Algebra, Geometry and Physics: From Feynman graphs to moduli spaces

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Dedicated to Yuri Ivanovich Manin
on the occasion of his 85th birthday
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Plan

1. Intro
   - Thanks
   - Setup
2. Feynman cats
   - Graphs
   - Definition
3. Decoration/Grothendieck
   - $\mathcal{F}_{dec\mathcal{O}}$
4. W-construction
   - W–construction
5. Geometry
   - Moduli space geometry
6. The End

Thank You!

Inspirations and aspirations

- We thank Yuri Ivanovich for his continued guidance and support.
- His example of regarding mathematical truths, formulating and sharing his insights have been a guiding light for mathematics.
- His unmistakeable style, as evidenced by the title of this conference, is an aspirational goal for the field.
- His character, vision and overarching influence in and outside of mathematics have been a constant inspiration for me.

The progression

Journey

This research has been a journey whose beginning were the moduli spaces and their operations, which I studied with Yuri Ivanovich during my PhD. [Man99]
Predating this his book on Quantum Groups and Noncommutative Geometry [Man18] that introduced me to his way of thinking before meeting him.

The progression

Steps

1. Graphs. (Borisov-Manin), Categories (Feynman categories)
2. Algebra. (Representations operads)
3. Geometry. Moduli spaces.

Physics

1. Feynman graphs
2. Renormalization Hopf algebras

Small Ontology: two inputs

Combinatorial categorical level

Trees $\subset$ Graphs.

Expressed categorically, this gives a diagram for push–forwards and pullbacks.

Planar Trees have cyclic orders

Expressed categorically cyclic orders are a type of Grothendieck construction called decoration.

Almost ribbon graphs/topological types are push–forwards

Putting the two things together, one can push–forward the decorations and obtains a type of decorated graph. These graphs give the topological type of surfaces.

Combinatorial $\rightsquigarrow$ topology/moduli spaces

Tool

W–constructions. [KW17].
This yields a cubical complex.

Application/Results with C. Berger

1. A cubical model for Igusa type complexes homotopy equivalent to moduli spaces of curves resulting from a derived push–forward.
2. A cubical complex which is the cone over the combinatorial compactification of Penner/Kontsevich. [Kon92, Pen87]

Borisov–Manin graph categories

Objects and morphisms

1. $(V, F, \partial, \imath)$. The involution $\imath : F \to F$ glues flags F to edges.
2. $\phi = (\phi_V, \phi^F, \imath_\phi)$.
3. $\phi_V : V \twoheadrightarrow V', \phi^F : F' \hookrightarrow F, \imath_\phi : F \setminus F' \to F \setminus F'$
4. Morphisms. Glue edges, contract edges, merge vertices.

Ghost graph

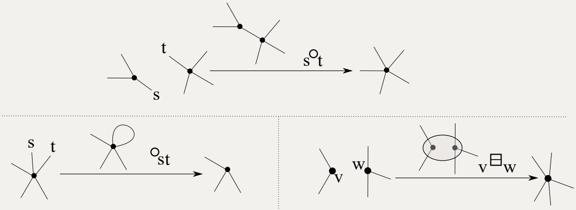
$(V, F, \partial, \imath_\phi)$. Keeps track of contracted edges and glued and then contracted edges.

Explanation in terms of Non–unique factorization

Work in progress with M. Monaco and C. Berger on graphs.

Example

Picture



Feynman Categories of graphs

Monoidal structure

Disjoint union gives a monoidal structure.

Feynman category of graphs [BK22]

- Basic objects. Irreducible objects. Connected graphs.
- Basic morphisms. Irreducible morphisms Those whose target is connected.

Feynman subcategories. Related to operadic things

- Basic objects are corollaries.
- Basic morphisms have connected underlying graphs $\mathfrak{F}^{\mathfrak{G}^{ctd}}$.
- Basic morphisms have trees as underlying graphs $\mathfrak{F}^{\mathfrak{T}} = \mathfrak{F}^{\mathrm{cyc}}$.

Feynman category

Basic morphisms $\mathfrak{F}^{\mathfrak{G}}$

- Non-self gluing or virtual edge contraction
- Self–gluing or virtual loop contraction (exclude for trees/forests)
- Merger (exclude for connected).
- Everything is labelled and kept track of.

Theorem [BK22]

The categories of graphs and aggregates from a Feynman double category, viz. a category internal to Feynman categories. Suitably restricted the double category has connections.

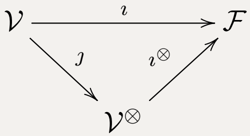
Definition of Feynman category

Data

1. $\mathcal{V}$ a groupoid
2. $\mathcal{F}$ a symmetric monoidal category
3. $\imath : \mathcal{V} \to \mathcal{F}$ a functor.

Notation

$\mathcal{V}^{\otimes}$ the free symmetric category on $\mathcal{V}$ (words in $\mathcal{V}$).



Feynman category

Definition: Data and Axioms

Such a triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ is called a Feynman category if

i. $\imath^{\otimes}$ induces an equivalence of symmetric monoidal categories between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.

ii. $\imath$ and $\imath^{\otimes}$ induce an equivalence of symmetric monoidal categories between $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.

iii. For any $* \in \mathcal{V}$, $(\mathcal{F} \downarrow *)$ is essentially small.

Operads and $\mathbb{S}$–modules in general: $\mathcal{O}ps$ and $\mathcal{M}ods$

Definition

Fix a symmetric cocomplete monoidal category $\mathcal{C}$, where colimits and tensor commute, and $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ a Feynman category.

- Consider the category of strong symmetric monoidal functors

$$\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} := Fun_{\otimes}(\mathcal{F}, \mathcal{C})$$

  which we will call $\mathcal{F}$–ops in $\mathcal{C}$ An element is called and $\mathcal{F}$–op.
- Functors form $\mathcal{V}$,

$$\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} := Fun(\mathcal{V}, \mathcal{C})$$

  will be called $\mathcal{V}$-modules in $\mathcal{C}$ with elements being called a $\mathcal{V}$–mod in $\mathcal{C}$.

The monoidal category of operations $\mathcal{O}ps$

Trival *op*

Let $\mathcal{T}_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$ be the functor which assigns $\mathbb{I} \in Obj(\mathcal{C})$ to any object, and which sends morphisms to the identity of the unit.

Proposition

With objectwise monoidal product $(\mathcal{P} \otimes \mathcal{O})(X) := \mathcal{P}(X) \otimes \mathcal{O}(X)$ $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ is a symmetric monoidal category with unit $\mathcal{T}_F$.

Structure Theorems

Theorem (Free/Monadicity/Triples)

The forgetful functor $G : \mathcal{O}ps \to \mathcal{M}ods$ has a left adjoint F (free functor) and this adjunction is monadic.

$$F : \mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} \leftrightarrows \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} : G$$

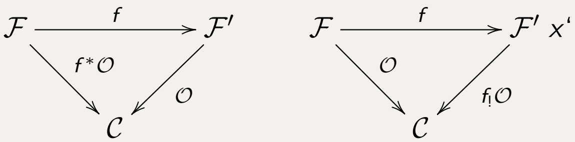
1. The endofunctor $\mathbb{T} = GF$ is a monad (triple).
2. $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$, are algebras over it.

Frobenius reciprocity

Theorem[KW17]

Feynman categories form a category. Morphisms are pairs of compatible functors. There is an adjunction

$$f_! : [\mathcal{F}, \mathcal{C}]_\otimes \leftrightarrows [\mathcal{F}', \mathcal{C}]_\otimes : f^*$$



The push–forward is given by a left Kan extension $f_! = Lan_f$. The theorem is that this functor is monoidal.

Remarks

Sometimes there is also a right adjoint $f_* = Ran_f$ which is "extension by zero". These will form part of a 6 functor formalism [War19].

Basic objects and basic morphisms

Unraveling the axioms: Consequences

1. $X \simeq \bigotimes_{v\in I} \imath(*_v)$.
   - $*_v \in \mathcal{V}$. The $\imath(*_v)$ are called the basic objects.
   - $Iso(X) \simeq \bigotimes_{v\in I} Iso(*_v)$
2. $\phi : Y \to X$,
   - $\phi \simeq \bigotimes_{v\in I} \phi_v$
   - $\phi_v : Y_v \to \imath(*_v)$, $Y \simeq \bigotimes_{v\in I} Y_v$.
   - The morphisms $\phi_v : Y \to \imath(*_v)$ are called basic morphisms.

Hereditary property

Hereditary diagram

1. In particular, fix $\phi : X \to X'$ and fix $X' \simeq \bigotimes_{v \in I} \imath(*_v)$: there are $X_v \in \mathcal{F}$, and $\phi_v \in Hom(X_v, *_v)$ s.t. the following diagram commutes.

(1)
$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ \downarrow \simeq & & \downarrow \simeq \\ \bigotimes_{v \in I} X_v & \xrightarrow{\bigotimes_{v \in I} \phi_v} & \bigotimes_{v \in I} \imath(*_v) \end{array}$$

2. For any two such decompositions $\bigotimes_{v \in I} \phi_v$ and $\bigotimes_{v' \in I'} \phi'_{v'}$ there is a bijection $\psi : I \to I'$ and isomorphisms $\sigma_v : X_v \to X'_{\psi(v)}$ s.t. $P_\psi^{-1} \circ \bigotimes_v \sigma_v \circ \phi_v = \bigotimes \phi'_{v'}$ where P_ψ is the permutation corresponding to ψ.
3. These are the only isomorphisms between morphisms.

Examples see e.g. [Kau21]

Set versions

1. Finite sets. Restriction to injections and surjections.
2. Ops are unital algebras, FI algebras and nonunital algebras,
3. If one considers the non–symmetric analogue, one obtains ordered sets, with order preserving surjections and associative algebras.
4. The $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ for $\mathcal{F}in\mathcal{S}et$ are unital commutative algebras. item $\Delta_{+}S$ crossed simplicial group. There are the skeleton of non-commutative sets: order on the fibers of morphisims

Examples based on $\mathfrak{G}$: morphisms have underlying graphs

$\mathfrak{F}$	Feynman category for	condition/additional decoration
$\mathfrak{F}^{operad}$	operads	rooted trees
$\mathfrak{O}^{pl}$	non-Sigma operads	planar rooted trees
$\mathfrak{F}^{operad,mult}$	operads with mult.	b/w rooted trees.
$\mathfrak{F}^{\Sigma} = \mathfrak{F}^{\mathrm{cyc}}$	cyclic operads	trees
$\mathfrak{F}^{\neg\Sigma\mathrm{cyc}}$	non Σ cyclic operads	planar trees
$\mathfrak{F}^{\mathfrak{G}^{ctd}}$	unmarked modular operads	connected graphs
$\mathfrak{F}^{\mathrm{mod}}$	modular operads	connected + genus marking
$\mathfrak{F}^{\neg\Sigma\ mod}$	non-sigma modular operads	connected +surface marking
$\mathfrak{F}^{\mathfrak{G}}$	unmarked nc modular operads	graphs
$\mathfrak{F}^{ncmod}$	nc modular operads	genus marking
$\mathfrak{F}^{nc\neg\Sigma mod}$	nc non Σ modular operads	surface marking
$\mathfrak{F}^{diop}$	dioperads	connected directed graphs w/o directed loops or parallel edges
$\mathfrak{F}^{PROP}$	PROPs	directed graphs w/o directed loops
$\mathfrak{F}^{properad}$	properads	connected directed graphs w/o directed loops

Table: List of Feynman categories with conditions and *decorations*

Adjunction

Classical Frobenius reciprocity

Taking $\mathcal{V}_1 = \underline{H}$, $\mathcal{F}_1 = \mathcal{V}_1^{\otimes}$, $\mathcal{V}_2 = \underline{G}, \mathcal{F}_2 = \mathcal{V}_2^{\otimes}$ with the standard inclusion, then for a morphism induced by an inclusion $f : \underline{H} \to \underline{G}$, the adjunction means that:

$$Hom_{k[G]}(ind_H^G \rho, \lambda) \leftrightarrow Hom_{k[H]}(\rho, res_H^G \lambda)$$

Application to graphs and moduli spaces

We will consider the inclusion $k : \mathfrak{F}^{\mathfrak{T}} \to \mathfrak{F}^{\mathfrak{G}^{ctd}}$.
This realizes the inclusion trees $\subset$ connected graphs.

$\mathfrak{F}_{dec\mathcal{O}}$ w/ J. Lucas, C. Berger. Grothendieck construction

Theorem

Given an $\mathcal{O} \in \mathcal{F}$–$\mathcal{O}ps$, then there is a Feynman category $\mathcal{F}_{dec\mathcal{O}}$ which is indexed over $\mathcal{F}$.

- *It objects are pairs $(X, dec \in \mathcal{O}(X))$*
- *$Hom_{\mathcal{F}_{dec\mathcal{O}}}((X, dec), (X', dec'))$ is the set of $\phi : X \to X'$, s.t. $\mathcal{O}(\phi)(dec) = dec'$.*

(This construction works a priori for Cartesian $\mathcal{C}$, but with modifications it also works for the non–Cartesian case.)

Example

$\mathfrak{F} = \mathfrak{F}^{\mathrm{cyc}}$, $\mathcal{O} = CycAss$, $CycAss(*_S) = \{\text{cyclic orders} \prec \text{on } S\}$. New basic objects of $\mathfrak{C}_{decCycAss}$ are planar corollas $*_{S,\prec}$. Basic morphisms have underlying trees with a planar structure.

Main result

Theorem

(2)

$$\begin{array}{ccc} \mathfrak{F}_{dec\mathcal{O}} & \xrightarrow{f^{\mathcal{O}}} & \mathfrak{F}'_{dec\, f_!(\mathcal{O})} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathfrak{F} & \xrightarrow{f} & \mathfrak{F}' \end{array} \qquad \begin{array}{ccc} \mathfrak{F}_{dec\mathcal{O}} & \xrightarrow{\sigma_{dec}} & \mathfrak{F}_{dec\mathcal{P}} \\ \downarrow{\scriptstyle f^{\mathcal{O}}} & & \downarrow{\scriptstyle f^{\mathcal{P}}} \\ \mathfrak{F}'_{dec f_!(\mathcal{O})} & \xrightarrow{\sigma'_{dec}} & \mathfrak{F}'_{dec f_!(\mathcal{P})} \end{array}$$

The squares above commute squares and are natural in $\mathcal{O}$. We get the induced diagram of adjoint functors.

(3)

$$\begin{array}{ccc} \mathcal{F}_{dec\mathcal{O}}\text{-}\mathcal{O}ps & \underset{f^{\mathcal{O}}{}_*}{\overset{f_!^{\mathcal{O}}}{\rightleftarrows}} & \mathcal{F}'_{dec\, f_!(\mathcal{O})}\text{-}\mathcal{O}ps \\ {\scriptstyle \pi_!}\downarrow\uparrow{\scriptstyle \pi^*} & & {\scriptstyle \pi'^*}\uparrow\downarrow{\scriptstyle \pi'_!} \\ \mathcal{F}\text{-}\mathcal{O}ps & \underset{f^*}{\overset{f_!}{\rightleftarrows}} & \mathcal{F}'\text{-}\mathcal{O}ps \end{array}$$

More $\mathcal{F}_{dec\mathcal{O}}$ [KL17]

Theorem

$\pi_!\pi^*(\mathcal{T}) = \mathcal{O}$.

Definition

We call a morphism of Feynman categories $i : \mathfrak{F} \to \mathfrak{F}'$ *a connected if* $i_!(\mathcal{T}_{\mathcal{F}}) = \mathcal{T}_{\mathcal{F}'}$ *in* $\mathfrak{F}'$-$\mathcal{O}ps_{\mathcal{C}}$*, where* $\mathcal{T}_{\mathcal{F}} : \mathcal{F} \to \mathcal{S}et$ *is the terminal* $\mathcal{S}et$ *operation.*

Proposition

If $f : \mathfrak{F} \to \mathfrak{F}'$ *is a connected then* $f^{\mathcal{O}} : \mathfrak{F}_{dec\mathcal{O}} \to \mathfrak{F}'_{decf_!(\mathcal{O})}$ *is as well.*

Factorization

Theorem (w/ C. Berger)

Any morphism of Feynman $f : \mathfrak{F} \to \mathfrak{F}'$ categories factors as connected morphism followed by a covering, viz. the projection of a set decoration.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & \mathcal{F}'_{dec\ f_!(\mathcal{T})} \\ & \searrow^{f} & \downarrow \\ & & \mathcal{F}' \end{array}$$

Furthermore, connected morphisms and covering form a factorization system. In particular, the factorization is essentially unique.

These come from a comprehension scheme of connected morphisms and coverings.

Theorem w. C. Berger [BK22]

Connecting the different categories

$$
\begin{array}{ccc}
\mathfrak{F}^{\mathrm{cyc}}_{dec\ CycAss} = \mathfrak{F}^{\neg\Sigma\mathrm{cyc}} & \xrightarrow{i^{CycAss}} & \mathfrak{F}^{\mathrm{mod}}_{dec\ i_!(CycAss)} = \mathfrak{F}^{\neg\Sigma\mathrm{mod}} \\
\downarrow \pi & & \downarrow \pi \\
\mathfrak{F}^{\mathrm{cyc}} & \xrightarrow{i} & \mathfrak{F}^{\mathrm{mod}} = \mathfrak{F}^{\mathfrak{G}^{ctd}}_{j_!(\mathcal{T})} \\
& \searrow j & \downarrow \pi \\
& & \mathfrak{F}^{\mathfrak{G}^{ctd}}
\end{array}
$$

1. The commutative square exists simply by the Theorems about decoration and factorization.
2. $\mathfrak{F}^{\mathrm{mod}}$ are the Getzler-Kapranov [GK98] modular operads. These now are *defined* by push-forward of the trivial functor.

Theorem w. C. Berger [BK22]

Connecting the different categories

$$
\begin{array}{ccccc}
\mathfrak{F}^{\mathrm{cyc}}_{dec\ CycAss} = \mathfrak{F}^{\neg\Sigma\mathrm{cyc}} & \xrightarrow{i^{CycAss}} & \mathfrak{F}^{\mathrm{mod}}_{dec\ i_!(CycAss)} = \mathfrak{F}^{\neg\Sigma\mathrm{mod}} & & \\
\downarrow \pi & & \downarrow \pi & & \\
\mathfrak{F}^{\mathrm{cyc}} & \xrightarrow{i} & \mathfrak{F}^{\mathrm{mod}} = \mathfrak{F}^{\mathfrak{G}^{ctd}}_{j_!(\mathcal{T})} & & \\
& \searrow^{j} & \downarrow \pi & & \\
& & \mathfrak{F}^{\mathfrak{G}^{ctd}} & &
\end{array}
$$

1. $\mathfrak{F}^{\neg\Sigma\mathrm{cyc}}$-$\mathcal{O}ps$ are non–Σ cyclic operads.
2. $j_!(\mathcal{T})(*_S) = \amalg_{g\in\mathbb{N}} *$. Accordingly the basic objects of $\mathfrak{F}^{\mathrm{mod}}$ are of the form $*_{g,S}$. These can be thought of as the topological types of an oriented surfaces: genus g with S boundaries.

Modular/Surface theory

Modular graphs and almost ribbon graphs

$$
\begin{array}{ccccc}
\mathfrak{F}^{\mathrm{cyc}}_{dec\ CycAss} = \mathfrak{F}^{\neg\Sigma \mathrm{cyc}} & \xrightarrow{i^{CycAss}} & \mathfrak{F}^{\mathrm{mod}}_{dec\ i_!(CycAss)} = \mathfrak{F}^{\neg\Sigma \mathrm{mod}} \\
\downarrow \pi & & \downarrow \pi \\
\mathfrak{F}^{\mathrm{cyc}} & \xrightarrow{i} & \mathfrak{F}^{\mathrm{mod}} = \mathfrak{F}^{\mathfrak{G}^{ctd}}_{j_!(\mathcal{T})} \\
& \searrow^{j} & \downarrow \pi \\
& & \mathfrak{F}^{\mathfrak{G}^{ctd}}
\end{array}
$$

4. $\mathfrak{F}^{\neg\Sigma \mathrm{mod}}$-$\mathcal{O}ps$ are non–sigma modular operads [Mar16, KP06]. Objects of $\mathcal{V}$ are $*_{g,s,S_1,\dots,S_b}$ where each S_i has a cyclic order. These can be thought of as oriented surfaces with genus g, s internal marked points, b boundaries where each boundary i has marked points labelled by S_i in the given cyclic order.

Example

Bootstrap

$$\begin{array}{ccccccc}
\mathfrak{F}^{\mathrm{cyc}}_{dec\ CycAss} & = & \mathfrak{F}^{\neg\Sigma\mathrm{cyc}} & \xrightarrow{i^{CycAss}} & \mathfrak{F}^{\mathrm{mod}}_{dec\ i_!(CycAss)} & = & \mathfrak{F}^{\neg\Sigma\mathrm{mod}} \\
\downarrow \pi & & & & \downarrow \pi & & \\
\mathfrak{F}^{\mathrm{cyc}} & & \xrightarrow{\quad i \quad} & & \mathfrak{F}^{\mathrm{mod}} & = & \mathfrak{F}^{\mathfrak{G}^{ctd}}_{j_!(\mathcal{T})}
\end{array}$$

$$\begin{array}{ccc}
\mathfrak{F}^{\mathrm{cyc}} & \xrightarrow{\quad j \quad} & \mathfrak{F}^{\mathfrak{G}^{ctd}} \\
& & \uparrow{\scriptstyle \pi} \ \text{from } \mathfrak{F}^{\mathrm{mod}}
\end{array}$$

5. This is now actually a *calculation*. A succinct proof uses the theorem that the spanning tree graph is connected and mutations act transitively.

More details in [BK22]

Remarks

- The calculation of the push-foward can be done in
  1. graphs, the calculation involving spanning trees (new and important for the following)
  2. cyclic words, classification of surfaces by labelling schemes.
  3. topological surfaces, classification of surfaces, [KP06], [CL07].
  4. chord diagrams, (nice formalism).
  5. other combinatorial gadgets [Mar16].
- Fobenius reciprocity can be used to give a new proof and understanding of the well known results that 2d TFT and OTFT being defined by Frobenius algebras.
- This can be used to recover the algebraic string topology ([CS99]) operations of [Kau08] purely from graphs and natural constructions.

Intermediate covers

Proposition

There is a tower of $\mathfrak{F}^{\mathfrak{G}}$-ops$_{\mathcal{S}}$et

$$\mathcal{O}_{Fl} \to \mathcal{O}_{Euler,poly} \to \mathcal{O}_{genus} \to *$$

and accordingly a tower of covers

$$\mathfrak{F}^{\neg\Sigma\, mod} \to \mathfrak{F}^{\mathfrak{G}^{ctd}}_{dec\,\mathcal{O}_{Euler,poly}} \to \mathfrak{F}^{\mathrm{mod}} = \mathfrak{G}^{ctd}_{dec\mathcal{O}_{genus}} \to \mathfrak{G}^{ctd}$$

□

Remarks

The functors

- $\mathcal{O}_{FI}(*_S)$ is the set of tuples $(g, p, \{S_1^{\circlearrowleft}, \ldots, S_b^{\circlearrowleft}\})$ of two natural numbers and a {partition of S into *non–empty sets* and a cyclic structure on each of the sets}. These signify the genus, number of punctures and marked points on the boundary. Alternatively, p can also be thought of as the number of emtpy partitions in the partition of S
- $\mathcal{O}_{Euler,poly}(*_S) = \mathbb{N}_0 \times$ {partitions of S into *non–empty sets* and a cyclic structure on each of the sets}. The natural number is the 1-Euler characteristic of the closed surface.

Intermediate coverings

Proposition

There are morphisms between the $\mathfrak{F}^{\mathfrak{G}}$ *– ops:*

$$\mathcal{O}_{Fl} \to \mathcal{O}_{poly_{\mathbb{N}}} \to \mathcal{O}_{poly} \to *$$

and accordingly a tower of covers

$$\mathfrak{F}^{\neg\Sigma\, mod} \to \mathfrak{F}^{\mathfrak{G}\, ctd}_{dec\, \mathcal{O}_{poly_{\mathbb{N}}}} \to \mathfrak{F}^{\mathfrak{G}\, ctd}_{dec \mathcal{O}_{poly}} \to \mathfrak{F}^{\mathfrak{G}\, ctd}$$

Functors

- $\mathcal{O}_{poly_{\mathbb{N}}}(*_S)$ is the set of possibly empty partitions of S with cyclic order on the parts.
- $\mathcal{O}_{poly}(*_S)$ is the set of non–empty partitions of S.
- Note: Neither $\mathcal{O}_{poly_{\mathbb{N}}}$ nor $\mathcal{O}_{poly}$ map to $\mathcal{O}_{genus}$.

History

Remark

1. The $\mathfrak{G}^{ctd}$-*op* $\mathcal{O}_{FI}$ first appeared in the gluing description with ${}_s\circ_t$ and $\circ_{ss'}$ in [KP06] as the open part of the c/o structure given by π_0 of $\mathcal{ARC}$. This is an extension of [KLP03].
2. $\mathfrak{F}^{\neg\Sigma\, mod}$ was first formalized in [Mar16].
3. The correlation functions of [Kau08] use the projection to $\mathcal{O}_{Euler,poly}$. (Application to string topology).
4. The full non–sigma modular structure is needed for open/closed string topology [Kau10].
5. Props or operads using $\mathcal{O}_{poly}$ and $\mathcal{O}_{poly_{\mathbb{N}}}$ arise in different situations.

W-construction [KW17]

Input: Cubical Feynman categories in a nutshell

- Ex: $\phi_{e_1} \circ \phi_{e_2} = \phi_{e'_2} \circ \phi_{e'_1}$, commutative square for two consecutive edge contractions.
- Generators and relations for basic morphisms.
- Additive length function $l(\phi)$, $l(\phi) = 0$ equivalent to ϕ is iso.
- Quadratic relations and every morphism of length n has precisely $n!$ decompositions into morphisms of length 1 up to isomorphisms.

Definition

Let $\mathcal{P} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{T}op}$. For $Y \in ob(\mathcal{F})$ we define

$$W(\mathcal{P})(Y) := colim_{w(\mathfrak{F},Y)} \mathcal{P} \circ s(-)$$

Technical Details

The category $w(\mathfrak{F}, Y)$, for $Y \in \mathcal{F}$ Objects:

Objects are the set $\coprod_n C_n(X, Y) \times [0,1]^n$, where $C_n(X, Y)$ are chains of morphisms from X to Y with n degree ≥ 1 maps modulo contraction of isomorphisms.
An object in $w(\mathfrak{F}, Y)$ will be represented (uniquely up to contraction of isomorphisms) by a diagram

$$X \xrightarrow[f_1]{t_1} X_1 \xrightarrow[f_2]{t_2} X_2 \to \cdots \to X_{n-1} \xrightarrow[f_n]{t_n} Y$$

where each morphism is of positive degree and where $t_1, \ldots, t_n$ represents a point in $[0,1]^n$. These numbers will be called weights. Note that in this labeling scheme isomorphisms are always unweighted.

Setup: cubical Feynman category $\mathfrak{F}$

The category $w(\mathfrak{F}, Y)$, for $Y \in \mathcal{F}$ Morphisms:

1. Levelwise commuting isomorphisms which fix Y, i.e.:

$$\begin{array}{ccccccccccc} X & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \dots & \longrightarrow & X_n & \longrightarrow & Y \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & & & \downarrow \cong & \nearrow & \\ X' & \longrightarrow & X_1' & \longrightarrow & X_2' & \longrightarrow & \dots & \longrightarrow & X_n' & & \end{array}$$

2. Simultaneous $\mathbb{S}_n$ action.
3. Truncation of 0 weights: morphisms of the form $(X_1 \xrightarrow{0} X_2 \to \cdots \to Y) \mapsto (X_2 \to \cdots \to Y)$.
4. Decomposition of identical weights: morphisms of the form $(\cdots \to X_i \xrightarrow{t} X_{i+2} \to \dots) \mapsto (\cdots \to X_i \xrightarrow{t} X_{i+1} \xrightarrow{t} X_{i+2} \to \dots)$ for each (composition preserving) decomposition of a morphism of degree ≥ 2 into two morphisms each of degree ≥ 1.

Nontechnical version

Nontechnical version for graphs

Glue together cubes. One n-cube for each graph with n edges. There are two boundaries per edge. Contract or mark. Glue along these edges.

Remark (Kreimer)

This is exactly what happens in Cutkosky rules. Only instead of marking edge as fixed, forget (aka. cut) the edge.

Example for an algebra, [GCKT20a]

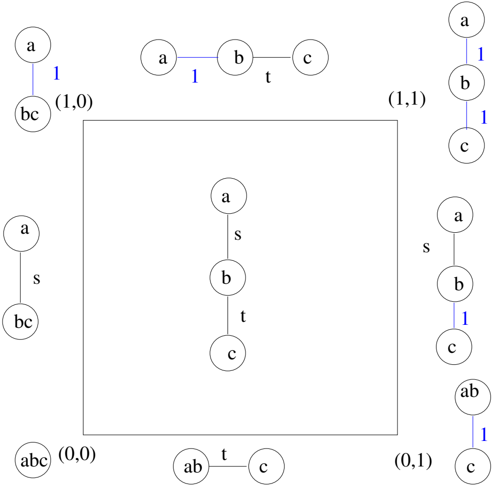


Figure: The cubical structure in the case of $n = 3$. One can think of the edges marked by 1 as cut.

Other interpretations of the same picture

Remark

The cubical structure also becomes apparent if we interpret $[n] = 0 \to 1 \to 2 \to \cdots \to n$ as the simplex.

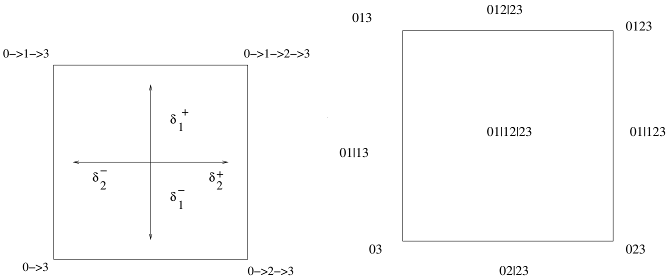


Figure: Two other renderings of the same square. Note: $0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{c} 3$

Cubical decomposition of associahedra

$W(Ass)$

The associative operad $Ass(n) = regular(\mathbb{S}_n)$. $W(Ass)(n)$ is a cubical decomposition of the associahedron.

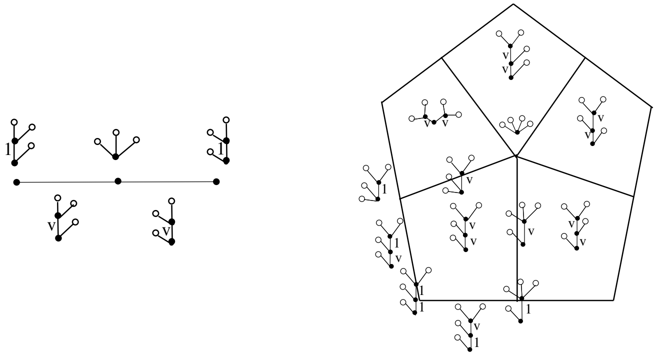


Figure: The cubical decomposition for K_3 and K_4, v indicates a variable height.

Models for moduli spaces and push–forwards

The square revisited

$$
\begin{array}{ccc}
\mathfrak{F}^{\neg\Sigma\mathrm{cyc}} & \xrightarrow{i^{CycAss}} & \mathfrak{F}^{\neg\Sigma\mathrm{mod}} \\
\downarrow \pi & & \downarrow \pi \\
\mathfrak{F}^{\mathrm{cyc}} & \xrightarrow{i} & \mathfrak{F}^{\mathrm{mod}}
\end{array}
$$

Theorem with C. Berger

1. $Wi_!(CycAss)(*_{g,n}) = Cone(\bar{M}^{K/P}_{g,n}) \supset \bar{M}^{K/P}_{g,n} \supset M_{g,n}$, metric almost ribbon graphs (emtpy graph is allowed).
2. $i_!^{cycAss}(W\mathcal{T})(*_{g,s,S_1\amalg\cdots\amalg S_b}) \simeq M_{g,s,S_1\amalg\cdots\amalg S_b}$. This is a generalization of Igusa's theorem $M_{g,n} \simeq Nerve(IgusaCat)$ [Igu02]

Details

Open moduli space

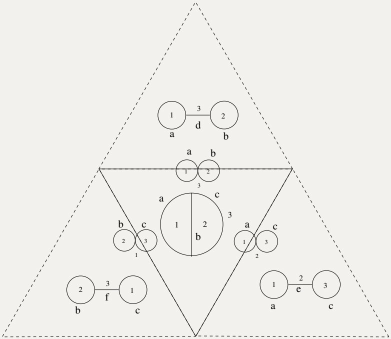
1. The first part of the homotopy equivalence is an identification of the nerve of the Igusa–type category with a simplicial decomposition of the cubical complex where each n–cube is subdivided into $n!$ simpilces.
2. The second part is the natural emedding into the combinatorial moduli space.
3. The third part is a retraction of moduli space as an open subset of the compactified combinatorial moduli space to the embedded nerve.

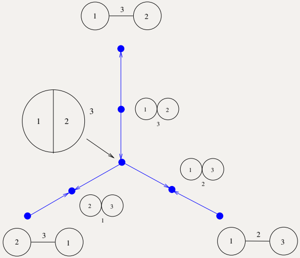
Compactified moduli space

This uses the fact that the cone point is $(0, \ldots, 0)$ corner of any cube and that any cube minus this point can be retracted to the simplex $\sum_i t_i = 1$.
This can be done coherently.

$M_{0,3}$

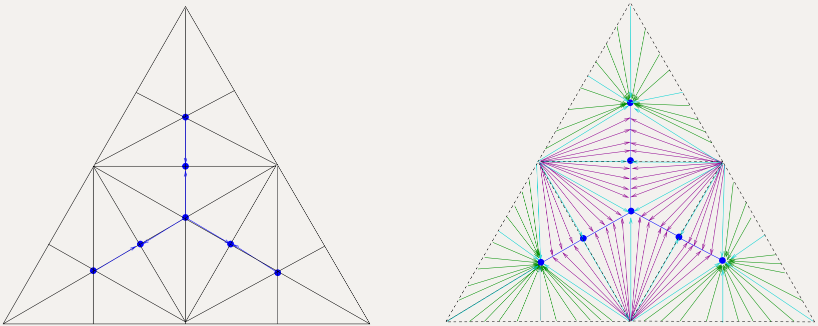
$M_{0,3}^{comb}$, its spine/nerve





$M_{0,3}$

$M_{0,3}^{comb}$, its spine/nerve and the retraction



Cutkosky/Outer space, w/ C. Berger

The cube complex $j_!(W(CycAss))(*_S)$

Is the complex whose cubical cells are indexed by pairs (Γ, τ), where

- Γ is a graph with S–labelled tails and τ is a spanning forest.
- The cell has dimension $|E(\tau)|$
- the differential ∂_e^- contracts the edge
- ∂_e^+, removes the edge from the spanning forest.

Remark

This complex and the differential are not defined by hand, but automatic!

Outer space

Without the almost ribbon structure, we obtain analogous results pertaining to Outer Space.

Blow-ups/Compactifications w/ J.J. Zuniga

Claim

1. There is a natural blow–up of the W–construction above, which is induced by the cubical structure of the Feynman category. This leads to new compactificaion of the moduli space.
2. There is a sequence of blow–downs which terminates in the final blow–down $\overline{M}^{KSV}_{g,n} \to \overline{M}^{DM}_{g,n} \to \overline{M}^{comb}_{g,n}$.
3. This can be modeled on both the analytic/algebraic side and the combinatorial side, giving the desired orbircell decomposition to all spaces.

Remark

This is driven by master-equations [KWZn15] and is directly related to the Jewels of Vogtman et. al. and the truncations in QFT (Kreimer group).

Quantum and Noncommutative

Hopf algebras and quadratic

1. There are Hopf and bialgebras hidden here [GCKT20a, GCKT20b, KY21].
2. Relations to [KM01].
3. This works for general cubical Feynman categories [KW21].
4. more . . .

The end

Thank you

I hope you enjoyed the tour of mathematical results which trace their being back to Yuri Ivanovich.

To Yuri Ivanovich

Happy birthday!

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