Geometric Structure and the Local Langlands Conjecture

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Arbeitstagung 2013

23 May, 2013


$1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4$
5 Eckp. des reg. 5-Ecks
A_5
$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{array}{l} ad - bc = 1, \\ a,b,c,d \in F_5 \end{array} \right\}$

extended quotient

extended quotient of the second kind

extended quotient of the second kind twisted by a 2-cocycle

Equivalence of categories

$$\left(\begin{array}{c} \text{Commutative unital finitely generated} \\ \text{nilpotent} - \text{free } \mathbb{C} \text{ algebras} \end{array} \right) \cong \left(\begin{array}{c} \text{Affine algebraic} \\ \text{varieties over } \mathbb{C} \end{array} \right)^{op}$$

$$\mathcal{O}(X) \longleftarrow X$$

The extended quotient

Let Γ be a finite group acting on an affine variety X.

X is an affine variety over the complex numbers $\mathbb{C}$.

$$\Gamma \times X \longrightarrow X$$

The quotient variety X/Γ is obtained by collapsing each orbit to a point.

For $x \in X$, Γ_x denotes the stabilizer group of x.

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\}$$

$c(\Gamma_x)$ denotes the set of conjugacy classes of Γ_x.

The extended quotient is obtained by replacing the orbit of x by $c(\Gamma_x)$.

This is done as follows:

Set $\widetilde{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$

$\widetilde{X} \subset \Gamma \times X$

$\widetilde{X}$ is an affine variety and is a sub-variety of $\Gamma \times X$.

Γ acts on $\widetilde{X}$.

$\Gamma \times \widetilde{X} \to \widetilde{X}$

$g(\gamma, x) = (g\gamma g^{-1}, gx) \qquad g \in \Gamma \qquad (\gamma, x) \in \widetilde{X}$

The extended quotient, denoted $X/\!/\Gamma$, is $\widetilde{X}/\Gamma$.

i.e. The extended quotient $X/\!/\Gamma$ is the ordinary quotient for the action of Γ on $\widetilde{X}$.

The extended quotient is an affine variety.

$\widetilde{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$

The projection $\widetilde{X} \to X$

$$(\gamma, x) \mapsto x$$

is Γ-equivariant and, therefore, passes to quotient spaces to give a map

$$\rho : X//\Gamma \to X/\Gamma$$

EXTENDED QUOTIENT OF THE SECOND KIND

Let Γ be a finite group acting as automorphisms of a complex affine variety X.

$$\Gamma \times X \to X.$$

For $x \in X$, Γ_x denotes the stabilizer group of x:

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

Let $\mathrm{Irr}(\Gamma_x)$ be the set of (equivalence classes of) irreducible representations of Γ_x. The representations are on finite dimensional vector spaces over the complex numbers $\mathbb{C}$.

The extended quotient of the second kind, denoted $(X//\Gamma)_2$, is constructed by replacing the orbit of x by $\mathrm{Irr}(\Gamma_x)$.
This is done as follows :
Set $\widetilde{X}_2 = \{(x,\tau) \,\big|\, x \in X \text{ and } \tau \in \mathrm{Irr}(\Gamma_x)\}$.
Γ acts on $\widetilde{X}_2$.

$$\Gamma \times \widetilde{X}_2 \to \widetilde{X}_2,$$
$$\gamma(x,\tau) = (\gamma x, \gamma_* \tau),$$

where $\gamma_* \colon \mathrm{Irr}(\Gamma_x) \to \mathrm{Irr}(\Gamma_{\gamma x})$. $(X//\Gamma)_2$ is defined by :

$$(X//\Gamma)_2 := \widetilde{X}_2/\Gamma,$$

i.e. $(X//\Gamma)_2$ is the usual quotient for the action of Γ on $\widetilde{X}_2$.

$(X//\Gamma)_2$ is not an affine variety, but is an algebraic variety in a more general sense.

$(X//\Gamma)_2$ is a non-separated algebraic variety over $\mathbb{C}$.

Notation. If A is a $\mathbb{C}$ algebra, Irr(A) is the set of (isomorphism classes of) irreducible left A-modules.

Example. X an affine variety over $\mathbb{C}$, Irr($\mathcal{O}(X)) = X$.

$$\Gamma \times X \longrightarrow X$$

$HP_* =$ periodic cyclic homology

$$HP_j(\mathcal{O}(X) \rtimes \Gamma) \;=\; \bigoplus_k H^{2k+j}(X//\Gamma\,;\,\mathbb{C}) \qquad\qquad j = 0, 1$$

$$\mathrm{Irr}(\mathcal{O}(X) \rtimes \Gamma) = (X//\Gamma)_2$$

Extended quotients are used to “lift” BC (Baum-Connes) from K-theory to representation theory.

Lie groups

p-adic groups

Lie groups — e.g. $SL(n,\mathbb{R})$ $GL(n,\mathbb{R})$

$$\mathrm{Irr}(G) \longleftrightarrow (\mathrm{Irr}(\mathcal{G}/\mathcal{K})//K)_2$$

See results of Nigel Higson and his students.
Uses theorems of David Vogan.

Let G be a reductive p-adic group.

Examples of reductive p-adic groups are $GL(n, F), SL(n, F)$ where n can be any positive integer and F can be any finite extension of the field $\mathbb{Q}_p$ of p-adic numbers.

The smooth dual of G is the set of (equivalence classes of) smooth irreducible representations of G. The representations are on vector spaces over the complex numbers $\mathbb{C}$. In a canonical way, the smooth dual of G is the disjoint union of countably many subsets known as the Bernstein components.

Various results —

- Proof by V. Lafforgue that the BC (Baum-Connes) conjecture is valid for any reductive p-adic group G.

$$K_* C^*_r G \;\cong\; K^G_*(\beta G)$$

- P. Schneider (N.Higson-V.Nistor) theorem on the periodic cyclic homology of any reductive p-adic group G.

$$HP_*(\mathcal{H}G) \;\cong\; \mathbb{C} \otimes_{\mathbb{Z}} K^G_*(\beta G)$$

$\beta G =$ the affine Bruhat-Tits building of G

- V. Heiermann (and many others) theorems on Bernstein's ideals in $\mathcal{H}G$ and finite type algebras.

- P.Baum-V.Nistor theorem on the periodic cyclic homology of affine Hecke algebras.

- M. Solleveld (and many others) theorems on the representation theory of affine Hecke algebras.

— indicate that a very simple geometric structure might be present in the smooth dual of G.

The ABPS (Aubert-Baum-Plymen-Solleveld) conjecture makes this precise by asserting that each Bernstein component in the smooth dual of G is a complex affine variety. These varieties are explicitly identified as certain extended quotients.

For connected split G, (granted a mild restriction on the residual characteristic) the ABPS conjecture has recently been proved for any Bernstein component in the principal series of G. A corollary is that the local Langlands conjecture is valid throughout the principal series of G.

The above is joint work with Anne-Marie Aubert, Roger Plymen, and Maarten Solleveld.

ABPS Conjecture

ABPS $=$ Aubert-Baum-Plymen-Solleveld

The conjecture can be stated at four levels :

- K-theory
- Periodic cyclic homology
- Geometric equivalence of finite type algebras
- Representation theory

ABPS Conjecture

ABPS = Aubert-Baum-Plymen-Solleveld

The conjecture can be stated at four levels :

- K-theory
- Periodic cyclic homology
- Geometric equivalence of finite type algebras
- Representation theory $\longleftarrow$

Let G be a reductive p-adic group. G is defined over a finite extension F of the p-adic numbers $\mathbb{Q}_p$. $\overline{F}$ denotes the algebraic closure of F. Shall assume that $G(\overline{F})$ is connected in the Zariski topology.

Examples are:

$$GL(n,F) \qquad SL(n,F) \qquad PGL(n,F) \qquad SO(n,F) \qquad Sp(n,F)$$

where n can be any positive integer and
F can be any finite extension of the p-adic numbers $\mathbb{Q}_p$.
These are connected split reductive p-adic groups.
"split" $=$ the maximal p-adic torus in G has the "correct" dimension.

Definition

A *representation* of G is a group homomorphism

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

where V is a vector space over the complex numbers $\mathbb{C}$.

The p-adic numbers $\mathbb{Q}_p$ in its natural topology is a locally compact and totally disconnected topological field. Hence G is a locally compact and totally disconnected topological group.

Definition

A representation

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

of G is *smooth* if for every $v \in V$,

$$G_v = \{g \in G \mid \phi(g)v = v\}$$

is an open subgroup of G.

Definition

Two smooth representations of G

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

and

$$\psi : G \to \mathrm{Aut}_{\mathbb{C}}(W)$$

are *equivalent* if $\exists$ an isomorphism of $\mathbb{C}$ vector spaces $T : V \to W$ such that for all $g \in G$ there is commutativity in the diagram

$$\begin{tikzcd} V \arrow[r, "\phi(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\psi(g)"'] & W \end{tikzcd}$$

The smooth dual of G, denoted $\widehat{G}$, is the set of equivalence classes of smooth irreducible representations of G.

$$\widehat{G} = \{\text{Smooth irreducible representations of } G\}/\sim$$

Problem: Describe $\widehat{G}$.

Since G is locally compact we may fix a (left-invariant) Haar measure dg for G.

The Hecke algebra of G, denoted $\mathcal{H}G$, is then the convolution algebra of all locally-constant compactly-supported complex-valued functions $f : G \to \mathbb{C}$.

$$\begin{array}{rcl} (f+h)(g) & = & f(g)+h(g) \\ & & \\ (f*h)(g_0) & = & \displaystyle\int_G f(g)h(g^{-1}g_0)dg \end{array} \quad \left\{ \begin{array}{l} g \in G \\ g_0 \in G \\ f \in \mathcal{H}G \\ h \in \mathcal{H}G \end{array} \right.$$

Definition

A *representation* of the Hecke algebra $\mathcal{H}G$ is a homomorphism of $\mathbb{C}$ algebras

$$\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$$

where V is a vector space over the complex numbers $\mathbb{C}$.

Definition

A representation

$$\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$$

of the Hecke algebra $\mathcal{H}G$ is *irreducible* if $\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$ is not the zero map and $\nexists$ a vector subspace W of V such that W is preserved by the action of $\mathcal{H}G$ and $\{0\} \neq W$ and $W \neq V$.

Definition

A *primitive ideal* I in $\mathcal{H}G$ is the null space of an irreducible representation of $\mathcal{H}G$.

Thus

$$0 \longrightarrow I \hookrightarrow \mathcal{H}G \xrightarrow{\psi} \mathrm{End}_{\mathbb{C}}(V)$$

is exact where ψ is an irreducible representation of $\mathcal{H}G$.

There is a (canonical) bijection of sets

$$\widehat{G} \longleftrightarrow \mathrm{Prim}(\mathcal{H}G)$$

where $\mathrm{Prim}(\mathcal{H}G)$ is the set of primitive ideals in $\mathcal{H}G$.

Bijection (of sets)

$$\widehat{G} \longleftrightarrow \mathrm{Prim}(\mathcal{H}G)$$

What has been gained from this bijection?

On $\mathrm{Prim}(\mathcal{H}G)$ have a topology — the Jacobson topology.

If S is a subset of $\mathrm{Prim}(\mathcal{H}G)$ then the closure $\overline{S}$ (in the Jacobson toplogy) of S is

$$\overline{S} = \{J \in \mathrm{Prim}(\mathcal{H}G) \mid J \supset \bigcap_{I \in S} I\}$$

$\mathrm{Prim}(\mathcal{H}G)$ (with the Jacobson topology) is the disjoint union of its connected components.

Point set topology. In a topological space W two points w_1, w_2 are in the same connected component if and only if :

Whenever U_1, U_2 are two open sets of W with $w_1 \in U_1, w_2 \in U_2$, and $U_1 \cup U_2 = W$, then $U_1 \cap U_2 \neq \emptyset$.

As a set, W is the disjoint union of its connected components. If each connected component is both open and closed, then as a topological space W is the disjoint union of its connected components.

$\widehat{G} = \mathrm{Prim}(\mathcal{H}G)$ (with the Jacobson topology) is the disjoint union of its connected components. Each connected component is both open and closed. The connected components of $\widehat{G} = \mathrm{Prim}(\mathcal{H}G)$ are known as the *Bernstein components*.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ denotes the set of connected components of $\mathrm{Prim}(\mathcal{H}G)$.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ is a countable set and has no further structure.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ is the *Bernstein spectrum* of G.

$\pi_o\mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$ where (M, σ) can be any cuspidal pair i.e.
M is a Levi factor of a parabolic subgroup P of G
and σ is an irreducible super-cuspidal representation of M.

$\sim$ is the conjugation action of G, combined with tensoring σ by unramified characters of M.

"unramified" $=$ "the character is trivial on every compact subgroup of M."

$\pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$
$(M, \sigma) \sim (M', \sigma')$ iff there exists an unramified character
$\psi\colon M \to \mathbb{C}^{\times} = \mathbb{C} - \{0\}$ of M and an element g of G, $g \in G$, with

$$g(M, \psi \otimes \sigma) = (M', \sigma')$$

The meaning of this equality is:

- $gMg^{-1} = M'$
- $g_*(\psi \otimes \sigma)$ and σ' are equivalent
  smooth irreducible representations of M'.

For each $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$,
$\widehat{G}_\alpha$ denotes the connected component of $\mathrm{Prim}(\mathcal{H}G) = \widehat{G}$.

The problem of describing $\widehat{G}$ now breaks up into two problems.

Problem 1 Describe the Bernstein spectrum
$\pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$.

Problem 2 For each $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$,
describe the Bernstein component $\widehat{G}_\alpha$.

Problem 1 involves describing the irreducible super-cuspidal representations of Levi subgroups of G. The basic conjecture on this issue is that if M is a reductive p-adic group (e.g. M is a Levi factor of a parabolic subgroup of G) then any irreducible super-cuspidal representation of M is obtained by smooth induction from an irreducible representation of a subgroup of M which is compact modulo the center of M. This basic conjecture is now known to be true to a very great extent.

For Problem 2, the ABPS conjecture proposes that each Bernstein component $\widehat{G}_\alpha$ has a very simple geometric structure.

Notation

$\mathbb{C}^\times$ denotes the (complex) affine variety $\mathbb{C} - \{0\}$.

Definition

A *complex torus* is a (complex) affine variety T such that there exists an isomorphism of affine varieties

$$T \cong \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times.$$

Bernstein assigns to each $\alpha \in \pi_o\mathrm{Prim}(\mathcal{H}G)$ a complex torus T_α and a finite group Γ_α acting on T_α.

T_α is a complex algebraic group and $\exists$ a non-negative integer r such that T_α as an algebraic group defined over $\mathbb{C}$ is (non-canonically) isomorphic to $(\mathbb{C}^\times)^r := \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$. $\mathbb{C}^\times := \mathbb{C} - \{0\}$

$$T_\alpha \cong \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$$

In general, Γ_α acts on T_α not as automorphisms of the algebraic group T_α but only as automorphisms of the underlying complex affine variety T_α.

Bernstein then forms the quotient variety T_α/Γ_α and proves that there is a surjective map π_α mapping $\widehat{G}_\alpha$ onto T_α/Γ_α .

$$\begin{array}{c} \widehat{G}_\alpha \\ \Big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha \end{array}$$

This map π_α is referred to as the infinitesimal character or the central character or the cuspidal support map.

In Bernstein's work $\widehat{G}_\alpha$ is a set (i.e. is only a set) so π_α

$$\begin{array}{c} \widehat{G}_\alpha \\ \Big\downarrow{\scriptstyle \pi_\alpha} \\ T_\alpha/\Gamma_\alpha \end{array}$$

is a map of sets.

π_α is surjective, finite-to-one and generically one-to-one.

$\pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$

Given a cuspidal pair (M, σ), let $W_G(M)$ be the Weyl group of M.

$$W_G(M) := N_G(M)/M$$

Bernstein's finite group Γ_α is the subgroup of $W_G(M)$:

$$\Gamma_\alpha := \{w \in W_G(M) \,|\, \exists \, \text{an unramified character} \, \chi \, \text{of} \, M \, \text{with} \, w_*\sigma \sim \chi \otimes \sigma\}$$

Bernstein's complex torus T_α is a finite quotient of the complex torus consisting of all unramified characters of M.

$\pi_o\mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$

Given a cuspidal pair (M, σ), the Bernstein component $\widehat{G}_\alpha \subset \widehat{G}$ consists of all irreducible constituents of $Ind_M^G(\chi \otimes \sigma)$ where Ind_M^P is (smooth) parabolic induction and χ ranges over all the unramified characters of M.

$$\begin{array}{c} \widehat{G}_\alpha \\ \Big\downarrow {\scriptstyle \pi_\alpha} \\ T_\alpha/\Gamma_\alpha \end{array}$$

π_α is surjective, finite-to-one and generically one-to-one.

Conjecture

Let G be a connected split reductive p-adic group.
Let $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$.
Then there is a certain resemblance between

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & & \widehat{G}_\alpha \\ \rho_\alpha \big\downarrow & \text{and} & \big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha & & T_\alpha/\Gamma_\alpha \end{array}$$

Conjecture

$$
\begin{array}{ccc}
T_\alpha//\Gamma_\alpha & & \widehat{G}_\alpha \\
\rho_\alpha \Big\downarrow & \text{and} & \Big\downarrow \pi_\alpha \\
T_\alpha/\Gamma_\alpha & & T_\alpha/\Gamma_\alpha
\end{array}
$$

are almost the same.

How can this conjecture be made precise?
What does "almost the same" mean?
The precise conjecture uses the extended quotient of the second kind.

The precise conjecture consists of four statements.

Conjecture

#1. The infinitesimal character

$$\pi_\alpha : \widehat{G}_\alpha \to T_\alpha/\Gamma_\alpha$$

is one-to-one if and only if the action of Γ_α on T_α is free.

#2. The extended quotient of the second kind $(T_\alpha//\Gamma_\alpha)_2$ is canonically in bijection with $\widehat{G}_\alpha$.

$$(T_\alpha//\Gamma_\alpha)_2 \longleftrightarrow \widehat{G}_\alpha$$

Conjecture

#3. There is a canonically defined commutative triangle

$$\begin{tikzcd}
 & (T_\alpha//W_\alpha)_2 \arrow[dl] \arrow[dr] & \\
\widehat{G}_\alpha \arrow[rr] & & \{\text{Langlands parameters}\}^\alpha/{}^L G
\end{tikzcd}$$

in which the left slanted arrow is bijective, the right slanted arrow is surjective and finite-to-one, and the horizontal arrow is the map of the local Langlands correspondence. The maps in this commutative triangle are canonical.

Conjecture

#4. A geometric equivalence

$$\mathcal{O}(T_\alpha/\!/\Gamma_\alpha) \sim \mathcal{O}(T_\alpha) \rtimes \Gamma_\alpha$$

can be chosen such that the resulting bijection

$$T_\alpha/\!/\Gamma_\alpha \longleftrightarrow (T_\alpha/\!/\Gamma_\alpha)_2$$

when composed with the canonical bijection $(T_\alpha/\!/\Gamma_\alpha)_2 \longleftrightarrow \widehat{G}_\alpha$ gives a (non-canonical) bijection

$$\nu_\alpha : T_\alpha/\!/\Gamma_\alpha \longleftrightarrow \widehat{G}_\alpha$$

with the following properties:

$\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$
Within the admissible dual $\widehat{G}$ have the tempered dual $\widehat{G}_{tempered}$.
$\widehat{G}_{tempered}$ = {smooth tempered irreducible representations of G}/ $\sim$
$\widehat{G}_{tempered}$ = Support of the Plancherel measure
K_α = maximal compact subgroup of T_α.
K_α is a compact torus. The action of Γ_α on T_α preserves the maximal compact subgroup K_α , so can form the compact orbifold $K_\alpha//\Gamma_\alpha$.

Conjecture : Properties of the bijection ν_α

- The bijection $\nu_\alpha : T_\alpha//\Gamma_\alpha \longleftrightarrow \widehat{G}_\alpha$ maps
  $K_\alpha//\Gamma_\alpha$ onto $\widehat{G}_\alpha \cap \widehat{G}_{tempered}$
  $K_\alpha//\Gamma_\alpha \longleftrightarrow \widehat{G}_\alpha \cap \widehat{G}_{tempered}$

Conjecture : Properties of the bijection ν_α

- For many α the diagram

$$\begin{tikzcd} T_\alpha//\Gamma_\alpha \arrow[r, "\nu_\alpha"] \arrow[d, "\rho_\alpha"'] & \widehat{G}_\alpha \arrow[d, "\pi_\alpha"] \\ T_\alpha/\Gamma_\alpha \arrow[r, "I"'] & T_\alpha/\Gamma_\alpha \end{tikzcd}$$

  does not commute.
  $I =$ the identity map of T_α/Γ_α.

Conjecture : Properties of the bijection ν_α

- In the possibly non-commutative diagram

$$\begin{tikzcd} T_\alpha//\Gamma_\alpha \arrow[r, "\nu_\alpha"] \arrow[d, "\rho_\alpha"'] & \widehat{G}_\alpha \arrow[d, "\pi_\alpha"] \\ T_\alpha/\Gamma_\alpha \arrow[r, "I"'] & T_\alpha/\Gamma_\alpha \end{tikzcd}$$

the bijection $\nu_\alpha : T_\alpha//\Gamma_\alpha \longrightarrow \widehat{G}_\alpha$ is continuous where $T_\alpha//\Gamma_\alpha$ has the Zariski topology and $\widehat{G}_\alpha$ has the Jacobson topology
AND the composition

$$\pi_\alpha \circ \nu_\alpha : T_\alpha//\Gamma_\alpha \longrightarrow T_\alpha/\Gamma_\alpha$$

is a morphism of algebraic varieties.

Conjecture : Properties of the bijection ν_α

- For each $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$ there is an algebraic family

$$\theta_t : T_\alpha // \Gamma_\alpha \longrightarrow T_\alpha / \Gamma_\alpha$$

of morphisms of algebraic varieties, with $t \in \mathbb{C}^\times$, such that

$$\theta_1 = \rho_\alpha \qquad \text{and} \quad \theta_{\sqrt{q}} = \pi_\alpha \circ \nu_\alpha$$

$\mathbb{C}^\times = \mathbb{C} - \{0\}$
q = order of the residue field of the p-adic field F over which G is defined
π_α = infinitesimal character of Bernstein

Conjecture : Properties of the bijection ν_α

- Fix $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$. For each irreducible component $Z \subset T_\alpha//\Gamma_\alpha$ (Z is an irreducible component of the affine variety $T_\alpha//\Gamma_\alpha$) there is a cocharacter

$$h_Z : \mathbb{C}^\times \longrightarrow T_\alpha$$

such that

$$\theta_t(x) = \lambda(h_Z(t) \cdot x)$$

for all $x \in Z$.

cocharacter = homomorphism of algebraic groups $\mathbb{C}^\times \longrightarrow T_\alpha$
$\lambda : T_\alpha \longrightarrow T_\alpha/\Gamma_\alpha$ is the usual quotient map from T_α to T_α/Γ_α.

Question

Where are these correcting co-characters coming from?

Answer

The correcting co-characters are produced by the $SL(2,\mathbb{C})$ part of the Langlands parameters.

$$\mathcal{W}_F \times SL(2,\mathbb{C}) \longrightarrow {}^L G$$

Example

$G = GL(2, F)$
F can be any finite extension of the p-adic numbers $\mathbb{Q}_p$.
q denotes the order of the residue field of F.
$\widehat{G}_\alpha = \{$ Smooth irreducible representations of $GL(2, F)$ having a non-zero Iwahori fixed vector$\}$

$$\begin{aligned} T_\alpha &= \{\text{unramified characters of the maximal torus of } GL(2,F)\} \\ &= \mathbb{C}^\times \times \mathbb{C}^\times \end{aligned}$$

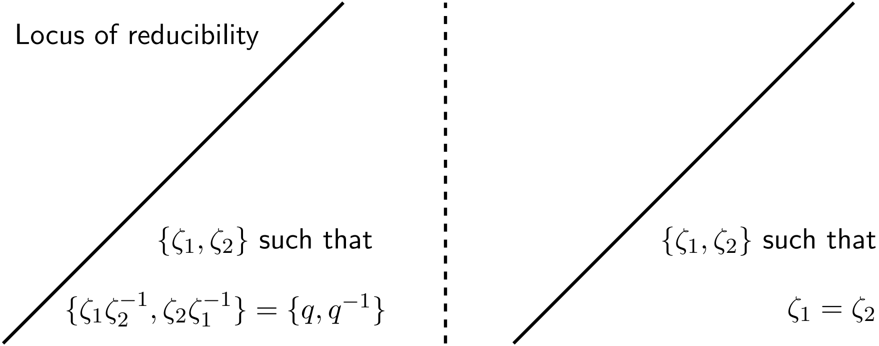
$\Gamma_\alpha =$ the Weyl group of $GL(2, F) = \mathbb{Z}/2\mathbb{Z}$

$0 \neq \gamma \in \mathbb{Z}/2\mathbb{Z} \qquad \gamma(\zeta_1, \zeta_2) = (\zeta_2, \zeta_1) \qquad (\zeta_1, \zeta_2) \in \mathbb{C}^\times \times \mathbb{C}^\times$

$$(\mathbb{C}^\times \times \mathbb{C}^\times)//(\mathbb{Z}/2\mathbb{Z}) = (\mathbb{C}^\times \times \mathbb{C}^\times)/(\mathbb{Z}/2\mathbb{Z}) \bigsqcup \mathbb{C}^\times$$

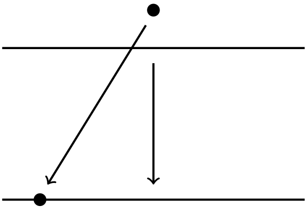
$$\mathbb{C}^\times \times \mathbb{C}^\times // (\mathbb{Z}/2\mathbb{Z}) = \mathbb{C}^\times \times \mathbb{C}^\times / (\mathbb{Z}/2\mathbb{Z}) \bigsqcup \mathbb{C}^\times$$

$$\mathbb{C}^\times \times \mathbb{C}^\times / (\mathbb{Z}/2\mathbb{Z})$$

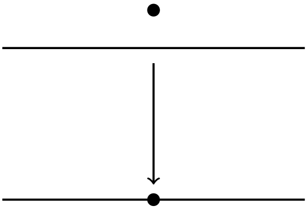


correcting cocharacter $\mathbb{C}^\times \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times$ is $t \mapsto (t, t^{-1})$

Infinitesimal character



Projection of the extended quotient on the ordinary quotient



QUESTION. In the ABPS view of $\widehat{G}$, what are the L-packets?

CONJECTURAL ANSWER. Fix $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$. In the list $h_1, h_2, \ldots, h_r$ of correcting cocharacters (one h_j for each irreducible component of the affine variety $T_\alpha // \Gamma_\alpha$) there may be repetitions — i.e. it may happen that for $i \neq j$, $h_i = h_j$. It is these repetitions that give rise to L-packets.

Fix $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$. Let
$Z_1, Z_2, \ldots, Z_r$ be the irreducible components of the affine variety $T_\alpha // \Gamma_\alpha$.
Let $h_1, h_2, \ldots, h_r$ be the correcting cocharacters.
Let $\nu_\alpha : T_\alpha // \Gamma_\alpha \longrightarrow \widehat{G}_\alpha$ be the bijection of ABPS.
CONJECTURE. Two points $[(\gamma, t)]$, $[(\gamma', t')]$ have

$$\nu_\alpha[(\gamma, t)] \text{ and } \nu_\alpha[(\gamma', t')] \text{ are in the same L} - \text{packet}$$

if and only if

$$h_i = h_j \qquad \text{where } [(\gamma, t)] \in Z_i \text{ and } [(\gamma', t')] \in Z_j$$

and

$$c_i = c_j$$

and

$$\text{For all } \tau \in \mathbb{C}^\times, \ \ \theta_\tau[(\gamma, t)] = \theta_\tau[(\gamma', t')]$$

WARNING. An L-packet might have non-empty intersection with more than one Bernstein component. The conjecture does not address this issue. The statement of the ABPS conjecture begins

$$\text{Fix } \alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G).$$

So the ABPS conjecture assumes that a Bernstein component has been fixed — and then describes the intersections of L-packets with this Bernstein component.

Example

$G = SL(2, F)$
F can be any finite extension of the p-adic numbers $\mathbb{Q}_p$.
q denotes the order of the residue field of F.
$\widehat{G}_\alpha = \{$ Smooth irreducible representations of $GL(2, F)$ having a non-zero Iwahori fixed vector$\}$

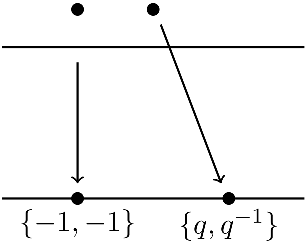
$$\begin{aligned} T_\alpha &= \{\text{unramified characters of the maximal torus of } SL(2, F)\} \\ &= \mathbb{C}^\times \end{aligned}$$

$\Gamma_\alpha =$ the Weyl group of $SL(2, F) = \mathbb{Z}/2\mathbb{Z}$

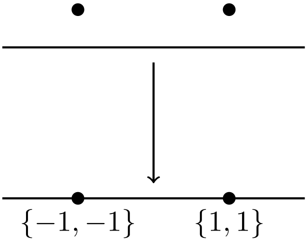
$0 \neq \gamma \in \mathbb{Z}/2\mathbb{Z} \qquad \gamma(\zeta) = \zeta^{-1} \qquad\qquad \zeta \in \mathbb{C}^\times$

$\mathbb{C}^\times//(\mathbb{Z}/2\mathbb{Z}) = \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) \bigsqcup \bullet \bigsqcup \bullet$

Infinitesimal character



Projection of the extended quotient on the ordinary quotient



Correcting cocharacter is $t \mapsto t^2$.

Preimage of $\{-1, -1\}$ is an L-packet.

Wiggly arrow indicates
"There is some interaction between the two conjectures."

$$\begin{tikzcd} \text{Baum-Connes} \arrow[d, leftrightsquigarrow] \\ \text{ABPS} \arrow[d, leftrightsquigarrow] \\ \text{Local Langlands} \end{tikzcd}$$

Theorem (V. Lafforgue)

Baum-Connes is valid for any reductive p-adic group G.

Theorem (Harris and Taylor, G.Henniart)

Local Langlands is valid for GL(n, F).

Theorem (ABPS)

ABPS is valid for GL(n, F).