Blow-up formula for quantum cohomology

(joint work in progress with L.Katzarkov, T.Pantev and T.Yu)

Brief reminder: Gromov-Witten invariants

X : smooth projective variety / k + ample $\mathcal{O}(1)$ (could be of char $\neq 0$)

or

compact symplectic mfd, $[\omega] \in$ image of $H^2(X,\mathbb{Z})$

$\rightsquigarrow$ $\underbrace{\text{genus} = 0}_{\text{for today's talk}}$ GW invariants:

classes $I_{n,\beta} \in \mathrm{Chow}_\bullet(X^n) \otimes \mathbb{Q}$

$\forall\, n \geq 0$, $\forall\, \beta \in CH_1(X)/\sim_{\text{numerical}}$

or

$\beta \in H_2(X,\mathbb{Z})$

(images of $[\mathcal{M}_{0,n}(X,\beta)]_{\text{virt}}$)

$\rightsquigarrow$ generating series

$$\mathcal{F}_0 = \sum_{\beta \in H_2(X,\mathbb{Z})} q^{\int_\beta [\omega]} \sum_{n \geq 0} \frac{1}{n!} \sum_{\alpha_1 \ldots \alpha_n} \pm \int_{I_{n,\beta}} \Delta^{\alpha_1} \otimes \ldots \otimes \Delta^{\alpha_n} \cdot t_{\alpha_1} \cdots t_{\alpha_n}$$

where (Δ^α) - graded basis of $H^\cdot(X, \mathbb{Q})$

(t_α) - dual to (Δ^α) super. variables

$$\mathcal{F}_0 \in \mathbb{Q}[[q]][[(t_\alpha)]]$$

Consider $\mathcal{F}_0$ as a function on the open unit ball in the affine space $H^\cdot(X,\mathbb{Q}) \otimes \mathbb{Q}((q))$ / non-archimedean field $\mathbb{Q}((q))$

Poincaré pairing $\rightsquigarrow$ flat metric $\eta^{\alpha\alpha'} dt_\alpha dt_{\alpha'}$

$\rightsquigarrow$ quantum product on $H^\cdot(X)$ parametrized by (t_α):

$$\Delta^{\alpha_1} * \Delta^{\alpha_2} := \sum_{\alpha_3, \alpha} \frac{\partial^3 \mathcal{F}_0}{\partial t_{\alpha_1} \partial t_{\alpha_2} \partial t_{\alpha_3}} (\eta^{-1})_{\alpha\alpha_3} \Delta^{\alpha}.$$

WDVV equation:

quantum product is associative
(automatically commutative and unital)

1993-1994 : M.K + Yu. Manin "Gromov-Witten classes....."

translated Witten's paper
"Two-dimensional gravity ....." (1991)
to mathematics

+ added new concepts and ideas.

"New" notion

$\mathcal{F}$-bundle

Data: a supermanifold B (formal, analytic, smooth, ...)
over field k of char $= 0$

\+ a supervector bundle $\mathcal{H} / B \times \mathrm{Specf}\, k[[u]]$

\+ a meromorphic connection ∇ on $\mathcal{H}$ s.t.

Axioms: ① $\nabla^2 = 0$ (flat)

② $\nabla_{u^2 \frac{\partial}{\partial u}}$ is a regular operator (no poles)

③ $\nabla_{u \xi}$ for $\forall \xi \in \Gamma(B, T_B)$ is also regular.

Example: $k = \mathbb{Q}((q))$, $B =$ open unit ball in $H^{\cdot}(X, k)$

$\mathcal{H}$ is trivialized, fibers $= H^{\cdot}(X)$

$$\nabla_{\frac{\partial}{\partial u}} = \frac{\partial}{\partial u} + \frac{1}{u^2} K + \frac{1}{u} G$$

$$\nabla_{\frac{\partial}{\partial t_\alpha}} = \frac{\partial}{\partial t_\alpha} + \frac{1}{u} A^\alpha$$

where

$$A^\alpha = \Delta^\alpha * .$$

$$G = \bigoplus_{i=0}^{2\dim X} \frac{i - \dim X}{2} \, id_{H^i(X)}$$

$$K = \Big[-c_1 T_X + \sum_{\alpha;\ \deg \Delta^\alpha \neq 2} \frac{\deg \Delta^\alpha - 2}{2} t_\alpha \Delta^\alpha \Big] * .$$

In the example coming from GW theory,
rk of bundle $\mathcal{H}$ = dim of B.

Encoding in general theory:
for $\forall$ F-bundle on B, $\forall\, b \in B$

$$T_b B \xrightarrow[\mathrm{res}_{u=0}]{} \mathrm{End}(\mathcal{H}_{b,0})$$

commuting operators

<u>Def</u>: F-bundle is maximal (at b) if $\exists$ even $v \in \mathcal{H}_{b,0}$
s.t. $T_b B \to \mathrm{End}(\mathcal{H}_{b,0}) \xrightarrow{\text{action on } v} \mathcal{H}_{b,0}$
is an isomorphism.

(from <u>maximal</u> family of commuting operators)

$\boxed{\text{open condition in } b \in B}$

Maximality $\rightsquigarrow$ $T_b B$ is a unital commutative associative superalgebra

$\rightsquigarrow$ B is an F-manifold (C. Hertling, Yu. Manin)

$\rightsquigarrow$ Euler field Eu on B (Coefficient of u^{-2} at $\nabla_{\frac{\partial}{\partial u}}$ commutes with operator in $T_b B \subset End(\mathcal{H}_{b,0})$)

in the GW example:

$$Eu = \frac{\partial}{\partial t_{\alpha_1}} + \sum_\alpha \frac{\deg(\Delta^\alpha) - 2}{2} t_\alpha \partial_{t_\alpha} \quad \text{where } \Delta^{\alpha_1} = -c_1(T_X)$$

is an <u>affine</u> vector field.

K_b = action of $Eu|_b$ on $\mathcal{H}_{b,0}$

(Almost) universal source of maximal $\mathcal{F}$-bundles:

(extended) moduli space for

a smooth proper $\mathbb{Z}/2\mathbb{Z}$-graded CY

A_∞ (or dg) category $\mathcal{C}$

with degeneration Hodge $\Rightarrow$ de Rham

Includes A, B-model (Fukaya category, $D^b(\mathrm{Coh}\,X)$)

Hence, homological mirror symmetry

explains Hodge-theoretic description of $\mathcal{F}_0$.

Quantum spectrum

for an F-bundle $/B$ and a (even) point $b \in B_{red}$

$\rightsquigarrow$ operator $Eu_b \in \mathrm{End}(\mathcal{H}_{b,0})$

it has a spectrum "$\subset \mathbb{C}$"

$$(\lambda_i) = \mathrm{Spec}\, Eu_b$$

<u>Decomposition</u>: for a <u>maximal</u> F-bundle,

a formal neighborhood of b in B $\overset{\text{canonically}}{\cong}$ $\coprod_{\lambda_i}$ (formal maximal F-bundles $/B_i$)

Facts and expectations in algebraic case:

Consider affine subspace in B given by linear combinations of algebraic classes

$$\left(\text{Image } Chow^{\cdot}(X) \to H^{\cdot}(X) \otimes \mathbb{Q}((q)) \right)$$

① (Fact) any point $\lambda_i \in Spec(Eu \cdot)$ gives a canonical iso. class of a non-commutative motive ($2\pi i = 1$, mod out by $\mathbb{Z}(1)$) with a non-degenerate possibly non-symmetric pairing

② (Hope) choose Gabrielov paths

$\Rightarrow$ SOD of $\left(H^{\cdot}(X), \substack{\text{Euler} \\ \chi Hom(\dots)}\right)$ ②′ : S.O.D. of $D^b(Coh\, X)$!

Blow-up formula

X smooth projective $\supset Y$ smooth closed

codim $Y = m \geq 2$

$\rightsquigarrow \tilde{X} := Bl_Y X$

Theorem (D. Orlov):

$D^b Coh\, \tilde{X} = \langle D^b Coh\, X, \underbrace{D^b Coh\, Y, \dots, D^b Coh\, Y}_{(m-1) \text{ times}} \rangle$

Blow-up formula identifies near some limiting points at ∞ the base $B_{\tilde{X}}$ of GW theory F-bundle

with $B_{X \cup \underbrace{Y \cup \dots \cup Y}_{(m-1) \text{ times}}} = B_X \times \underbrace{B_Y \times \dots \times B_Y}_{(m-1)}$

About "limiting points":

variable q is redundant, can be replaced by

$t_{\alpha_0} := \log(q)$ where $\delta^{\alpha_0} = [\omega]$ ample class

In a sense, we expand $\mathcal{F}_0$ near

"$(-\infty)\cdot[\omega]$" in $H^\cdot(X,\mathbb{Q})$

In general, one can expand at a "point"

"$(-\infty)[\omega'] + (\delta_0 + \delta_4 + \delta_6 + \ldots)$" where $\exists\, \varepsilon > 0$ s.t.

$\omega' - \varepsilon K_X > 0$

and $\delta_0 \in H^0(X)$, $\delta_4(X) \in H^4(X)$ ...

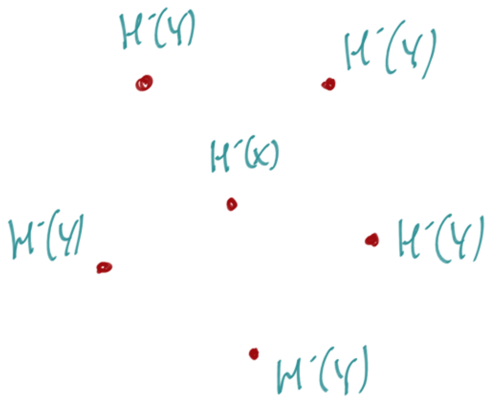
are arbitrary even classes.

Limiting point for $\tilde{X} = Bl_Y X$

$$\begin{array}{c} \tilde{X} \\ \downarrow \pi \\ X \end{array}$$

"$-\infty \cdot \underbrace{\pi^*(\omega_X)}_{\text{ample class on } X}$"

Quantum spectrum at this limiting point is



$$\{0\} \cup \{(m-1)\sqrt[m-1]{-1}\}$$

Limiting point for $X \cup Y \cup \dots \cup Y$ is much more trickier

given by "$-\infty [\omega_X] + \delta_4^{(X)} + \delta_6^{(X)} + \dots$"

$+$ "$-\infty (\text{pullback}_{Y \subset X})^* [\omega_Y] + \delta_0^{(Y)} + \delta_4^{(Y)} + \delta_6^{(Y)} + \dots$" for copies of Y

where $\delta_\cdot^{(Y)}$ are certain universal polynomials in $\mathbb{Q}[c_1, c_2, \dots, c_m]$ $\quad c_i = c_i(N_{Y \subset X})$

$\delta_\cdot^{(X)} = (\text{pushforward}_{Y \subset X})_*$ of another collection of universal elements of $\mathbb{Q}[c_1, c_2 \dots]$

depending on codimension $m \geq 2$ only

Example: $m = 2$

$$\delta^{(X)} = (Y \hookrightarrow X)_* \left(-\frac{1}{3} c_2 + \frac{1}{4} c_1 c_2 - \frac{1}{5} (c_1^2 c_2 + c_2^2) + \dots \right)$$

$$\delta^{(Y)} = -1 + c_2 - \frac{1}{2} c_1 c_2 + \frac{1}{3} (c_1^2 c_2 + c_2^2) + \dots$$

Details + explicit description of the hypothetical iso $B_{\tilde{X}} \sim \Delta_X \times \underbrace{B_Y \times \dots \times B_Y}_{m-1}$ near limiting points:

look at my talk (November 2021):

simonsfoundation.org/event/simons-collaboration-on-homological-mirror-symmetry-annual-meeting-2021/

About the (potential) proof:

non-explicit reconstruction of $GW_{g=0}$ invariants of $\tilde{X} = Bl_Y$ in terms of

1. $GW_{g=0}$ invariants of X, Y
2. restriction map $H^\cdot(X) \to H^\cdot(Y)$
3. classes $c_i = c_i(N_{Y \subset X}) \in H^\cdot(Y)$ $i = 1, \dots, m$

in "Birational cobordism invariance of uniruled symplectic manifolds" by J. Hu, T-J. Li, Y. Ruan Invent. Math. 172, 231-275 (2008)

based on relative GW invariants. Translating to the language of F-bundles

Applications:

(Assuming blow-up formula):

Very general cubic 4-fold $\left(\begin{matrix}\deg = 3\\ \text{in } \mathbb{P}^{6-1}\end{matrix}\right)$
is **not** rational

General argument:

Let X, $\dim X = d$ be a smooth projective variety

Define elementary pieces as noncommutative motives with (non)symmetric pairings associated to eigenvalues of Eu· at the generic point of the subvariety of B_X given by linear combinations of algebraic cycles

(in the ideal world: smooth proper dg. categories)

Blow-up formula

$\Rightarrow$ elementary pieces for $\tilde{X}$

= elementary pieces for X

$\sqcup$ $(m-1)$ copy of elementary pieces for Y

$\Rightarrow$ if an elementary piece for X does not appear as an elementary piece of $\leq (d-2)$-dimensional variety

one representative for each birational type

$\rightsquigarrow$ X is not rational.

Returning to the cubic 4-fold.

$\boxed{d=4}$

One of elementary pieces:

Kuznetsov component $\sim \begin{pmatrix} 1 & & 1 \\ & 20 & \\ 1 & & 1 \end{pmatrix}$

(like $H^{\cdot}(K3)$)

depends on 20 parameters.

We understand $\leq 2 = d-2$ - dimensional varieties / birational $\sim$

$\rightsquigarrow$ list of possible elementary pieces.

Kuznetsov component can not come from K3 (19 parameters)

can not come from general type

(pairing is not symmetric)

Another application:

Let X_0 be a singular variety $\left(\begin{array}{c}\text{normal}\\ \text{reduced}\end{array}\right)$

Consider any resolution of singularities

X and use in GW theory

$\downarrow \pi$ only curves in X which project

X_0 to a point in X_0

Consider quantum multiplication near limiting point "$-\infty\, \pi^* \omega_{x_0}$" in $H^\cdot(X)$ (or even near 0...)

Restrict to subvariety in B_X given by linear combinations of algebraic classes excluding $H^0(X,\mathbb{Q}) = \mathbb{Q}$ $\qquad H^{\neq 0}_{alg}(X)$

Consider the elementary piece corresponding to eigenvalue $\lambda_i = 0$ of $(E_u \cdot)$

<u>Theorem</u> (assuming blow-up formula)
This elementary piece at the generic point of $\mathcal{H}^{\bullet\neq 0}_{alg}(X)$
does not depend on resolution of singularities $X \to X_0$.

The proof does not need a detailed knowledge of blow-up formula.

Several key ideas:

1. In the whole story one can forget about pairing (recover only at the last step)

Example: maximal F-bundle can be defined for noncompact X such that the map $ev_1 : \overline{\mathcal{M}}_{0,2}(X,\beta) \to X$ is proper $\forall \beta \in H_2(X,\mathbb{Z})$

$\Rightarrow$ Blow-up formula is certain morphism of operads (not PROPs)

Only Trees! no general graphs.

2. Quantum multiplication by anything is an operator in $End(H^\bullet(X))$ coming from certain universal "convolution algebra":

$X^2 \supset$ closed subset Z, $Z \times_X Z \subset Z$ s.t. $Z \xrightarrow{pr_1} X$ is proper

(in our example $Z = X' \times_{X_0} X'$ where $X' \subset X$ is the closed locus where $\pi : X \to X_0$ is not $\cong$)

$\Rightarrow$ graded associative algebra

$$H^\bullet(X,\mathbb{Q}) \oplus H_\bullet^{BM}(Z,\mathbb{Q})$$

(acts on $H^\bullet(X,\mathbb{Q})$)

Expectations :

The canonical elementary piece is H^0 (a canonical smooth proper category)

Also, there is a canonical Fukaya category associated to $\lambda_i = 0$

Compare with intersection cohomology:
Our theory gives a larger space.