathfrak{k}_1 = \varnothing$ then the corresponding vector field on G/K does not have zeros. Using localization we can prove that $S_x \mathcal{O}(G/K) = 0$.

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Goal: Find small detecting subgroup in G.

- Let $G = GL(m|n)$. When $K = GL(p|q) \times GL(m-p|n-q)$ is detecting?
- Geometric necessary condition implies $\min(p,q) + \min(m-p, n-q) = \min(m,n)$ or, equivalently, $p-q$, $m-n$ and $(m-p)-(n-q)$ are all non-negative or all non-positive.
- Unitary trick. The group G has a compact real form U, unitary supergroup. Then the supergrassmannian $M = U/(K \cap U) = Gr(p|q, m|n)$ is a closed U-orbit in G/K. It has U-invariant volume form. The integral $\int_M$ defines a U-equivariant map $\mathcal{O}(G/K) \to \mathbb{C}$ and hence a G-equivariant map. This map defines a splitting iff the volume of the supergrassmannian M is not zero. We can prove that it is not zero exactly when the necessary condition holds.

Theorem (V.S.-A. Sherman)

The subgroup $GL(p|q) \times GL(m-p|n-q)$ is a detecting subgroup in $GL(m|n)$ if and only if $Gr(p|q, m|n)$ has a non-zero volume.

Supergrassmannians

$Gr(p|q, m|n)$ will denote the supermanifold of $(p|q)$-dimensional subspaces in $\mathbb{C}^{m|n}$.

Properties

- Underlying manifold $Gr(p, m) \times Gr(q, n)$ is compact.
- $D := \dim_{\mathbb{C}} Gr(p|q, m|n) = ((m-p)p + (n-q)q|(m-p)q + p(n-q))$.
- $Gr(p|q, m|n) \simeq Gr(m-p|n-q, m|n)$.
- Homogeneous supermanifold $U(m|n)/U(p|q) \times U(m-p|n-q)$ where $U(a|b)$ is the unitary supergroup: preserves Hermitian form on $\mathbb{C}^{a|b}$.
- Lie algebra $\mathfrak{u}(m|n) = \{X \in \mathfrak{gl}(m|n) \mid X^* = X\}$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} \bar{A}^t & \mathbf{i}\bar{B}^t \\ \mathbf{i}\bar{C}^t & D^t \end{pmatrix}.$$

  Let $x \in \mathfrak{u}(m|n)_{\bar{0}}$ of rank $(p|q)$ and $x^2 = x$. Then $Gr(p|q, m|n)$ is isomorphic to the $U(m|n)$-orbit of x.
- $Gr(p|q, m|n)$ is a symplectic supermanifold, the action of $U(m|n)$ is Hamiltonian.

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Volume forms and Berezin integral

Berezinian:

$$\mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\det D^{-1}.$$

Volume form $\omega = f(x,\xi)d\xi dx$ on a supermanifold $\mathcal{M}$:

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Example

(Rudakov) Let $\dim \mathcal{M} = (1|2)$, $\mathcal{M}_0 = (0,1)$, two coordinate systems (x, ξ_1, ξ_2) and $y = (y, \eta_1, \eta_2)$ related by

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$$\int_{\mathcal{M}} y d\eta_1 d\eta_2 dy = 0, \quad \int_{\mathcal{M}} (x + \xi_1 \xi_2) d\xi_1 d\xi_2 dx = -1.$$

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Volumes of supergrassmannians

Theorem

Let $\mathcal{M}$ be a symplectic supermanifold with Hamiltonian action of the Lie superalgebra $\mathfrak{g}$. Then there exists a canonical $\mathfrak{g}$-invariant volume form ω on $\mathcal{M}$.

In Darboux coordinates

$$\Omega = \sum_{i=1}^{\frac{m}{2}} dx_i \wedge dy_i + \sum_{j=1}^{n} d\xi_j \wedge d\xi_j,$$

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Problem. Compute the volume of $Gr(p|q, m|n)$.

Very often the volume of the compact supermanifold is zero. For example the volume of the supergroup $U(m|n)$ (with respect to Haar volume form) is 0 unless $mn = 0$.

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Theorem (V.S.-A.Sherman)

The supergrassmannian $Gr(p|q, m|n)$ *has a non-zero volume if and only if* $\min(p,q) + \min(m-p, n-q) = \min(m,n)$ *or, equivalently,* $p-q$, $m-n$ *and* $(m-p)-(n-q)$ *are all non-negative or all non-positive.*

Main tool is Schwarz-Zaboronsky localization formula.

Introduce some notations:

- For a vector field X on a supermanifold $\mathcal{M}$ we set $Z(X)$ to be the zero locus of X.
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Theorem (A.Schwarz-O. Zaboronsky)

Let $\mathcal{M}$ be an oriented compact supermanifold with volume form ω. Assume that X is an odd vector field such that

- $X^2 = \frac{1}{2}[X, X]$ *is a compact vector field,*
- $X\omega = 0$,
- $Z(X)$ *is finite and every $p \in Z(X)$ is an isolated zero of X.*

Then there exist an odd function f on M such that

- $X^2 f = 0$,
- *the set of singular points of Xf coincides with $Z(X)$.*

For any such f we have

$$\int_{\mathcal{M}} \omega = \pi^n \sum_{p \in Z(X)} \gamma(H_p(Xf), \omega_p).$$

Note that if $Z(X) \neq \varnothing$ then $\dim \mathcal{M} = (2n|2n)$. If $Z(X) = \varnothing$ for some X then $\int_{\mathcal{M}} \omega = 0$.

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Example

Let (M_0, Ω_0) be a symplectic manifold and $M := \Pi T M_0$.

1. De Rham differential d is an odd vector field on M,
2. Functions on M are differential forms on M_0,
3. M has a canonical volume form ω via identification $TM \simeq \Pi TM$.

Let X_0 be a compact vector field on M_0 with Hamiltonian h. Then $X = d + i_{X_0}$ is an odd vector field with $X^2 = 2X_0$ and $f := h + \Omega$ is a function on M. It is easy to check that $Xf = 0$. Then

$$\int_{\mathcal{M}_0} e^{\mathrm{i}h}\Omega_0^n = \frac{\mathrm{i}^n}{n!}\int_{\mathcal{M}} e^f \omega = \pi^n \frac{\mathrm{i}^n}{n!} \sum_{p \in Z(X_0)} \gamma(H_p(h), \Omega_0^n),$$

and the Schwarz-Zaboronsky formula yields to the Duistermaat-Heckman formula.

Example

Let (M_0, Ω_0) be a symplectic manifold and $M := \Pi T M_0$.

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Dream

Conjecture

(T. Voronov)

$$\operatorname{Vol} Gr(p|q, m|n) = g_D \frac{G(a+1)G(b+1)}{G(a+b+1)},$$

where $g_{(d_0|d_1)} := \pi^{d_0} 2^{\frac{d_1}{2}}$, $a = p - q$, $b = (m - n) - (p - q)$ and $G(z)$ is the Barnes function:

$$G(n) = \begin{cases} 0, & n \leqslant 0, \\ 1, & n = 1, \\ (n-2)!!, & n > 1. \end{cases}$$

- Up to normalization the volume of supergrassmannian $Gr(V, W)$ depends only on superdimension of V and W.
- Universal Deligne's category $GL(t)$, for $t \in \mathbb{C}$ "covers" the categories $\operatorname{Rep} GL(m|n)$ with $m - n = t$.
- The supergrassmannian can be defined in the category $GL(r) \times GL(s)$.
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Some consequences

- $SL(1|1)^d$ is a detecting subgroup of $GL(m|n)$ where $d = \min(m, n)$. We conjecture that it a minimal detecting subgroup.
- We can prove that the volume of Q-grassmannian $QGr(p, n)$ is not zero iff $p(n-p)$ is even. Thus, $Q(2)^d$ or $Q(2)^d \times Q(1)$ is a detecting subgroup for $Q(n)$ with $n = 2d$ or, respectively, $n = 2d + 1$.
- We can prove that $\mathcal{X}_M = 0$ iff M is projective for $GL(m|n)$ and $Q(n)$.

We are close to proving

Conjecture

If G is a quasireductive algebraic group and M is a G-module then M is projective if and only if $\mathcal{X}(M) = \{0\}$.

Green correspondence

For a moment let G be a finite group, we work over a ground field of positive characteristic p. Let H be a p-Sylow subgroup and K be the normalizer of H. There is a bijection between indecomposable representations of G of non-zero dimension and indecomposable representations of K of non-zero dimension. This, as noticed by Etingof and Ostrik implies equivalence of semisimplifications of $\operatorname{Rep} K$ and $\operatorname{Rep} G$.

Now go back to supergroups. Let $G = GL(m|n)$, $m \leqslant n$, $H = GL(1|1)^m$, K be the normalizer of H, K is a semidirect product of H and the symmetric group $S_m \times GL(n-m)$.

Theorem (V.S.-A. Sherman-I. Entova-Aizenbud)

Let V be an indecomposable representation of G of nonzero superdimension. Then exactly one among indecomposable K-components has a non-zero superdimension.

Open questions

- The functor between $\operatorname{Rep} \bar{G} \to \operatorname{Rep} \bar{K}$ is not an equivalence (although equivalence in defect 1 case). It defines a surjective morphism of pro-reductive supergroups $\bar{K} \to \bar{G}$.
- Generalize unitary trick for other quasireductive supergroups.
- Borel-Weil-Bott theorem. New results: Kapranov–Pimenov, Sam-Snowden.

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Happy Birthday!

Вся моя интеллектуальная жизнь была сформирована тем, что я условно стал называть Просвещенческим проектом. Его основная посылка состояла в вере, что человеческий разум имеет высшую ценность, а распространение науки и просвещения само по себе неизбежно приведет к тому, что лучшие, чем мы, люди, будут жить в лучшем, чем мы, обществе.
Ничто из того, что я наблюдал вокруг себя в течение двух третей прошлого века и подходящего к концу десятилетия нового века, не оправдывало этой веры.
И все же я верю в Просвещенческий проект.