# Uniquely minimal spaces

Ľubomír Snoha

*Department of Mathematics, Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia*
[Lubomir.Snoha@umb.sk]

Joint work with: Tomasz Downarowicz (Institute of Mathematics of the Polish Academy of Science and Wroclaw University of Technology), Dariusz Tywoniuk (Wroclaw University of Technology)

If $X$ is a compact metric space and $T : X \to X$ a homeomorphism, the dynamical system $(X, T)$ or the homeomorphism $T$ itself is *minimal* if every orbit (equivalently, every forward orbit) is dense. A space $X$ admitting a minimal homeomorphism is called a *minimal space*.

In known examples of infinite compact metric spaces admitting a minimal homeomorphism (the circle, the torus, the Cantor set, ...) always there are uncountably many such homeomorphisms. On the other hand, some other spaces admit no minimal homeomorphisms. Do there exist "intermediate" infinite compact metric spaces which admit, but at most countably many, minimal homeomorphisms?

It is known, see [dG], that for every abstract group $G$ there exists a topological space $X$ such that $H(X, X) \simeq G$, i.e. the group $H(X, X)$ of all homeomorphisms $X \to X$ is algebraically isomorphic to $G$. Such a space $X$ always exists in the class of one-dimensional, connected, locally connected, complete metric spaces and always exists in the class of compact, connected, Hausdorff spaces. (However, such a space need not exist in the class of compact metric spaces, because a compact metric space has at most cardinality $\mathfrak{c}$, while there are groups of arbitrary cardinalities.) Moreover, as proved in [dGW], if $G$ is at most countable then $X$ can be chosen to be a Peano continuum of any positive dimension. In particular, there is a Peano continuum $X$ with $H(X, X)$ being the trivial group (then $X$ is called a *rigid space for homeomorphisms*). Also, there is a Peano continuum $X$ such that $H(X, X) \simeq \mathbb{Z}$, i.e. $H(X, X) = \{T^n : n \in \mathbb{Z}\}$, the elements of $H(X, X)$ being pairwise distinct. In view of these facts, we are interested in whether there exists a compact metric space $X$ such that $H(X, X) \simeq \mathbb{Z}$ and the generating homeomorphism $T$ is minimal.

We adopt the following definition.

A (nonempty) compact metric space $X$ is called *uniquely minimal* if $X$ admits a minimal homeomorphism $T$ and $H(X, X) = \{T^n : n \in \mathbb{Z}\}$.

It is easy to see that if $X$ is a uniquely minimal space with $H(X, X) = \{T^n : n \in \mathbb{Z}\}$, then there are three possibilities:

(1) $\operatorname{card} X = 1$, $H(X, X)$ is the trivial group;
(2) $\operatorname{card} X = 2$, $H(X, X) \simeq \mathbb{Z}_2$;
(3) $\operatorname{card} X = \mathfrak{c}$, $X$ has no isolated point, $T$ is totally minimal (i.e. $T^n$ is minimal for all $n \in \mathbb{Z} \setminus \{0\}$), $H(X, X) \simeq \mathbb{Z}$.

Notice that whenever $H(X, X)$ is cyclic, positivity or finiteness of the topological entropy of the dynamical system $(X, T)$ does not depend on the choice of $T \in H(X, X)$, $T$ different from the identity. Thus we can talk about *zero, finite positive* and *infinite entropy* uniquely minimal spaces.

Apriori, it is unclear whether there exists an infinite uniquely minimal space. In the lecture we construct a family of such spaces in the class of one-dimensional continua. Some of them have zero entropy, some finite positive and some infinite entropy.

Let us describe an application of our result.

Given a compact metric space $X$ and a homeomorphism $T : X \to X$, by the functional envelope of the system $(X, T)$ we mean here the system $(H(X, X), F_T)$ where $H(X, X)$ is endowed with the uniform metric and $F_T : H(X,X) \to H(X,X)$ is defined by $F_T(\varphi) = T \circ \varphi$, $\varphi \in H(X,X)$ (cf. [AKS]).

If $X$ is an infinite compact metric space with $H(X,X) = \{T^n : n \in \mathbb{Z}\}$, then the topological entropy of the (countable) system $(H(X,X), F_T)$ is zero. The fact that we have constructed uniquely minimal spaces with positive entropy thus gives negative answer to the question from [KS], whether always the entropy of $(H(X,X), F_T)$ is larger than or equal to the entropy of $(X,T)$. (Nevertheless, under the additional assumption that $X$ is homogeneous, the answer is positive.)

## References

[AKS] J. Auslander, S. Kolyada and Ľ. Snoha, *Functional envelope of a dynamical system*, Nonlinearity **20** (2007), no. 9, 2245–2269.

[dG] J. de Groot, *Groups represented by homeomorphism groups*, Math. Ann. **138** (1959), 80–102.

[dGW] J. de Groot, R. J. Wille, *Rigid continua and topological group-pictures*, Arch. Math. **9** (1958), 441–446.

[KS] S. Kolyada, J. Semikina, *On topological entropy: when positivity implies +infinity*, Max-Planck-Institut für Mathematik, Preprint Series 2013(8), pages 19.