Reflections on moduli space

In preparation for my Hirzebruch Lecture in Bonn (15 May 2023), I sent several of my friends, collaborators, and colleagues the following message:

Why are you interested in moduli spaces? If you could write a quick paragraph, I would be grateful.

I received a flood of interesting replies varying in perspective from the philosophical to the technical. Instead of keeping all of the answers to myself, I have presented them below in an anonymous form (slightly edited in some cases). Each bullet point represents a distinct person (none of whom are me). I hope some sense of the motivations of the community will come through.

I have been in part inspired by the masterpiece *Working* by Studs Terkel which I found on the bookshelf of a friend more than 30 years ago.

Rahul, May 2023

• What do I mean by moduli space? The sets of all algebro-geometric objects of a certain type are themselves algebro-geometric objects, called moduli spaces, whose geometric structure depends on, and hence reveals, properties of the objects they parametrize. This is a fascinating phenomenon whose enchantment never ends: the more you understand the parametrized objects, the more you know their moduli space, and the more you understand the moduli space, the more you know the parametrized objects.

• Geometry before moduli theory can be compared to the past fascination with cabinets of curiosities, as the wonder for the discovery and exhibition of new species were the main motivations of geometers. In the same way that *Enlightenment thinkers contributed to the declining influence of cabinets of curiosities by placing greater emphasis on patterns and systems within nature*, moduli theory offers a universal approach to geometry that permits the encompassment of all general features and anomalies at once. The anomalies are no longer isolated wonder objects, but they are studied to capture the nature of general objects. The resulting collection provided by moduli theory turns out to have a nature of its own, as if the catalogue of all butterflies were a new, exotic animal.

• People have always been fascinated with transcendent ideas, among them the paradoxes and beauty of self-referential concepts. Such concepts give one the feeling one is close to God, even if one is a total atheist. Moduli spaces give geometers a mathematical playground in which self-referential concepts are concrete, immensely fruitful, and not at all paradoxical: the space of all lines in the plane is another plane, the space of circles in the plane is *half of a 3-space*, and so on. How could one resist?

• If you studied individual fish swimming in the sea, you might learn a thing or two about how they move and navigate their surroundings, but fish usually swim in schools, and if you study the movement of entire schools of fish, you will probably be led to a far greater understanding of how the individual fish move and communicate with one another (not to mention the beauty of watching fish swim in schools). In the same way, mathematical objects often arise in families, and studying the families can often lead to a better understanding of the individual objects. For example, to understand why each individual cubic surface contains a line, it is, perhaps, easiest to study the geometry of the moduli space parametrizing pairs of a cubic surface and a line on it.

• Moduli spaces are encrypted treasure maps. We know there is treasure somewhere, but we must decipher the map to find it. It is often harder to point out specific objects. It can be helpful to first consider a space which parametrizes all possibilities and then put conditions to understand original problem (as in Schubert calculus). Some moduli spaces, such as the moduli space of curves, are fundamentally important objects, so that the comparison to a treasure map is an underestimation. Maybe I should say that a moduli space is a treasure map which is itself a treasure.

• Moduli spaces can be viewed as a Noah's ark, a collection of species, except for the poor unicorns who didn't make it (perhaps they were too singular?). The ark minimally contains what is needed to recreate the world we live in after the storm.

• Moduli spaces are a way to conjure up a whole world to explore, a whole country with interesting inhabitants, with just a few magic words "What about the moduli space of ...". Usually in mathematics, when you want a space to study, you must build it up yourself: you start with something known, take products, blow-ups, projective bundles, cut it down again with equations, and so on. All of that can be hard work, and often in the end there is no mystery: you get what you ordered. But with moduli spaces, you can just name what you are interested in (triangles in the plane, ordered bases of a vector space, Riemann surfaces of genus 5), and a whole landscape is summoned for you to explore. After travelling in these moduli spaces, you start to get a feel for the interesting questions to ask on arrival: Is the country connected? Is it compact or can you walk on and on without ever returning? What is the dimension? Are there any regions with singularities I should avoid? Often the answer to these questions is directly tied to properties of the inhabitants of the moduli space, and this interplay is another part of the charm. When starting to study mathematics, one typically looks at objects in isolation and gets to know them better. The moduli space puts them in perspective: you learn which ones are general and which ones are very special. And sometimes you then manage to prove a property of a general object by looking at a special one (or vice versa). This encourages a certain cosmopolitan approach for your studies: even if you are just interested in a single object and its features, it might be helpful to travel around, come back and then understand better the place that you started from.

• In many intellectual disciplines one tries to balance universality and peculiarity. Universal notions provide context for particular situations, and the most interesting phenomena occur when the particular fights back against the universal. In our corner of mathematics, moduli theory is the setting best suited for understanding universal properties of our favorite mathematical objects. At the same time, moduli theory reveals how special, exciting, and weird some of our favorite objects are. A basic example occurs in my favorite space, the moduli space of curves. When first learning algebraic geometry, one might try to write down a curve a genus g for every natural number g. The easiest way to do this is to write down equations for a hyperelliptic curve. As one starts to study moduli, one realizes how special hyperelliptic curves are and how weird it is that they were the ones we wrote down in the first place.

• If you want to understand human psychology, you had better study how humans live and interact with each other. If you want to understand curves, you must also study where they live and interact. But perhaps what attracts me most to moduli spaces is the fact that they often unexpectedly and beautifully form bridges to different mathematical islands. You can go to the island of number theory, modular forms, theta series, and lattices via the bridge of the moduli spaces of abelian varieties and $K3$ surfaces by seeing them as Siegel/orthogonal Shimura varieties. You can go to the island of representation theory via the moduli of points on surfaces and the moduli of quiver representations. You can go to the island of combinatorics via $\overline{\mathcal{M}}_{0,n}$. I imagine math as a bunch of islands in the north of Norway, and there is a crazy architect – not very concerned with efficiency, but with beauty and probably using magic mushrooms – who is designing all these bridges that take you to surprising places.

• Two aspects of mathematics that I have always found intriguing are *translation* and *connection*, and moduli spaces embody both concepts. They are meta-spaces, in that their points represent different geometric objects, which is, in and of itself, philosophically satisfying. Problems such as enumerative questions are then naturally *translated* into intersection theoretic problems on appropriate moduli spaces. Moduli spaces create connections among different parts of mathematics in at least two different ways, both from the fact that the geometric objects we study in moduli spaces end up being relevant in other disciplines (string theory, representation theory, knot theory, and so on) and from the fact that working with moduli spaces requires tools and intuition from topology, combinatorics, and more. I find it very interesting that even though the geometry of moduli spaces can be very sophisticated and complex, some things can be deduced by taking a concrete perspective on simple properties of the objects parameterized (for example, the stratification of $\overline{\mathcal{M}}_{g,n}$ depends on topological concepts presented in a second year undergraduate course). Finally, and still in the realm of *connection*, I find it interesting that when one studies moduli spaces, one never studies just one space at a time, but a whole network of spaces connected by a wealth of tautological morphisms, and this gives rise to a lot of structure.

• My curiosity about the hidden symmetries governing the processes observable in nature drove me to learn different ways to characterize the possible evolutions of diverse physical systems. Often the mathematical formulation of this question is phrased in terms of studying solutions of (partial) differential equations living on geometries meant to capture some part of our world. Working with PDEs is a daunting task as one often needs to develop new techniques for each case and there are only a few universally applicable methods. By transforming the problem into an algebro-geometric one, it becomes more concrete and gains a more manageable form. At this point, one can use the robust theory of algebraic geometry developed clearly and precisely by many famous mathematicians. This solid foundation makes it easier to apply tools from algebraic topology, combinatorics, representation theory, and other fields to the study of moduli spaces, and it leaves us with almost unlimited possibilities to introduce new theories while finding connections between the existing ones.

• Nature likes equilibria, which are typically minima of some energy-like function (like minimal surfaces). However, when such minima are not unique, she is forced to explore the moduli space of all possible equilibria.

• Most mathematical problems are moduli problems. Often the moduli questions are infinite-dimensional or do not have a reasonable topology or are much too complicated to be useful. But sometimes we are lucky and the moduli problem in question can be studied by algebro-geometric or more general geometric tools. Amazingly, in many of these questions, a link with theoretical physics can be found. These types of moduli spaces usually lie on crossroads of many mathematical fields. These moduli spaces are guiding stars that help me to study new mathematics (and, if I am extremely lucky, to discover new mathematics).

• Algebraic geometers are interested in moduli spaces because we want to understand what exists. It's one thing to know that there is a curve of any genus, but we also feel a responsibility to understand how many curves there are. And it's a particularly beautiful feature of algebraic geometry that the space of all varieties of some sort can often be viewed as a variety itself. Beyond that, it is striking how complicated $\mathcal{M}_g$ is, and it has taken heroic efforts to understand it as much as we do. In part, the complexity has to do with the surprising difficulty of describing a general curve (or varieties in general) by equations.

• My first papers which were explicitly about moduli spaces concerned the moduli of polarized abelian surfaces of low degree. We proved that an assortment of moduli spaces were rational. The original motivation, however, came from studying explicit equations for abelian surfaces in projective space. We often wanted to be able study these equations via Macaulay on a computer. Of course, the equations involve coefficients which are determined by the modulus of the abelian surface, and you immediately have to start to think about what the moduli spaces are to have a hope of writing down equations. Rationality becomes fundamental: if the moduli space is rational, then it is usually easy to write down equations. If the moduli space is general type, then you have to work very hard to find even one point in the moduli space and to write down the corresponding equations. So the structure of the moduli space essentially told us how easy it was to study abelian surfaces in an effective way.

• To be interested in a type of geometric object is to be interested in the moduli of those objects. For example, the stack of genus g curves is just a category whose objects are families of genus g curves: can you really be interested in one without being interested in the others? Presumably some people manage, but it feels hard. My other main motivation comes from trying to make precise an aesthetic sense of which constructions are clear/natural. I tried various things along the lines of *does not depend on choice of coordinates/equations*, but (after I learnt what it meant) I found *makes sense on the universal object* a more precise and accurate metric.

• Moduli spaces do a good job of encoding information of various types, both directly as moduli spaces and indirectly via their Euler characteristic, cohomology, and other invariants. For example the Hilbert schemes of n points on a surface don't just solve the problem of parametrizing families of n points on the surface, they also are a geometric realization of the combinatorics of partitions. Maybe another way to think about this is that moduli spaces are the most complicated geometric spaces that I know, so if you want to focus on studying specific spaces then they are going to be the most interesting ones.

• Associated to a geometric space X of some type are usually various moduli spaces. These are parameter spaces, parametrizing the solutions of an equation on X, or all possible geometric structures of some type on X, or all possible geometric objects (of a certain type) contained in X. The geometry of such a moduli space $\mathcal{M}$ often reflects that of X in various ways. It is also usually a very sophisticated (nonlinear) invariant of X. Simple (linear) algebraic invariants of $\mathcal{M}$ (like its cohomology) are (nonlinear) invariants of X, containing subtle and deep information.

• Moduli spaces are very useful tools to extract interesting numbers and structures from your variety. How would you otherwise know that there is an Igusa cusp form hiding in the $K3$ surface? Or how would you find the automorphic forms of Borcherds (coming from moduli spaces of $K3$ surfaces)? The moduli spaces of stable sheaves on $K3$ surfaces are hyperkähler manifolds, which is a very rare class of varieties, so moduli spaces are also good for construction.

• Algebraic geometry is the study of algebraic varieties: we are interested in describing their geometry, topology, cohomology, and so on. Varieties of high dimension are usually very hard for us to study. Moduli theory can help us construct examples of higher dimensional varieties from low dimensional data (moduli of curves, sheaves, and $K3$ surfaces are all examples). The theory creates interesting examples of higher dimensional varieties and many exciting problems.

• For a family of varieties, what is the best possible limit we can choose? Naturally for an algebraic geometer, that is translated into asking whether we can write a good functor, such that the corresponding moduli space is compact. In the case $K_X > 0$, the Kollár-Shepherd-Barron limit given by the minimal model program provides a satisfactory answer. While in the case $K_X < 0$, K-stability provides the right one. The latter was once beyond the vision of algebraic geometers, since it comes from the not-yet-completely-proven speculation that Kähler-Einstein metrics form a compact space.

• I think of moduli spaces as the classification scheme for varieties, with the remarkable property that they are themselves solutions to systems of equations. For me, invariant theory is the heart of the subject.

• I'm interested in moduli spaces just like I'm interested in the Hodge conjecture: if a geometric object is defined by algebraic equations, it feels good to be able to read its geometry (the cohomology) from algebraic structures (the algebraic cycles). If that geometric object is a moduli space, it feels equally good to be able to understand the geometry of the moduli space through its modular interpretation.

• The moduli spaces we work with are (generally) provably as complicated as nature allows them to be by Murphy's Law. In this sense, they contain infinite complexity. Therefore, they have the potential of encoding a huge amount of information, which we are able to extract in some cases using virtual intersection theory.

• One shouldn't be able to say anything about moduli spaces. They appear as huge impenetrable objects at first. But they end up being so highly structured that almost every meaningful construction you can do in geometry implies some nontrivial result about moduli. The cell structure of the Grassmannian is a great example: the basic idea is high school geometry, but, down the road, it tells you things about lines on the cubic surface. In fact, once you throw away the technical stuff, many of the techniques we use are in the spirit of the geometry of lines and planes in space that you learn in school. You just sit there and think "what can I do to these spaces" and something wonderful pops out. I find this mix of being very mysterious, but also amenable to naive geometric thinking, very attractive. The other thing that makes them interesting is that the answers to the questions we ask are unexpectedly beautiful.

• When I learned about moduli spaces, it was at a time when I generally liked all sorts of stuff (literature/cinema) that was self-referential, so a statement like *the set of all geometric objects of a certain type is itself a geometric object* was really music to my ears.

• It just so happened that my first mathematical research was on moduli spaces. But it made me realize that it is a wonderful part of mathematics, at the crossroads of main areas of mathematics. It involves algebra, geometry, analysis and number theory and has already quite a history. Although general theories apply, the field is much more than general theories and exhibits many objects with intricate and beautiful properties whose existence already looks like a miracle. The various interpretations of these as moduli give their properties extra meaning. Often geometry and number theory seem intricately connected here as if by magic. Time and again surprises appear in this field and completely unexpected connections between seemingly unrelated objects are discovered.

• The first paper that brought me to moduli spaces was Kontsevich's proof of Witten's conjecture (besides the main result, the appendix contains a simplification of Harer and Zagier's computation of the Euler characteristic of $\mathcal{M}_{g,n}$). As I was studying it, the ELSV formula appeared. Both papers made a huge impression on me. The reason is, I think, that they used a lot of what I had just learned and liked most in mathematics: intersection theory, generating series in combinatorics, the saddle point method, the asymptotic expansion of the Γ-function, and the representation theory of the symmetric group. I don't really know why I liked these topics more than, for instance, functional analysis or PDEs. But I certainly wasn't expecting that there was a topic mixing them all in such a marvelous way.

• I think nobody really knows why the cohomology of $\overline{\mathcal{M}}_{g,n}$ decides to structure itself as a universal source of integrable systems. It's a miracle worth investigating and a powerful thing to know how to manipulate, and everything cool about this garden of elegant geometry that is the the moduli theory of algebraic curves, from mirror symmetry, to Abel Jacobi theory, to intersection theory, has an integrable counterpart that is as beautiful and rich.

• The moduli space that I study the most is that of algebraic curves, and even the objects it parametrizes, Riemann surfaces, are absolutely central to mathematics. They can be studied geometrically (as 1-dimensional varieties), analytically (as Riemann surfaces, via Fuchsian groups), or algebraically (via Galois theory or group theory). The insights one obtains in these ways are highly complementary. When it comes to the space of essential parameters for curves of genus g, that is, to $\mathcal{M}_g$, all these different aspects are still present, but in addition combinatorics, modular forms, and theoretical physics also come to the fore. The resulting object is therefore extremely rich, and, in fact, very often really important progress in the field is achieved by mixing nontrivially these perspectives. This feature of moduli space mixing various fields of mathematics has been very appealing to me.

• Before going into algebraic geometry, I was interested in the topology of 4-manifolds and gauge theory (Donaldson invariants and Seiberg-Witten invariants). Moduli spaces appear in these fields as well, and I enjoyed seeing the interactions between topology, analysis (which enters the constructions of the moduli spaces), and algebraic geometry (via explicit realizations of the moduli spaces). I particularly liked the fact that the algebro-geometric approach can be used to carry out many concrete computations. The transition to other moduli spaces and to algebraic geometry seemed quite natural to me at the time, so I arrived at moduli theory because of my own trajectory/background as well as my personal interest. Why study moduli theory now? Here are four reasons: many moduli spaces exhibit interesting/unexpected geometric features that are fun to explore and of great interest to understand, the study of moduli carries rich and unexpected connections with other nearby fields, moduli theory is broad enough that it allows for many points of view (there are questions of abstract nature as well as concrete computations), and there are many very interesting open questions.

• Moduli spaces have four main attractions for me. First, they allow us to study objects by specialization. We can prove properties about a general object by specializing to simpler objects at the boundary. Almost all my work involves computing the rank of a matrix or counting the solutions to a system of polynomials by specializing the system to a more manageable system. Second, among all varieties moduli spaces have extra structure: a modular interpretation of the points which gives us an additional handle to study them. The interaction between families of objects and properties of the moduli space makes the geometry of moduli spaces richer and more accessible. Third, moduli spaces allow us to define interesting invariants. Topological and geometric invariants of the moduli spaces influence and constrain the geometry of the objects they parameterize. Finally, moduli spaces allow us to access positivity that is not apparent from a single object. For example, it is almost impossible to understand rational curves on a variety or the algebraic hyperbolicity of a variety without varying it in a family to make use of the positivity of certain tangent and normal bundles of the universal family.

• I studied nonlinear PDEs on manifolds, such as harmonic maps, minimal surfaces, and Kähler-Einstein metrics, when I was a student, following a suggestion of my supervisor. Around the same time, a topologist gave a course on Donaldson's work, namely the application of gauge theory to topology. His course motivated me to study PDE aspects of gauge theories. While writing several papers in this direction, I was also fascinated by the beauty of moduli spaces through Donaldson's work and also Mukai's result for $K3$ surfaces. Therefore, I decided to study moduli spaces. Relations to representation theory came to me afterwards.

• I am interested in moduli spaces (in particular moduli of sheaves on algebraic varieties) because they produce many interesting and deep geometric/algebraic structures from a given variety: holomorphic symplectic structures on the moduli of sheaves on $K3$ surfaces, Heisenberg action for Hilbert schemes of points, symmetries from derived equivalences, and so on. And these structures connect algebraic geometry to other research fields, such as representation theory and mathematical physics.

• Having been raised in the Oxford school of gauge theory, I was taught that the geometry of moduli spaces is meant to shed light on the geometry, or even the topology, of the original space, as it did in Donaldson's famous theorem. Decades later, I have come to understand that such pursuits in mathematics often become ends in themselves. The study of moduli spaces turns out to be like a vast gallery or labyrinth, with many wings or corridors that recede to infinity.

• The two things that come to mind are the following. First, just what everyone would say: the idea of not just studying one instance of a geometric object, but instead studying all instances of a certain class of geometric objects by looking at a space that parametrizes them is just a very cool idea. Second, I like the way that the same moduli space shows up in wildly different contexts. For example, the moduli space of instantons shows up in physics, algebraic geometry, differential geometry, and representation theory: the space of minima of the Yang-Mills functional in quantum field theory is the same as the moduli space of algebraic bundles on the projective plane (trivialized on a line) is the same as the same as the space of anti-self dual connections on a principal bundle up to gauge equivalence is the same as quiver representations of the ADHM quiver.

• Moduli spaces of very simple objects have rich geometry. It is surprising how much one can understand about moduli spaces of complicated objects by analogy. Moduli problems in different branches of mathematics shed light on each other: instantons, monopoles, stable maps, objects in a derived category, special Lagrangian submanifolds. Studying moduli problems has led to breakthroughs on a priori unrelated questions. For example, instantons changed our understanding of smooth structures, stable maps revealed symplectic rigidity phenomena, and so on.

• Moduli spaces straddle categories. One Riemann surface is an object of algebraic geometry, but the moduli space is just as much an object of homotopy theory. One hypersurface X in projective space is an object of algebraic geometry, but the moduli space $\mathcal{M}$ is an object of symplectic geometry via the map

$$\mathcal{M} \to \mathsf{BSymp}(X)\,.$$

Moduli spaces eliminate unnecessary choice. A single instanton on a 4-manifold is of possible interest in physics, if you had a good reason to choose it. The space of all of them is intrinsic to the underlying geometry, and, in some ways, even topology.

• My interest comes from attempting to classify mathematical objects. Classically this means describing the set of mathematical objects of some type modulo isomorphism in some nontautological, and perhaps even useful, way. But homotopy theory teaches us that one should avoid strictly identifying isomorphic objects that are not identical, because they can be isomorphic in different ways, so instead one should try to describe the groupoid (*mathematical objects of some type*, *isomorphism*). In algebraic geometry, this change of view is quite modest: it promotes consideration of the stack $\mathcal{M}_g$ over its coarse moduli space. But in geometric topology, the change of view is drastic: *almost all coarse moduli spaces are a single point*, and the groupoid is then the symmetry group of such an object. In other words

$$\text{classify families} \;\Leftrightarrow\; \text{moduli spaces} \;\Leftrightarrow\; \text{classify symmetries}\,.$$

The only caveat is that *group* may have to be interpreted in a homotopical and not-quite-literal way.

• Manifolds form a category and studying them amounts to more than classification up to isomorphism, one must also understand the morphism spaces, and, in particular, automorphisms. In homotopy theory, a group G with its group structure contains the same information as its classifying space $\mathsf{B}G$, hence the study of $\mathsf{BDiff}(M)$ up to homotopy naturally accompanies classification questions of any type. Or you can take a *functor of points* perspective, for each base manifold S there is a groupoid of smooth fiber bundles $E \to S$ with fiber M. Up to homotopy, this functor is represented by $\mathsf{BDiff}(M)$. Typically one imposes conditions and extra structure on the bundles, and obtains many variations of moduli spaces, but it's the same idea. Classical classification questions (for example, the Kervaire-Milnor classification of *exotic spheres* up to diffeomorphism) relate to the case where S is a point, but that's only part of the story. Any reason to be interested in manifolds one at a time should also be a reason for trying to understand families of manifolds. It seems to me that pursuing that perspective more fully has somewhat revived high-dimensional manifold theory.

• I'm very fond of how the moduli space of curves is so closely connected to very disparate areas of math (algebraic geometry, homotopy theory, number theory, mathematical physics, geometric group theory, dynamical systems). It makes for a very fun corner of mathematics to work in. And all of these are actual, honest connections, not of the superficial type where someone studying subject X says "X is interesting because of close links to Y", while the people actually working on Y have no interest in X. From a homotopy theory perspective, a topological group G contains exactly the same information as its classifying space $\mathsf{B}G$. For $\mathsf{Diff}(M)$, the diffeomorphism group of a manifold M, $\mathsf{BDiff}(M)$ is the moduli space of M-bundles.

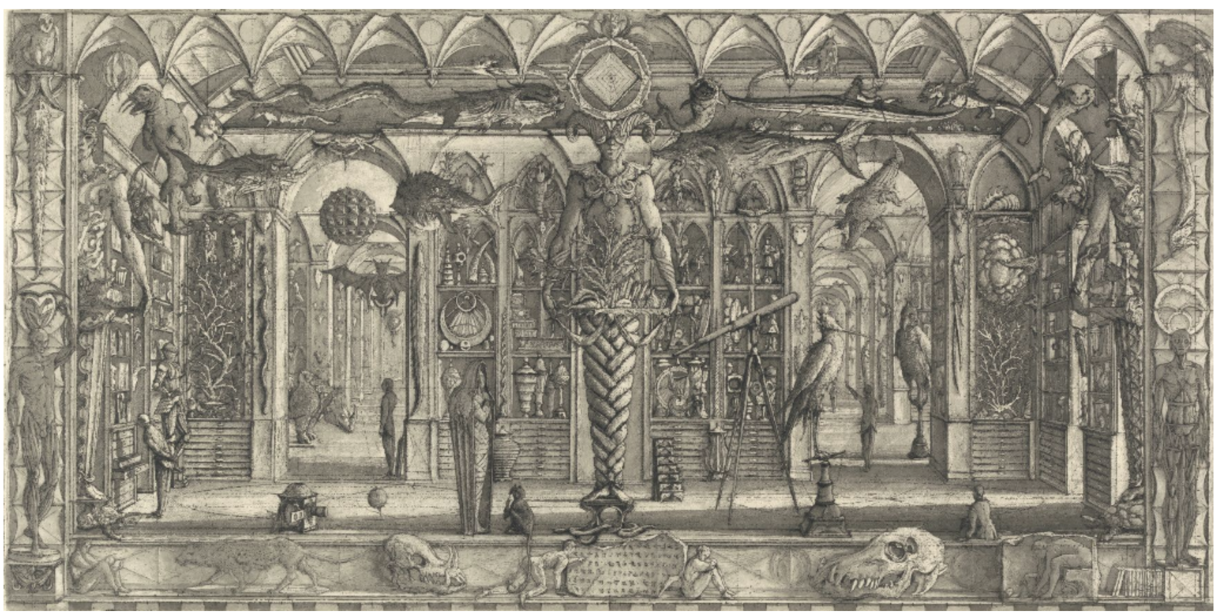
So studying the moduli space of manifolds is the same as studying spaces of diffeomorphisms, which is obviously a very fundamental mathematical problem. But I think the moduli space perspective is the more psychologically helpful one.

• Most moduli spaces I work with are essentially algebraic objects (keeping track of all algebraic structures on some space, or morphisms, or deformations thereof) and are not nice geometric objects like the moduli space of curves. My interests in those moduli spaces stems from the attempt to classify the underlying objects they encode. For example, local coordinates around a point could give a complete set of invariants of deformations of some algebraic structure or morphisms thereof (except that the spaces are often very singular, so it is not clear what is meant by *local coordinates*). However, specifically for the moduli space of curves, my interest does not come from classifying curves (I am not an algebraic geometer), but rather from the algebraic structures it carries, and the fact that the resulting operads have many links to algebra and topology. Furthermore, I am interested in computing $H^*(\mathsf{BDiff}(M))$ which produces a complete set of characteristic classes for M-bundles. The moduli space of curves,

$$\mathcal{M}_g \sim_Q \mathsf{BDiff}(S_g)\,,$$

where S_g is a surface of genus g, is the lowest dimensional interesting example.

• I am not so much interested in specific moduli spaces as geometric objects themselves. I agree they are natural objects to study, but that's just not my cup of tea.



Erik Desmazières, *Wunderkammer*, Musea Brugge