Eilenberg MacLane Spaces in Algebraic Surface Theory

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Outline

This talk is based on 2 joint papers; one in collaboration with R.V Gurjar and Buddhadev Hajra and another in collaboration with Poonam Pokhale.

The main aim of the talk is to classify non-singular complex surfaces (affine and projective) which are Eilenberg MacLane (EM). I'll first discuss the affine case and then the projective case. The classification is done by a case by case analysis depending on the (logarithmic) Kodaira dimension. We also discuss some properties of the universal cover of EM smooth projective surfaces.

The proofs are a mix of algebro-geometric and topological techniques.

EM Spaces

It is well-known that given an integer $n > 0$ and a group G which we assume is abelian if $n > 1$, there exists a connected CW complex X, unique upto homotopy type, with the property that $\pi_n(X) \simeq G$ and $\pi_m(X) = 0$ for any positive integer $m \neq n$. X is called an Eilenberg MacLane space, denoted $K(G, n)$.

The simplest examples are S^1 and the contractible spaces. The former is $K(\mathbb{Z}, 1)$ and the latter is $K(e, 1)$. The infinite-dimensional complex projective space $\mathbb{CP}^\infty$ is a model of $K(\mathbb{Z}, 2)$. The infinite-dimensional real projective space $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2, 1)$.

By taking products and coverings of such spaces, we get more examples. Similarly by considering fiber bundles with base and fibers EM, we get more examples.

First Obervations

Theorem

If a smooth quasi-projective variety X is $K(G, n)$, then $n = 1$ and G is torsion-free.

Proof:

It follows from the work of Serre and Cartan from the 50's that for a non-trivial group G and an integer $n > 1$, $H^*(K(G, n))$ is non-zero in infinitely many degrees. Since the homology groups of quasi-projective variety vanish beyond dimension, it forces that $n = 1$.

If G has torsion, let $\widetilde{X}$ be a universal cover of X. Since X is a $K(G, 1)$ space, $\widetilde{X}$ is contractible. If possible let G have a non-trivial torsion element, say σ. Consider the action of the cyclic subgroup $H =< \sigma >$ of G on $\widetilde{X}$ and let $f : \widetilde{X} \to \widetilde{X}/H$ be the quotient map. Since $|H|$ is finite, f is a finite covering map with deg $f = |H|$.

Proof cont..

Since f is finite, by a theorem of Giesecke, the induced map $H_i(\widetilde{X};\mathbb{Q}) \to H_i(\widetilde{X}/H;\mathbb{Q})$ is surjective, for all $i > 0$. Hence $H_i(\widetilde{X}/H;\mathbb{Q}) = (0)$ for all $i > 0$ and thus $\chi(\widetilde{X}/H) = \chi(\widetilde{X}) = 1$. Also,

$$\chi(\widetilde{X}) = (deg\ f) \cdot \chi(\widetilde{X}/H) = |H| \cdot \chi(X).$$

This implies $|H| = 1$, a contradiction. □

Thus from now on we will be interested in $K(G, 1)$ surfaces where G is torsion-free. This is equivalent to demanding that $\widetilde{X}$ is contractible. We will also assume that $G \neq \{e\}$ since classifying contractible non-general type surfaces seems impossibly difficult presently.

Nori's Lemma

M. V. Nori proved an important result about exactness of a sequence involving fundamental groups of smooth algebraic varieties:

Nori's lemma:
Let X and Y be smooth connected complex algebraic varieties and $f : X \to Y$ a dominant morphism with a connected general fiber F. Assume that there is a codimension two subset S of Y outside which all the fibers of f have at least one smooth point (i.e., $f^{-1}(p)$ is generically reduced on at least one irreducible component of $f^{-1}(p)$ for all $p \in Y - S$). Then the natural sequence $\pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to 1$ is exact.

Remark

With the notations as in Nori's Lemma, the proof of this lemma indicates that instead of the condition on every fiber over points in $Y - S$ given in the hypothesis, if the multiplicity of each fiber is one then too Nori's lemma holds. Here, for a fiber $F = \sum_i m_i F_i$, where F_i are the irreducible components of F and $m_i \geq 0$, the multiplicity of F is defined as $gcd(m_1, \cdots, m_r)$.

If the base curve B of a fibration $f : X \to B$ on a smooth surface is isomorphic to $\mathbb{P}^1$ and either there is only one multiple fiber or exactly two multiple fibers with relatively prime multiplicities, then Nori's Lemma is still valid.

The proof of Nori's Lemma also works for a holomorphic map $f : M \to N$ of complex manifolds with the additional condition that there exists suitable compactifications to complex manifolds $\bar{M}$ and $\bar{N}$ of M and N respectively and a holomorphic map $\bar{f} : \bar{M} \to \bar{N}$ which extends f.

Xiao Gang's Generalization of Nori's Result

The following lemma is a useful generalization of Nori's Lemma due to Xiao Gang.

Lemma : Let $f : X \to C$ be a surjective morphism from a smooth algebraic surface to a smooth algebraic curve such that a general fiber F of f is smooth and irreducible. Let $p_1, \ldots, p_r$ be the images of the multiple fibers of f in C with multiplicities $m_1, \ldots, m_r$ respectively. Let the genus of C be g and $\tilde{C}$ be the smooth completion of C with $\tilde{C} - C = \{p_{r+1}, \ldots, p_{r+\ell}\}$. Then there is an exact sequence

$$\pi_1(F) \to \pi_1(X) \to \Gamma \to (1)$$

where Γ is the group with generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_r, \gamma_{r+1}, \ldots, \gamma_{r+\ell}$ with the relations

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_r \cdot \gamma_{r+1} \cdots \gamma_{r+\ell} = 1 = \gamma_1^{m_1} = \cdots = \gamma_r^{m_r}.$$

Here $[a, b] = aba^{-1}b^{-1}$. If f has no multiple fibers, the group Γ is equal to the fundamental group of C. □

Logarithmic Kodaira dimension

This is a very important invariant for affine varieties defined by S. Iitaka. Let X be a smooth affine variety of dimension d. We can embed $X \subset V$, where V is a smooth projective variety such that $D := V - X$ is a divisor with simple normal crossings.
If $H^0(V, n(K_V + D)) = 0$ for all $n \geq 1$ then we write $\bar{k}(X) = -\infty$. Otherwise $H^0(V, n(K + D)) \neq (0)$ for some $n \geq 1$. We can prove that $dim_k H^0(V, n(K + D))$ is $O(n^r)$ for some r with $0 \leq r \leq d$. Then we say $\bar{k}(X) = r$.

$\bar{k}(X)$ is called the logarithmic Kodaira dimension of X. Thus for an affine surface X, $\bar{k}(X)$ can be either $-\infty, 0, 1, 2$. If $\bar{k}(X) = 2$, we call the affine surface, a general type surface.

Lemma

No smooth complex algebraic $K(G,1)$ surface can admit a smooth C^∞-fiber bundle structure over $\mathbb{P}^1$.

Proof.

If possible let X be a smooth complex algebraic $K(G,1)$ surface which admits a fiber bundle structure over $\mathbb{P}^1$. Let $f : X \to \mathbb{P}^1$ be a fiber bundle with fiber F. Since $\mathbb{P}^1$ is simply-connected, by Nori's lemma, the homomorphism $\pi_1(F) \to \pi_1(X)$ induced from the inclusion $F \hookrightarrow X$ is a surjection. Let $p : \tilde{X} \to X$ be a universal covering. Then $\tilde{X} \to \mathbb{P}^1$ is a fiber bundle with fiber $\tilde{F} := p^{-1}(F)$. It follows from elementary covering space theory that $\tilde{F}$ is connected. Now it follows that $\chi(\tilde{X}) = \chi(\tilde{F}) \cdot \chi(\mathbb{P}^1) = 2 \cdot \chi(\tilde{F})$. Therefore $\chi(\tilde{X})$ is an even integer, or infinite. However this contradicts the fact that $\chi(\tilde{X}) = 1$, since $\tilde{X}$ is contractible as X is a $K(G,1)$ space. This completes the proof. □

$\bar{k} = -\infty$

By a well-known result due to Fujita-Miyanishi-Sugie, X has an $\mathbb{A}^1$-fibration. Let $f : X \to B$ be the $\mathbb{A}^1$-fibration. Here B can be a smooth affine or a smooth projective curve.
Furthermore every fiber of f is a disjoint union of $\mathbb{A}^1$'s, if taken with reduced structure.

Theorem

Let X be a $K(G, 1)$ surface admitting an $\mathbb{A}^1$-fibration. Then:

1. *The $\mathbb{A}^1$-fibration f has no multiple fiber.*
2. *$\pi_1(X) \cong \pi_1(B)$ and B is not simply-connected.*

Proof.

Suppose f has a multiple fiber. Then using Xiao Gang's result, we can see that $\pi_1(X)$ is isomorphic to the group Γ (the group Γ is as in Xiao Gang's theorem), since $\mathbb{A}^1$ is simply-connected. Then clearly either Γ is trivial or else Γ has a non-trivial torsion element. But $\pi_1(X)$ cannot be trivial, since X being simply connected is equivalent to X being contractible, as X is a $K(G, 1)$ surface. So Γ is non-trivial and contains a non-trivial torsion element. This implies $\pi_1(X)$ has torsion; a contradiction again. This completes the proof of the first part.

For the second part, since f has no multiple fiber, Nori's lemma implies immediately that the fundamental groups of X and B coincide. Since X is a $K(G, 1)$ surface, simply-connectedness of B is equivalent to contractibility of X. This completes the proof of this proposition. $\square$

Theorem

Every fiber of f is irreducible and reduced.

Proof: Firstly irreducibility implies reduced here since we have shown f has no multiple fibers. If there exist reducible fibers, then they are all isomorphic to disjoint union of $\mathbb{A}^1$'s with gcd of their multiplicites equal to 1.

Now with quite a bit of effort we show that the existence of reducible fibers implies $b_2(\widetilde{X})$ is infinite, a contradiction.

Since all the fibers of f are irreducible and reduced, they are isomorphic to $\mathbb{A}^1$ scheme-theoretically and hence its well known that f is an $\mathbb{A}^1$-bundle (Zariski locally). As discussed before the base of this fibration is not isomorphic to $\mathbb{P}^1$. The base cannot be $\mathbb{A}^1$ either since $\mathbb{A}^1$ bundle over $\mathbb{A}^1$ is $\mathbb{A}^2$ which is contractible.

Conclusion : $K(G, 1)$ (with $G \neq e$) $\bar{\kappa} = -\infty$ smooth affine surfaces admit a fiber bundle over a smooth curve, not isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$ and with fiber $\mathbb{C}$.

Singular fibers of $\mathbb{C}^*$-fibration

We now discuss EM surfaces with $\bar{k} = 1$. By a theorem of Kawamata, X admits a unique $\mathbb{C}^*$-fibration over a smooth algebraic curve. Miyanishi described the singular fibers of such a fibration. A singular fiber F_s of such a fibration has the form $\Gamma \cup \Delta$ where,

1. Γ is either empty, or $m\mathbb{C}^*$ for $m \in \mathbb{N}$, or $m_1 A_1 \cup m_2 A_2$ with $\gcd(m_1, m_2) = 1$, where A_i is isomorphic to $\mathbb{A}^1$ for $i = 1, 2$ and A_1, A_2 meet each other transversally in one point. We call the last curve C, a cross.
2. Δ is either empty, or a disjoint union of $\mathbb{A}^1$'s possibly occuring with multiplicities.

Here one has the following classification theorem for $\bar{k} = 1$ surfaces:

Theorem

Let X be a non-contractible smooth affine $K(G, 1)$ surface which admits a $\mathbb{C}^*$-fibration $f : X \to B$ onto a smooth algebraic curve B. Then either f has no singular fiber at all or the singular fibers of f are only multiple $\mathbb{C}^*$. This implies that after going to a finite Galois etale cover X is a $\mathbb{C}^*$ bundle over a smooth algebraic curve not isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$.

Proof of classification theorem:

The proof is rather technical and involves analysis of many cases separately. As an illustration of the proof, consider the case where the base B of the $\mathbb{C}^*$-fibration has infinite fundamental group. By the ramified covering trick, we can assume that f has no multiple fibers. Let $\widetilde{B} \to B$ be the universal cover of B and let $X' = \widetilde{B} \times_B X$. One then has the following exact sequence :

$$\pi_1(\mathbb{C}^*) \to \pi_1(X') \to \pi_1(\widetilde{B}) \to (1)$$

As $\widetilde{B}$ is simply-connected $\pi_1(X')$ can either be trivial or isomorphic to $\mathbb{Z}$. If $\pi_1(X') \cong \mathbb{Z}$, the universal cover $\widetilde{X}$ of X' (and hence also of X) is a $\mathbb{C}$-fibration over $\widetilde{B}$. If f has a singular fiber with reduced structure, this $\mathbb{C}$-fibration will have a fiber infinitey many cross and/or infinitely many curves isomorphic to $\mathbb{C}$. We then show (as in the the earlier theorem but more technical to prove here) that $b_2(\widetilde{X})$ is infinite. The case where X' is simply-connected has to be handled quite differently. Likewise the case where the base is $\mathbb{P}^1$ has to be handled separately.

$\bar{k} = 0$

This analysis is more complicated and uses the work of Kojima. Firstly it is a theorem of Kojima that such surfaces are neccessarily rational. A large number of these surfaces have torsion in the fundamental group and they can be ruled out immediately in the classification. Each of the remaining surfaces admits a $\mathbb{C}^*$-fibration. Finally after a case by case analysis, one has the following classification theorem.

Theorem

Let X be a non-contractible smooth complex affine $K(G, 1)$ surface with $\bar{k}(X) = 0$. Then either $X \cong \mathbb{C}^* \times \mathbb{C}^*$, or $X \cong H[-1, 0, -1]$.

A two sheeted covering of $H[-1, 0, -1]$ is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$.

Description of $H[-1,0,-1]$

Start with $V_0 = \mathbb{F}_n\,; (n \geq 1)$, the Hirzebruch surface of degree n. Let M_n and $\overline{M}_n$ respectively denotes the minimal section of $\mathbb{F}_n$ and a section of the ruling on $\mathbb{F}_n$ with $(\overline{M}_n \cdot M_n) = 0$.
Let l_0, l_1 and l_2 be three distinct fibers of the ruling on V_0. Put $p_i := l_i \cap \overline{M}_n$, for $i = 1, 2$.

Let $\alpha : V_1 \to V_0$ be the blowing-up centering at p_1 and p_2. Let $D_i := \alpha^{-1}(p_i)$ and assume l_i' denote the proper transform of l_i under α, for $i = 1, 2$. Put $q_i := l_i' \cap D_i$, for $i = 1, 2$. Let $\beta : V_2 \to V_1$ be the blowing-up centering at q_1 and q_2. Let $E_i := \beta^{-1}(q_i)$. Assume that l_i'' and D_i' respectively denotes the proper transform of l_i' and D_i under β, for $i = 1, 2$.

Let $D := l_0 + l_1'' + l_2'' + M_n + \overline{M}_n + D_1' + D_2'$. Therefore it can be observed easily that $X' = V_2 - D$ has an induced untwisted $\mathbb{C}^*$-fibration over a curve B which is obtained from $\mathbb{P}^1$ by removing one point

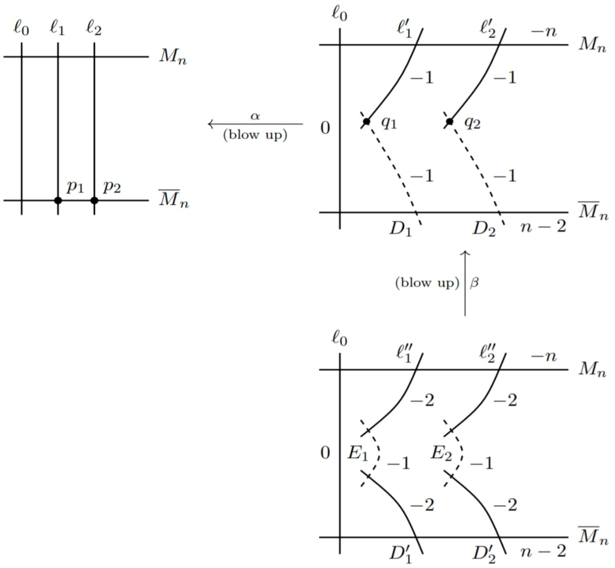

FIGURE 1. Fujita's $H[-1,0,-1]$

Projective surfaces

In a joint paper with Poonam, we computed the second homotopy groups of all smooth, projective surfaces of non-general type; i.e, Kodaira dimension $-\infty, 0, 1$. The surfaces covered under this assumption are Rational surfaces and Ruled surfaces ($\kappa = -\infty$) ; K3 surfaces, Abelian surfaces, Enriques surfaces and Hyperelliptic surfaces ($\kappa = 0$) and the $\kappa = 1$ surfaces which neccessaily admit an elliptic fibration. The fundamental groups of these surfaces were already well-known. Elliptic surfaces is the most interesting and non-trivial case here in the computation of π_2.

Theorem :

Let $f : X \to C$ be a relatively minimal elliptic fibration on a smooth projective surface X. Then :

1. If $\chi(\mathcal{O}_X) = \chi(X) = 0$

(This happens exactly when all the fibers are elliptic, possibly with multiplicity)

a) If there are at most 2 multiple fibers, $\pi_2(X) \cong \mathbb{Z}$ (resp. $\pi_2(X) = 0$) if genus of C is 0 (resp. genus of $C > 0$).

b) If $C \cong \mathbb{P}^1$ and there are 3 multiple fibers whose multiplicities form a platonic triple, then $\pi_2(X) \cong \mathbb{Z}$.

c) If there are 3 or more multiple fibers with the additional conditions that either $C \ncong \mathbb{P}^1$, or $C \cong \mathbb{P}^1$ and if there are 3 multiple fibers then their multiplicities do not form a platonic triple, then $\pi_2(X)$ is 0.

Theorem cont..

2. If $\chi(\mathcal{O}_X) > 0$ (or equivalently $\chi(X) > 0$) and $C \cong \mathbb{P}^1$

a) If there are no multiple fibers or only 1 multiple fiber, then X is simply-connected and hence $\pi_2(X) \cong \mathbb{Z}^{b_2(X)} \cong \mathbb{Z}^{10+12p_g(X)}$.

b) If there are exactly 2 singular multiple fibers with d being the gcd of their multiplicities, then $\pi_2(X) \cong \mathbb{Z}^{10+12(d-1+dp_g(X))}$.

c) If there are 3 multiple singular fibers and their multiplicities form a platonic triple, then $\pi_1(X)$ is finite of order, say d and $\pi_2(X) \cong \mathbb{Z}^{10+12(d-1+dp_g(X))}$.

d) If there are 3 or more multiple singular fibers with the additional assumption that if there are exactly 3 multiple singular fibers then their multiplicities do not form a platonic triple, then $\pi_2(X)$ is infinitely generated.

3. If $\chi(\mathcal{O}_X) > 0$, $C \ncong \mathbb{P}^1$, then $\pi_2(X)$ is infinitely generated free abelian group.

EM surfaces

As a consequence we can classify all compact, complex surfaces which are not of general type and EM. The relevant theorem is :

Theorem

Let X be a smooth compact complex $K(G, 1)$ surface of non-general type i.e., Kodaira dimension ≤ 1. Then the following assertions hold:

1. *If $\kappa(X) = -\infty$, then X is Inoue surface.*
2. *If $\kappa(X) = 0$ then X is either Abelian Surface, Hyperelliptic surface or Kodaira surface.*
3. *If $\kappa(X) = 1$ then X admits elliptic fibration with all fibers isomorphic to an elliptic curve when taken with reduced structure. Moreover there is an unramified cover of X which is an elliptic fiber bundle over a smooth algebraic curve, not isomorphic to $\mathbb{P}^1$.*

Thank You