Mahler measure, Fuglede-Kadison determinants and entropy of algebraic systems

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Abstract

Lind-Schmidt-Ward proved the logarithmic Mahler measure of an integral polynomial equals the entropy of a corresponding dynamical system (building on pioneering work of Yuzvniskii). After a breakthrough due to Deninger, this equality has been generalized greatly over recent years by several authors to an equality between the Fuglede-Kadison determinant of an element of the integral group ring of a group G and the sofic entropy of a corresponding action of G by automorphisms on a compact abelian group.

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My plan for this talk is to give a broad overview to new results in computing the entropy of certain dynamical systems of algebraic origin. To explain, let

- $f \in \mathbb{Z}\Gamma$
- $\mathbb{Z}\Gamma f \subset \mathbb{Z}\Gamma$ is a left-ideal and $\Gamma \curvearrowright \mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ by left-multiplication.
- $X_f := \widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f} = \mathrm{Hom}(\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f, \mathbb{R}/\mathbb{Z})$
- $\Gamma \curvearrowright X_f$ is a *principal algebraic action* $gx = x \circ g^{-1}$.
- Focus of the talk: Express the topological entropy of $\Gamma \curvearrowright X_f$ in terms of f.

1 General context

This problem fits into a larger class of problems as follows. Suppose M is a countable abelian and $\Gamma \curvearrowright M$ acts by automorphisms. Then $\Gamma \curvearrowright \hat{M}$ also by automorphisms. However, $\hat{M}$ is a compact abelian group and the action of Γ preserves the Haar measure. The general problem is to relate the dynamical properties of $\Gamma \curvearrowright \hat{M}$ to algebraic properties of $\Gamma \curvearrowright M$

2 The $\Gamma = \mathbb{Z}$ case

In the special case $\Gamma = \mathbb{Z}$, Yuzvinskii obtained a complete solution to this problem in the 60s. To motivate the answer:

Theorem 2.1 (Yuzvniskii). *If $T \in Aut(G)$ is an auto of a compact metriz group and $N < G$ is a closed normal subgroup (so*

$$1 \to N \to G \to G/N \to 1$$

is an exact sequence commuting with T then

$$h(T, G) = h(T, N) + h(T, G/N).$$

If $f, g \in \mathbb{Z}[x]$ have no roots in common then

$$1 \to X_g \to X_{fg} \to X_f \to 1$$

is exact.

This is dual to the exact sequence

$$0 \to \mathbb{Z}[x]/\mathbb{Z}[x]f \to \mathbb{Z}[x]/\mathbb{Z}[x]fg \to \mathbb{Z}[x]/\mathbb{Z}[x]/g \to 0.$$

(the first map is $r + \mathbb{Z}[x]f \to rg + \mathbb{Z}[x]fg$, the second map is $r + \mathbb{Z}[x]fg \to r + \mathbb{Z}[x]g$.)

So $f \mapsto \exp h(X_f)$ looks like a multiplicative function on polynomials.

What are the multiplicative functions on polynomials? Well, the top coefficient is one. Also if $J \subset \mathbb{C}$ then we can send $f \in \mathbb{Z}[x]$ to the product of all of its roots that are contained in J. These are essentially all of them.

Theorem 2.2 (Yuzvinskii, 1969)**.** *If $f(x) = c_s x^s + \cdots + c_0 \in \mathbb{Z}[x]$ with $c_s c_0 \neq 0$ then $\log h(X_f) = |c_s|$ times the product of the roots outside the unit circle:*

$$h(X_f) = \log|c_s| + \sum_{j=1}^{s} \log^+ |r_j|$$

where $r_1, \ldots, r_s$ are the roots of f and $\log^+|x| = \max(0, \log|x|)$.

Jensen $\Rightarrow \log^+|x| = \int_0^1 \log|e^{2\pi i t} - x| \; dt$

$$\Rightarrow h(X_f) = \int_0^1 \log|f(e^{2\pi i t})| \; dt = \log M(f).$$

2.1 Proof sketch

- $X_f = \{(x_i) \in \mathbb{T}^{\mathbb{Z}} : \; \sum_{k=0}^{s} c_k x_{m+k} \; \forall m \in \mathbb{Z}\}$
- Let ρ_N be the pseudometric on X_f given by

$$\rho_N(x, y) = \sup_{0 \le i \le N-1} |x_i - y_i|.$$

- $Y \subset X_f$ is (ρ_N, ϵ)-separated if for every $y^1, y^2 \in Y$,

$$|y_i^1 - y_i^2| < \epsilon \; \forall 0 \le i < N.$$

- Rufus Bowen $\Rightarrow h(X_f) = \sup_{\epsilon > 0} \limsup_{N \to \infty} \frac{1}{N} \log \max \mathrm{card}(\rho_\mathrm{N}, \epsilon)-$separated set
- $h(X_f) = \sup_{\epsilon > 0} \limsup_{N \to \infty} \frac{-1}{N} \log \mathrm{Haar}_{\mathrm{X_f}}(\mathrm{B}(\mathrm{N}, \epsilon))$
- $B(N, \epsilon) = \{x \in X_f : \; |x_i| < \epsilon \; \forall 0 \le i < N\}$

- Suppose $N \geq s$. If $x \in X_f$ and $(x_0, \ldots, x_{N-1})$ is given then there are $|c_s|$ choices for x_N.

- $\Rightarrow \frac{\mathrm{Haar}_{\mathrm{X_f}}(\mathrm{B(N-1,\epsilon)})}{\mathrm{Haar}_{\mathrm{X_f}}(\mathrm{B(N,\epsilon)})} = |c_s| \frac{\mathrm{vol}_{\mathbb{T}^s}(\pi_s(\mathrm{B(N-1,\epsilon)}))}{\mathrm{vol}_{\mathbb{T}^s}(\pi_s(\mathrm{B(N,\epsilon)}))}$

- $\pi_s(x) = (x_0, \ldots, x_{s-1})$

- $\epsilon > 0$ small enough $\Rightarrow \frac{\mathrm{Haar}_{\mathrm{X_f}}(\mathrm{B(N-1,\epsilon)})}{\mathrm{Haar}_{\mathrm{X_f}}(\mathrm{B(N,\epsilon)})} = |c_s| \frac{\mathrm{vol}_{\mathbb{R}^s}(\bar{\pi}_s(\mathrm{B(N-1,\epsilon)}))}{\mathrm{vol}_{\mathbb{R}^s}(\bar{\pi}_s(\mathrm{B(N,\epsilon)}))}$

$$A := \left(\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0/c_s & -c_1/c_s & -c_2/c_s & \cdots & -c_{n-1}/c_s \end{array} \right).$$

Then

$$\bar{\pi}_s(B(N,\epsilon)) = \{t \in \mathbb{R}^s : \ \|A^k t\|_\infty < \epsilon \quad \forall k = 0 \ldots N-s-1\}.$$

So it suffices to show

$$-\frac{1}{N} \log \mathrm{vol}_{\mathbb{R}^s}(\{\mathrm{t} \in \mathbb{R}^\mathrm{s} : \ \|\mathrm{A}^\mathrm{k}\mathrm{t}\|_\infty < \epsilon \quad \forall \mathrm{k} = 0 \ldots \mathrm{N} - \mathrm{s} - 1\}) \to \sum_{\mathrm{j}=1}^{\mathrm{s}} \log^+ |\mathrm{r_j}|$$

3 Mahler measure

$$M(f) := \exp \int_0^1 \log |f(e^{2\pi i t})| \ dt.$$

- (Lehmer) Is 1 an accumulation point of $\{M(f) : \ f \in \mathbb{Z}\Gamma\}$?

- (Lind-Schmidt-Ward): Equivalently, is the set of entropies of algebraic actions of $\mathbb{Z}$ equal to $[0, \infty]$? (If not then this set is a countable additive semigroup)

Theorem 3.1 (Lind-Schmidt-Ward,1990)**.** *For $f \in \mathbb{Z}[\mathbb{Z}^d]$ nonzero, $h(X_f) = \log M(f)$ where*

$$M(f) = \exp \int_{\mathbb{T}^d} \log |f(z)| \ dz$$

where $\mathbb{T}^d \subset \mathbb{C}^d$ is the d-dimensional unit torus.

4 von Neumann algebras

Q: What can be said about nonabelian groups? There is a generalization of Mahler measure to nonabelian groups. To explain it, we need to embed the group ring $\mathbb{Z}\Gamma$ into a larger algebra.

Let $\lambda, \rho : \Gamma \to B(\ell^2(\Gamma))$ be the left and right regular representations. So

$$(\lambda_g \phi)(f) = \phi(g^{-1}f), \quad (\rho_g \phi)(f) = \phi(fg).$$

- $\mathcal{N}\Gamma$=the weak closure of $\rho(\Gamma)$. Equiv., $\mathcal{N}\Gamma$ are all bounded linear operators that commute with $\lambda(\Gamma)$.
- $\mathrm{tr}_{\mathcal{N}\Gamma} : \mathcal{N}\Gamma \to \mathbb{C}$, $\mathrm{tr}_{\mathcal{N}\Gamma}(T) = \langle T(1_\Gamma), 1_\Gamma \rangle$.
- For $T \in \mathcal{N}\Gamma^\times$
$$\det_{\mathcal{N}\Gamma}(T) := \exp\left(\frac{1}{2}\mathrm{tr}_{\mathcal{N}\Gamma}(\log TT^*)\right) \in \mathbb{R}^\times$$
- In general,
$$\det_{\mathcal{N}\Gamma}(T) = \exp\left(\int_{[0,\infty)} \log t \; d\mu_{|T|}(t)\right).$$
- (FK, 1950s) $\det_{\mathcal{N}\Gamma}$ is a homomorphism.

If $\Gamma = \mathbb{Z}^d$ and $A : L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d)$ is a bounded operator,

$$\begin{array}{ccc} \ell^2(\mathbb{Z}^d) & \xrightarrow{A} & \ell^2(\mathbb{Z}^d) \\ \updownarrow \mathcal{F} & & \updownarrow \mathcal{F} \\ L^2(\mathbb{T}^d) & \xrightarrow{\hat{A}} & L^2(\mathbb{T}^d) \end{array}$$

If $A \in \mathcal{N}\Gamma$ (so it commutes with left-multiplication by $\mathbb{Z}^d$) then $\hat{A} \in L^\infty(\mathbb{T}^d)$ is the multiplication operator $\mathcal{F}(A(0))$ ($0 \in \ell^2(\mathbb{Z}^d)$ is the dirac mass at 0) so $A(0) \in \ell^2(\mathbb{Z}^d)$ so $\mathcal{F}(A(0)) \in L^2(\mathbb{T}^d)$ and this makes sense). Then the trace of A is the integral of $\mathcal{F}(A(0))$ over $\mathbb{T}^d$ which leads to:

For $f \in \mathbb{Z}[\mathbb{Z}^d]$, $\quad \det_{\mathcal{N}\Gamma}(\rho_f) = M(f)$.

We get an isomorphism of vn Algebras $\mathcal{N}\mathbb{Z}^d \to L^\infty(\mathbb{T}^d)$ taking the trace to integration.

5 Entropy

Deninger (2006): (implicitly in his paper but not directly stated) If Γ is amenable and $f \in \mathbb{Z}[\Gamma] \cap \mathcal{N}\Gamma^\times$ then $h(X_f) = \log \det_{\mathcal{N}\Gamma}(\rho_f)$?

If Γ is finite and f is not a unit in $\mathcal{N}\Gamma$ then it fails. (Using coinduction, we can build such examples for any group that contains a nontrivial finite subgroup). This is why we restrict to invertible elements of $\mathcal{N}^\Gamma$.

Answers:

- (Deninger, 2006) Yes if Γ has a strong Følner sequence, $f^{-1} \in L^1(\Gamma)$ and $f \geq 0$ in $\mathcal{N}\Gamma$.
- (Deninger-Schmidt, 2007) Yes if Γ is residually finite and $f^{-1} \in L^1(\Gamma)$.
- (Li, 2010) Yes.

5.1 Outline of Li's proof

- Show that if $f \geq 0$ then $h(X_f) = \log \det_{\mathcal{N}\Gamma}(\rho_f)$. This uses an ℓ^2-version of Bowen's entropy via spanning sets and an approximation formula of Deninger for $\det_{\mathcal{N}\Gamma}(f)$ when $f \geq 0$.
- Generalize Yuzvinskii's addition formula to arbitrary amenable groups:

$$0 \to N \to G \to G/N \to 0 \Rightarrow h(G) = h(N) + h(G/N).$$

  (uses various fiber and conditional entropies with Rudolph-Weiss' OE method)
- Prove $h(X_f) \geq \log \det_{\mathcal{N}\Gamma}(f)$ in general via perturbing the compression of f to an invertible linear operator. Uses Lück approximation type arguments and Ornstein-Weiss quasi-tiling.
- $h(X_{f^*}) \geq \log \det_{\mathcal{N}\Gamma}(f^*)$ so

$$h(X_{f^*f}) = h(X_f) + h(X_{f^*}) \geq \log \det_{\mathcal{N}\Gamma}(f) + \log \det_{\mathcal{N}\Gamma}(f^*)$$
$$= 2 \log \det(f) = \log \det(f^* f) = h(X_{f^*f})$$

  shows $h(X_f) = \log \det_{\mathcal{N}\Gamma}(f)$.

5.2 Li-Thom

Hanfeng Li and Andreas Thom. Entropy, determinants, and L2-torsion, *J. Amer. Math. Soc.* **27** (2014), no. 1, 239–292.

Li-Thom study the general case $\Gamma \curvearrowright \hat{M}$ where Γ is amenable and relate entropy to L^2-torsion. They show, for example that $h(\hat{M}) = \rho^{(2)}(M)$ if M has a finite free resolution (type FL) and $\chi(M) = 0$. It is not even obvious why $\rho^{(2)}(M)$ should be nonnegative apriori.

A corollary of this is that the L^2-torsion of the trivial module is zero when it has type FL which had been conjectured by Lück.

6 Sofic groups

Definition 1. Γ is sofic if there are a collection $\{V_i\}_{i \in I}$ of finite sets and maps $\sigma_i : \Gamma \to \mathrm{Sym}(V_i)$ such that for every $g, h \in G$,

$$\lim_{i \to \infty} |V_i|^{-1} \#\{v \in V_i : \ \sigma_i(g)\sigma_i(h)v = \sigma_i(gh)v\} = 1$$

and if $g \neq h$ then

$$\lim_{i\to\infty} |V_i|^{-1}\#\{v \in V_i:\ \sigma_i(g)v \neq \sigma_i(h)v\} = 1.$$

- Amenability $\Rightarrow$ sofic
- Residually finite $\Rightarrow$ sofic
- The class of sofic groups is closed under: subgroups, direct limits, inverse limits, direct products, extensions by amenable groups and free products with amalgamation over an amenable subgroups. (Elek-Szabo, Dykema-Kerr-Pichot, Paunescu)
- Soficity has been used to study group rings and group algebras... If G is sofic then G satisfies Gottshalk's surjunctivity conjecture, Connes embedding conjecture, the Determinant conjecture, Kaplansky's direct finiteness conjecture. (Gromov 1999, Weiss 2000, Elek-Szabo 2005)
- The sofic property enables one to associate invariants of actions on topological spaces, measure spaces, Banach spaces.
- OPEN: Is every countable group sofic?

7 Topological sofic entropy

Given

- $\Gamma \curvearrowright X$ cts action on cp metric space
- Σ=a sofic approximation to Γ
- ρ, continuous pseudo-metric on X

define

$$h_\Sigma(G \curvearrowright X, \rho) := \sup_{\epsilon>0} \inf_{W\subset G} \inf_{\delta>0} \limsup_{i\to\infty} |V_i|^{-1} \log N_\epsilon(\mathrm{Map}(W,\delta,\sigma_i),\rho_\infty).$$

where

- for $W \subset \Gamma$, $\delta > 0$ and $\sigma : \Gamma \to \mathrm{Sym}(V)$

  $$\mathrm{Map}(\mathrm{W},\delta,\sigma,\rho)$$

  to be the set of all maps $\phi : V \to X$ such that

  $$\rho_2(\phi\circ\sigma(w), w\circ\phi) < \delta,\ \forall w \in W.$$

  $$= \left(\frac{1}{|V|}\sum_{v\in V} \rho(\phi(\sigma(w)v), w\phi(v))^2\right)^{1/2}.$$

- $N_\epsilon(\mathrm{Map}(\mathrm{W},\delta,\sigma,\rho),\rho_\infty)$ denotes the max. card. of a ρ_∞-separated subset where $\rho_\infty(\phi_1,\phi_2)=\max_{v\in V}\rho(\phi_1(v),\phi_2(v))$.
- The elements of $\mathrm{Map}(\mathrm{W},\delta,\sigma,\rho)$ are the "approximately periodic" points for the action. They are also called "microstates" in analogy with Voiculescu's free entropy theory.

Theorem 7.1 (Kerr-Li, 2011)**.** *If ρ_1,ρ_2 are dynamically generating, then $h_\Sigma(G\curvearrowright X,\rho_1)=h_\Sigma(G\curvearrowright X,\rho_2)$. So $h_\Sigma(G\curvearrowright X):=h_\Sigma(G\curvearrowright X,\rho_1)$. Moreover, if G is amenable, then this coincides with classical topological entropy.*

Conjecture 1 (Gottschalk's surjunctivity conjecture)**.** *For any countable discrete group G, any finite set K and any continuous G-equivariant map $\phi:K^G\to K^G$, if ϕ is injective then must also be surjective.*

This was proven by Gromov for sofic groups. Kerr-Li give a new proof based on topological sofic entropy.

8 Sofic entropy of algebraic actions

Theorem 8.1 (Hayes)**.** *Let Γ be a sofic group, Σ a sofic approximation, $f\in M_n(\mathbb{Z}\Gamma)$ injective on $\ell^2(\Gamma)^{\oplus n}$. Then*

$$h_\Sigma(X_f)=\log\det(f).$$

If f is not injective then $h_\Sigma(X_f)=\infty$. Also if $f\in M_{m,n}(\mathbb{Z}\Gamma)$ then

$$h_\Sigma(X_f)\le\log^+\det(f).$$

There are also measure-theoretic counterparts to these statements.

Special cases were proven earlier by B.-, Kerr-Li, B.-Li.

Theorem 8.2 (Hayes)**.** *With Γ as above, if $f\in M_n(\mathbb{Z}\Gamma)\cap GL_n(\mathcal{N}\Gamma)$, $f\notin GL_n(\mathbb{Z}\Gamma)$ then* $\det(f)>1$.

8.1 Outline

We consider the case only when $f\in\mathbb{Z}\Gamma$.

Lemma 0:

$$h_\Sigma(X_f)=\sup_\epsilon\inf_{F,E,\delta}\limsup_i|V_i|^{-1}\log S_\epsilon(Map(\rho|A,F,\delta,\sigma_i),\rho_2)$$

where $F\subset\Gamma$ is finite, $A\subset\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ is finite, $\delta>0$ and $Map(\rho|A,F,\delta,\sigma_i)$ consists of all approximately equivariant maps $\phi:V_i\to\mathbb{T}^\Gamma$ such that

$$\frac{1}{|V_i|}\sum_{v\in V_i}|\phi(v)(a)|^2<\delta$$

(ρ here is the usual time 0 pseudo-metric).

Lemma 1:

- $h_\Sigma(X_f) = \sup_{\epsilon>0} \inf_\delta \limsup_i \frac{1}{|V_i|} \log S_\epsilon(\Xi_\delta(\sigma_i(f)), \theta_{2,\mathbb{Z}^{V_i}})$.
- $S_\epsilon(\cdot)$ is the max. card. of an ϵ-separated subset
- $\Xi_\delta(\cdot)$ is the set of $\xi \in \mathbb{R}^{V_i}$ such that $\min_{\lambda\in\mathbb{Z}^{V_i}} \|\sigma_i(f)\xi - \lambda\|_2 < \delta$ (we are also using this distance to measure the separation)

The idea is that any such ξ is associated to a map from V_i to X_f that is close in terms of the usual time 0 coordinate and vice versa.

In order to control the kernel of $\sigma_i(f)$ better, let $A = A_i, B = B_i \subset V_i$ be such that

$$P_{im(\sigma_i(f))^\perp}|_{\mathbb{R}^{A^c}}, P_{ker(\sigma_i(f))^\perp}|_{\mathbb{R}^{B^c}}$$

are isomorphisms onto $im(\sigma_i(f))^\perp, ker(\sigma_i(f))^\perp$. Set

$$x_i = \chi_A \sigma_i(f) \chi_B.$$

Because f is injective on $\ell^2(\Gamma)$, the density of A_i and B_i in V_i tend to 1 as $i \to \infty$. Therefore

$$\|x_i - \sigma_i(f)\|_2 \to 0, \quad \mu_{|x_i|} \to \mu_{|f|}$$

and

$$\sup_\xi \|x_i\xi - \sigma_i(f)\xi\|_{2,\mathbb{Z}^{V_i}} \to 0.$$

In particular, we can replace $\sigma_i(f)$ with x_i in the formula for entropy:

$$h_\Sigma(X_f) = \sup_\epsilon \inf_\delta \limsup_i \frac{1}{|V_i|} \log S_\epsilon(\Xi_\delta(x_i), \theta_{2,\mathbb{Z}^{V_i}}).$$

The matrix x_i is nicer than $\sigma_i(f)$ because $im(x_i) = \mathbb{R}^A, \ker(x_i) = \mathbb{R}^{B^c}$. So we can multiply x on the right and left by permutation matrices so that it has the form

$$x_i = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}$$

and T has entries in $\mathbb{Z}$ and is invertible over $\mathbb{R}$.

8.1.1 The upper bound

We will construct a product space $\mathcal{M} \times \mathcal{N} \times \mathcal{O}$ with a natural map into $\mathbb{R}^{V_i}$ and show that its image 2ϵ-covers $\Xi_\delta(x_i)$ (wrt $\|\cdot\|_{2,\mathbb{Z}^{V_i}}$. It then suffices to estimate the cardinalities of $\mathcal{M}, \mathcal{N}$ and $\mathcal{O}$.

Let

- $\mathcal{M} \subset x_i^{-1}(\delta \mathrm{Ball}(\ell^2_{\mathbb{R}}(\mathrm{V_i}))) \cap \mathbb{R}^{\mathrm{B}}$ be a maximal ϵ-separated subset. (These are points in the "domain" $\mathbb{R}^B$ that map into a small ball).

- $\mathcal{N} \subset x_i^{-1}(\mathbb{Z}^A) \cap \mathbb{R}^B$ be a section for the quotient map

$$x_i^{-1}(\mathbb{Z}^A) \cap \mathbb{R}^B \to \frac{x_i^{-1}(\mathbb{Z}^A) \cap \mathbb{R}^B}{\mathbb{Z}^B}$$

  These are points that map into $\mathbb{Z}^A$. Intuitively, this is the important part.

- $\mathcal{O} \subset Ball(\ell^2_{\mathbb{R}}(V_i)) \cap \mathbb{R}^{B^c}$ be a maximal ϵ-separated subset. This are the points not in the domain that we need to get a covering.

Lemma 2. The map

$$(m, n, o) \in \mathcal{M} \times \mathcal{N} \times \mathcal{O} \to \mathbb{R}^{V_i}$$

defined by

$$(m, n, o) \mapsto m + n + o$$

2ϵ-covers $\Xi_\delta(x_i)$ (wrt $\| \cdot \|_{2,\mathbb{Z}^{V_i}}$.

We observe that $|\mathcal{N}| = \det^+(x_i)$, $|\mathcal{M}| \leq \det_{4\delta/\epsilon}(x_i)^{-1}$ (where $\det_{4\delta/\epsilon}(x_i)$ is the product of the small eigenvalues of x_i) and

$$|\mathcal{O}| \leq \left(\frac{3 + 3\epsilon}{\epsilon}\right)^{|B^c|}.$$

From these inequalities we obtain that

$$\log S_\epsilon(\Xi_\delta(x_i), \theta_{2,\mathbb{Z}^{V_i}}) \leq \int_{4\delta/\epsilon}^{\infty} \log t \ d\mu_{|x_i|}$$

plus an error term that disappears in the limit. This is enough for the result because we consider $i \to \infty$ then $\delta \to 0$ then $\epsilon \to 0$.

8.1.2 The lower bound

We will construct sets $\mathcal{N}, \mathcal{M}$ (different from above) and an injective natural map $\mathcal{N} \times \mathcal{M} \to \mathbb{R}^{V_i}$ such that the image is an ϵ-separated subset of $\Xi_\delta(x_i)$ (wrt $\| \cdot \|_{2,\mathbb{Z}^{V_i}}$. Then it suffices to estimate the cardinalities of $\mathcal{N}$ and $\mathcal{M}$.

Let

- $\mathcal{N} \subset x^{-1}(\mathbb{Z}^A) \cap \mathbb{R}^B$ be such that $\{x_i\xi : \xi \in \mathcal{N}\}$ is a maximal η-separated subset wrt $\theta_{2,x_i\mathbb{Z}^B}$ where $\eta = 2\epsilon\|\hat{f}\|_1$. This should be the set that matters most.

- $p = \chi_{0,\delta/\epsilon}(|x_i|), W = p\mathbb{R}^{V_i}$

- $\mathcal{M} \subset W \cap x_i^{-1}(\delta\mathrm{Ball}(\ell^2(\mathrm{V_i})))$ be a maximal ϵ-separated subset wrt $\theta_{2,\mathbb{Z}^V}$. This is the set of points which are mapped to points of small size.

Consider the map $\mathcal{M} \times \mathcal{N} \to \mathbb{R}^{V_i}/\mathbb{Z}^{V_i}$ by

$$(m, n) \mapsto m + n$$

Because $\mathcal{M}, \mathcal{N}$ are appropriately separated, this map is injective. Also their definitions imply that $m + n \in \Xi_\delta(x_i)$.

We obtain

$$|\overset{+}{\det}(x_i)| \leq C|\mathcal{N}|$$

where $C > 0$ is a constant depend only of f.

Similarly,

$$\det_{\delta/\epsilon}(x_i)^{-1}(\delta/\epsilon)^{|W|} \leq C|\mathcal{M}|.$$

From this we obtain the lower bound as before.

9 Measure-theoretic

There is a measure-theoretic sofic entropy theory as well and Ben extended all of the results of the Theorem mentioned above to the measure-theory case (with one exception: it is not known if f being noninjective on $\ell^2(\Gamma)^{\oplus n}$ implies the measure-entropy is $+\infty$).

The variational principle reduces the problem to proving the lower bound. This follows as in the proof of the lower bound for topological entropy, once we know that most of the good maps constructed there are approximately equidistributed. This proof of this is similar to work that I did earlier in the residuallly finite case (and ℓ^1-invertible case). It relies on a result of Li-Peterson-Schmidt showing that the action is ergodic (wrt Haar measure).

10 Questions

- Given a group G, what can be said about $\mathrm{Det(G)} := \{\mathrm{det(f)} : \ \mathrm{f} \in \mathbb{Z}\mathrm{G}\} \subset [1, \infty)$? Lehmer's problem asks whether $\mathrm{Det}(\mathbb{Z})$ is discrete or equal to $[1, \infty)$.

- Is there a characterization of $f \in \mathbb{Z}G$ with $\det(f) = 1$? (So we 'should' have $h(X_f) = 0$). For example, if $G = \mathbb{Z}$ then these are products of cyclotomic polynomials.