RETURN TIMES AND SYNCHRONOUS RECURRENCE

EXTENDED ABSTRACT

PIOTR OPROCHA

Let (X, f) be a discrete dynamical system and let $\mathcal{F}$ be a hereditary upward set of subsets of $\mathbb{N}$ (the so-called *Furstenberg family*). A point x is $\mathcal{F}$-recurrent, if for any open neighborhood U of x, return times of x to U are in $\mathcal{F}$, that is $\{n : f^n(X)\} \in \mathcal{F}$.

A point x is $\mathcal{F}$-PR if for any $\mathscr{F}$-recurrent point y in any dynamical system (X, g) the pair (x, y) is recurrent for $(X \times Y, f \times g)$.

When $\mathscr{F}$ consists of all infinite sets of positive integers, then $\mathscr{F}$-recurrence is exactly the standard definition of recurrent point and $\mathscr{F}$-PR is *product recurrence* as defined in Furstenberg book more than 30 years ago [3]; using another standard families of sets we obtain some natural generalizations of this old concept.

The aim of this my talk is to present main ideas and advances related to the concept of product recurrence obtained since [3]. Especially, during last few years various new results appear, putting a new insight into the topic of synchronization of recurrent points.

A characterization of product recurrence can be found in the book of Furstnberg [3], where it is proved that a point is distal iff it is recurrent in pair with any recurrent point in any dynamical system.

If in place of recurrence in pair with any recurrent point we demand recurrence in pair with points in a smaller class of dynamical systems, it can lead to a wider class of points than the class of all distal points. For example, Auslander and Furstenberg in [1] asked about points which are recurrent in pair with any minimal point. While there is no known full characterization of points with this property, it was proved in [4] that class of such points is much larger than distal points, in particular it contains many points which are not minimal. Other sufficient conditions for this kind of product recurrence were provided in [2] and [5]. Moreover, [2] defines product recurrence in terms of Furstenberg families (i.e. upward hereditary sets of subsets of $\mathbb{N}$), which is a nice tool for a better classification of product recurrence. It is worth emphasizing that the concepts of [2] are not artificial, since it is possible almost immediately to relate these new types of product recurrence with some older results on disjointness.

After surveying some older results and relations between various concepts, we aim to present in more detail results contained in [2] and [6]. Among other things, we will present results leading to the following diagram, presenting relations between different types of $\mathscr{F}$-recurrence.

$$\begin{tikzcd}
\mathscr{F}_{\text{inf}} - \text{PR} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathscr{F}_{\text{pubd}} - \text{PR} \arrow[r, leftrightarrow] & \mathscr{F}_{\text{ps}} - \text{PR} \arrow[r, shift left] & \mathscr{F}_{\text{s}} - \text{PR} \arrow[l, shift left, "\text{not}"] \arrow[d] \arrow[dl, "\text{not}"] \\
\mathscr{F}_{\text{inf}} - \text{PR}_0 \arrow[r, shift right] & \mathscr{F}_{\text{pubd}} - \text{PR}_0 \arrow[l, shift right, "\text{not}"'] \arrow[r, shift right] \arrow[urr, "\text{not}"] & \mathscr{F}_{\text{ps}} - \text{PR}_0 \arrow[l, shift right, dotted, "?"'] \arrow[r, shift right] & \mathscr{F}_{\text{s}} - \text{PR}_0 \arrow[l, shift right, "\text{not}"']
\end{tikzcd}$$

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(P. Oprocha) AGH UNIVERSITY OF SCIENCE AND TECHNOLOGY, FACULTY OF APPLIED MATHEMATICS, AL. A. MICKIEWICZA 30, 30-059 KRAKÓW, POLAND

E-mail address: oprocha@agh.edu.pl