Surgery applied to 4-manifolds.

(CCOT)

<u>Aim</u>: Classify closed, connected, oriented, topological 4-manifolds with $\pi_1 \cong \mathbb{Z}$ up to homeomorphism

<u>Literature</u>

- Freedman-Quinn '90: statements, outline, "no formal proof ... extended exercise".
- Kreck, Surgery and Duality: Spin case. '00
  Hambleton-Kreck-Teichner: Geom dim ≤ 2 groups '09
  completed odd case.

I will present a third proof using surgery exact sequence.

M closed $\iff$ M compact and $\partial M = \emptyset$.

$$\mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$$
$$\sum n_g g \mapsto \sum n_g g^{-1}.$$

<u>Theorem A</u> (Existence) [FQ]

Let (H, λ) be a nonsingular, Hermitian, sesquilinear form over $\mathbb{Z}[\mathbb{Z}]$.
$= \mathbb{Z}[t, t^{-1}]$

$$\lambda : H \times H \longrightarrow \mathbb{Z}[\mathbb{Z}]$$

with $H \cong \bigoplus^n \mathbb{Z}[\mathbb{Z}]$ for some n

(ie. H is a free $\mathbb{Z}[\mathbb{Z}]$-module)

Let $k \in \mathbb{Z}/2$.

If λ is even (ie $\forall x \in H$, $\lambda(x,x) = q + \bar{q}$ for some $q \in \mathbb{Z}[\mathbb{Z}]$)

thus assume $k \equiv \operatorname{sign}(\lambda \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{R})/8 \; (2)$.

Then, there exists a CCOT 4-manifold M with $\pi_1(M) \cong \mathbb{Z}$, an isometry $\theta : \lambda_M \cong \lambda$ and Kirby-Siebenmann invariant $ks(M) = k \in \mathbb{Z}_2$.

$$
\begin{tikzcd}
\underset{\cong\, \pi_2(M)}{H_2(M;\mathbb{Z}[\mathbb{Z}])} \times H_2(M;\mathbb{Z}[\mathbb{Z}]) \arrow[dr, "\lambda_M"] & \\
& \mathbb{Z}[\mathbb{Z}] \\
H \times H \arrow[uu, "\theta\ \cong"] \arrow[ur, "\lambda"'] &
\end{tikzcd}
$$

Kirby-Siebenmann invariant

Here $ks(M) \in H^4(M; \pi_3(TOP/O)) \cong \mathbb{Z}/2$

is the (high dim smoothing thy) obstruction to a smooth structure on $M \times \mathbb{R}$, or equivalently on $M \#^{\ell} S^2 \times S^2$ for some ℓ.

In dim d, $ks(M^d) = 0 \Rightarrow \begin{cases} M \text{ PL-able} & d \geq 5 \\ M \text{ smoothable} & d = 5, 6, 7. \end{cases}$

in dim 4, $ks(M)$ is an obstruction, not complete,

$TOP/O \leftarrow$ homotopy fibre of u.

ks arises from bundle lifting problem:

$$\begin{tikzcd} & BO \arrow[d, "u"] \\ M \arrow[ur, "?\exists?"] \arrow[r, dashed] & BTOP \end{tikzcd}$$

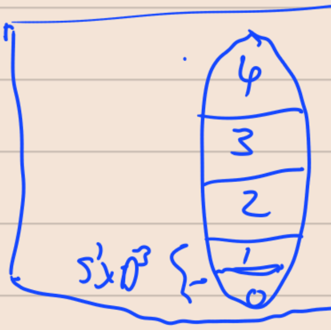
classifying map of universal bundle/BO thought of as a top bundle.

top tangent bundle classifying map.

Proof of Thm A

Start with $S^1 \times D^3$ |  $S^1 \times D^3$

$S^1 \times D^2 \subseteq D^2 \times D^2$

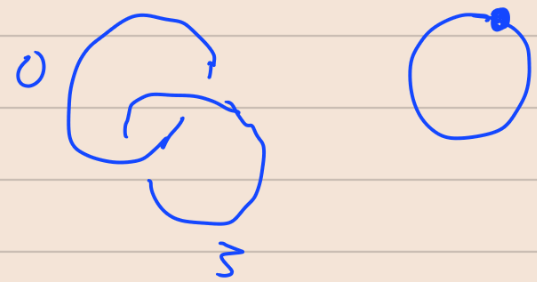


H smooth, handle decomp.

Add n 2-handles to $\partial(S^1 \times D^3) = S^1 \times S^2$ with framing-linking matrix $= \lambda(0)$

eg $\lambda = \begin{pmatrix} 0 & 1 \\ 1 & 3-t-t^{-1} \end{pmatrix}$

$n = 2$



1h

(0-surgery on this circle gives $S^1 \times S^2$)

Add clasps for $\pm t^k$ summands

2-h cores ∪ null-homotopies in D^4 give $H_2(W;\mathbb{Z}[\mathbb{Z}])$ generators

Get W, compact 4-mfld w. ∂, $\pi_1(W) = \mathbb{Z}$, $\lambda_W = \lambda$.

Want to fill in ∂W with $N \simeq S^1 \times D^3$.
(substitute for 3-4 handles)

Compute: ALL $\mathbb{Z}[\mathbb{Z}]$-coefficients.

$$\begin{array}{ccccccccccc}
H_3(W,\partial) & \to & H_2(\partial W) & \to & H_2(W) & \xrightarrow{j} & H_2(W,\partial) & \to & H_1(\partial W) & \to & H_1(W) \\
\| & & \| \text{ PD} & & & & \wr\| \text{ PD} & & & & \| \\
 & & H^1(\partial W) & & & & H^2(W) & & & & 0 \\
 & & \| \text{ UC} & & & & \wr\| \text{ UC} & & & & H_1 \text{ of uni cover.} \\
\mathbb{Z} & \to & \mathbb{Z} \oplus \operatorname{Hom}(H_1(\partial W), \mathbb{Z}[\mathbb{Z}]) & & & & \operatorname{Hom}(H_2(W), \mathbb{Z}[\mathbb{Z}]) & & & &
\end{array}$$

$\tilde{\lambda}$ $\cong$ $H_2(W) \to \operatorname{Hom}(H_2(W), \mathbb{Z}[\mathbb{Z}])$

λ nonsingular

j is an IM

$$\Rightarrow H_1(\partial W; \mathbb{Z}[\mathbb{Z}]) = 0.$$

ie ∂W is a $\mathbb{Z}[\mathbb{Z}]$-homology $S^1 \times S^2$.

$$H_k(S^1 \times S^2; \mathbb{Z}[\mathbb{Z}]) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{else.} \end{cases}$$

__Theorem__ (FQ) Application of surgery 1.

Let M be a closed, oriented 3-mfld.

Suppose $\exists$ HM $\pi_1(M) \xrightarrow{\varphi} \mathbb{Z}$

with $H_1(M; \mathbb{Z}[\mathbb{Z}]) = 0$.

Then $\exists$ $N^4 \simeq S^1$, COT 4-mfld with $\partial N = M$,

__Proof__ $\Omega_3^{fr}(B\mathbb{Z}) \cong \Omega_3^{fr} \oplus \Omega_2^{fr}$

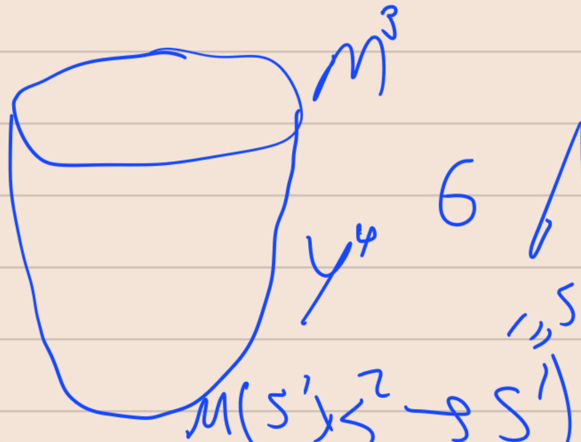
$$\Omega_3^{fr} \cong \mathbb{Z}/24, \qquad \Omega_2^{fr} \cong \mathbb{Z}/2 \text{ Arf}$$

can choose framing on M so this is 0.

Arf invariant detected by

$\operatorname{ord}(H_1(M;\mathbb{Z}[\mathbb{Z}]))(-1) \mod 8 = 0$

$\therefore \exists$ DONM



mapping cylinder

$$\in \mathcal{N}(M_\phi(M \xrightarrow{\phi} S^1), M)$$

$$X \simeq S^1$$

DONM = degree one normal map

This makes a target homotopy type to try to realise.

Poincaré pair (uses $H_1(M;\mathbb{Z}[\mathbb{Z}]) = 0$)

Surgery theory says (using that $\mathbb{Z}$ is a good group)

$$\mathcal{S}(X, M) \neq \emptyset \iff \sigma^{-1}(\{0\}) \neq \emptyset.$$

structure set

where σ is the surgery obstruction map:

$$\sigma : \mathcal{N}(X, M) \to L_4(\mathbb{Z}[\mathbb{Z}])$$

Best to ignore "normal" to this talk

DONMs up to normal bordism rel ∂.

$$\begin{array}{ccc} M & \to & Y \\ \downarrow & & \downarrow \\ M & \to & X \end{array}$$

Intersection pairing

nonsingular, Hermitian forms

$$\lambda \oplus^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \lambda' \oplus^l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Shaneson splitting

$$L_4(\mathbb{Z}[\mathbb{Z}]) \cong L_4(\mathbb{Z}) \oplus \underline{L_3(\mathbb{Z})} \cong L_4(\mathbb{Z}) \xrightarrow[\text{sign}]{\cong} 8\mathbb{Z}$$

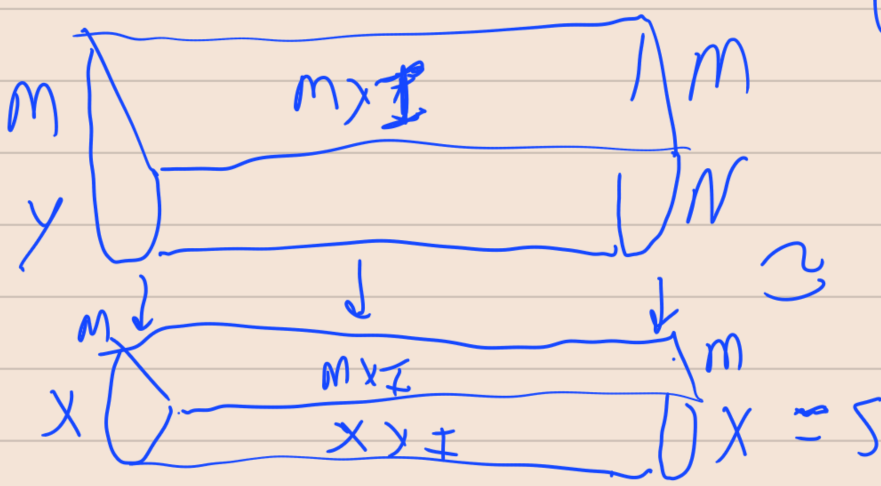
Theorem (Freedman)

$\exists$ closed 4-mfld Z with intersection form E_8, and $\therefore$ $\operatorname{sign}(Z) = 8$.

So $\sigma\left(Y \#^{|\ell|} -\left(\frac{\ell}{|\ell|}\right) Z \longrightarrow X\right) = 0 \in L_4$

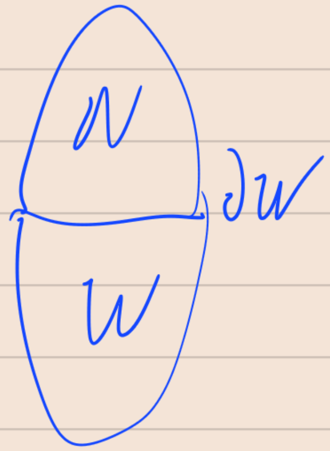
where $\ell := \operatorname{sign} \lambda_Y / 8$.

$\therefore$ $\exists$ normal bordism to a homotopy equivalence $(N, M) \xrightarrow{\simeq} (X, M)$ as desired



$\square$

Apply Thm with $M = \partial W$.



$W \cup_M -N$ is a ccot 4-mfld with $\pi_1 = \mathbb{Z}$ and $\lambda_M = \lambda$.

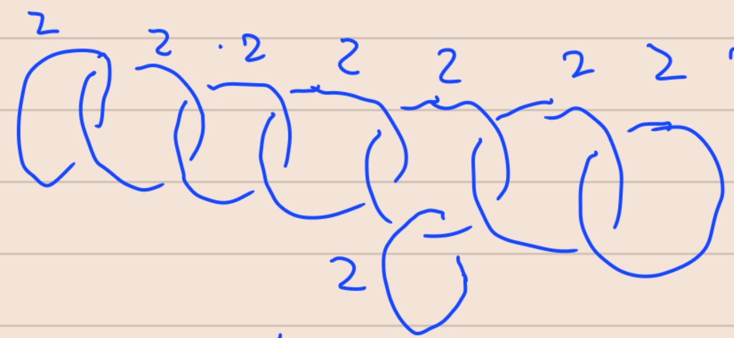
Aside: Apply with $M = S^3_0(K)$ where $K \subseteq S^3$, $\Delta_K \doteq 1$, to prove K is top slice.

Remains to realise KS for λ odd.

For λ even, Rochlin's thm $\Rightarrow$ $ks(M) = \frac{\operatorname{sign} \lambda \otimes \mathbb{R}}{2}$ (2).

Thm (Freedman)
Let Σ^3 be a $\mathbb{Z}HS^3$. Then $\exists$ contractible, compact, top 4-mfld V with $\partial V = \Sigma$.

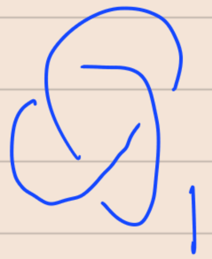
In fact, this was used to show $\exists$ E_8 mfld Z above.

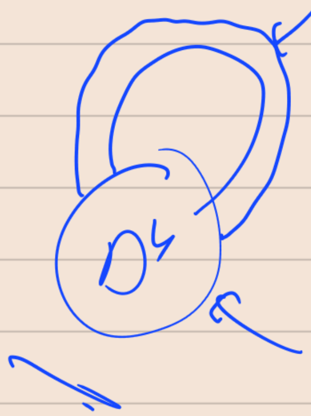
$\partial\Big($  $\Big) = $ Pé hom S^3.

$\overset{\shortparallel}{W} = D^4 \cup 8$ 2-handles
$D^2 \times D^2$

Z 

Construct $*\mathbb{CP}^2$





$\cup\ V$

$\partial = \mathbb{Z}HS^3$

$= *\mathbb{CP}^2$

$\not\simeq \mathbb{CP}^2$.

(not homeomorphic)

$ks(*\mathbb{CP}^2) = 1$.

For comparison

$\cup D^4$

$= \mathbb{CP}^2$

Given M with $\pi_1(M) \cong \mathbb{Z}$, $\lambda_M \cong \lambda$, λ odd.

$M \# *\mathbb{CP}^2$

$\uparrow f$ representing $1 \in \mathbb{Z} \cong \pi_2(*\mathbb{CP}^2)$

S^2

M odd $\Rightarrow$ $f \underset{\text{reg hom}}{\sim}$ embedding

sphere embedding thm (FQ, Stong)

$\cong \underbrace{*M}_{\text{det}} \# \mathbb{CP}^2$

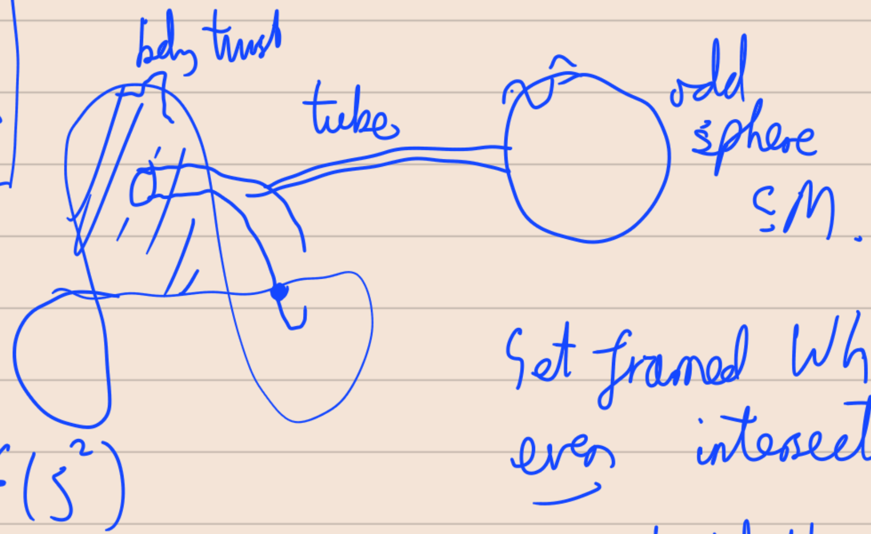
$\lambda_{*M} \cong \lambda_M$

$ks(*M) \neq ks(M)$ by additivity of ks. $\square$

Theorem A

SKIP!

SET explanation



Get framed Wh disc with even intersections with $f(S^2)$

$\Rightarrow$ embeddable by a regular homotopy.

much more work

Theorem B (FQ)

Let M, N be CCOT 4-manifolds

$\pi_1(M) \cong \mathbb{Z} \cong \pi_1(N)$ $\quad$ $ks(M) = ks(N)$

$\in \mathbb{Z}/2$.

Let $h : \underset{\underset{\pi_2(M)}{\text{\rotatebox{90}{$\cong$}}}}{H_2(M;\mathbb{Z}[\mathbb{Z}])} \longrightarrow H_2(N;\mathbb{Z}[\mathbb{Z}])$

be an isometry of the intersection forms

$h : \lambda_M \xrightarrow{\cong} \lambda_N$.

Then $\exists$ an orientation-preserving homeomorphism $f : M \xrightarrow{\cong} N$ inducing h.

In particular, $\exists$ exactly 2/1 4-mfds with $\pi_1 \cong \mathbb{Z}$ and given λ; if λ is odd/even up to homeomorphism.

Proof via surgery sequence

(Note $Wh(\mathbb{Z}[\mathbb{Z}]) = 0 \Rightarrow$ simple surgery automatically)

$$\begin{array}{ccccccccc}
\text{relative DONMs} & & \text{algebra} & & \text{structure set} & & \text{DONMs} & & \text{algebra} \\
\mathcal{N}(N\times I, \partial) & \to & L_5(\mathbb{Z}[\mathbb{Z}]) & \to & \mathcal{S}(N) & \to & \mathcal{N}(N) & \to & L_4(\mathbb{Z}[\mathbb{Z}]) \\
& & \| & & & & \cong & & \cap \\
& & L_5(\mathbb{Z}) \oplus L_4(\mathbb{Z}) & & & & [N, G/TOP] & & 8\mathbb{Z} \\
& & \| & & & & \cong & & \\
& & L_4(\mathbb{Z}) & & & & H^2(N;\mathbb{Z}/2) \oplus H^4(N;\mathbb{Z}) \cong \mathbb{Z}/2^n \oplus \mathbb{Z} & & \\
& & \cong & & & & & & \\
& & 8\mathbb{Z} & & & & & &
\end{array}$$

$0 \mapsto 8\mathbb{Z}$, $\cong$

$\Downarrow$ $\quad\quad$ $\Downarrow$

$N \times I \;\#_{S^1}\; \mathbb{Z}\times S^1 \longmapsto 8$

conn sum along 1-skeleton

$\Rightarrow L_5$ action on $\mathcal{S}(N)$ is trivial

$\exists\ G/TOP \to K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4)$ 5-equivalence

So $\mathcal{S}(N) \cong (\mathbb{Z}/2)^n$.

How big is it after modding out by $hAut(N)$?

Prop Let M, N as in Theorem B.

Then $\exists\ f: M \to N$ homotopy equivalence inducing $h: H_2(M;\mathbb{Z}[\mathbb{Z}]) \to H_2(N;\mathbb{Z}[\mathbb{Z}])$.

Will show f is homotopic to a homeomorphism.

Note:

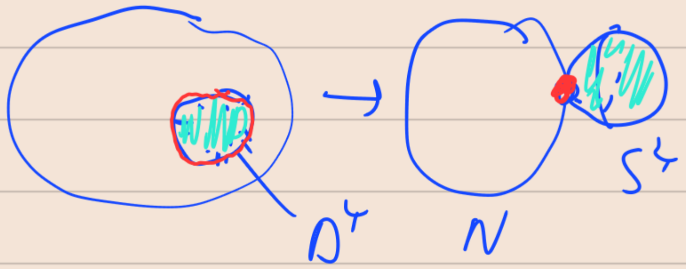
$(M, f) \in \mathcal{S}(N)$

Given $x \in \pi_2(N)$ with $x \cdot x \equiv 0 \ (2)$

$x \neq 0 \in H_2(N; \mathbb{Z}_2)$

there is a homotopy equivalence.

$$\Theta_x : N \longrightarrow N \vee S^4 \xrightarrow{(\mathrm{Id} \vee \eta \circ \Sigma \eta)} N \vee S^2 \xrightarrow[\mathrm{Id} \vee x]{} N$$



η Hopf map

$\eta \circ \Sigma \eta : S^4 \to S^3 \to S^2$

generator of $\pi_4(S^2) \cong \mathbb{Z}/_2$.

Θ_x Induces Id on $H_2(N; \mathbb{Z}[\mathbb{Z}])$

<u>and</u>

$$\begin{tikzcd} M \arrow[r, "f", "\simeq"'] \arrow[rr, bend right, "f'"'] & N \arrow[r, "\Theta_x", "\simeq"'] & N \end{tikzcd}$$

changes image in $\mathcal{N}(N)$ under $\mathcal{S}(N) \to \mathcal{N}(N)$

by dual to x in $H^2(N; \mathbb{Z}_2)$.

If λ even, can kill all classes.

$\rightsquigarrow$ $[f' : M \xrightarrow{\simeq} N] = [\mathrm{Id} : N \to N]$

$\in \mathcal{S}(N)$

Note
$\pi_2(N) \twoheadrightarrow H_2(N; \mathbb{Z}_2)$

i.e. $\exists$ homeo g

$$\begin{tikzcd} M \arrow[rr, "g", "\cong"'] \arrow[dr, "f'"'] & \arrow[d, phantom, "\simeq"] & N \arrow[dl, "\mathrm{Id}"] \\ & N & \end{tikzcd}$$

so f' is homotopic to a homeomorphism.

i.e. $$\mathcal{S}(N) / \mathrm{hAut}(N) = \{N\}.$$

If λ odd, one class in $H^2(N; \mathbb{Z}_2)$ cannot be killed. But $\exists \geq 2$ manifolds in the homotopy class by realisation of KS from thm A. $\therefore$

$$\mathcal{S}(N) / \mathrm{hAut}(N) = \{M, *M\}.$$

In thm, we assumed $ks(M) = ks(N)$, so again $f' \simeq$ homeo. $\square$

Remark

Conway - Powell extended $\pi_1 \cong \mathbb{Z}$ classification to 4-mflds with boundary provided $H_1(\partial M; \mathbb{Q}(t)) = 0$.

and $\pi_1(\partial M) \twoheadrightarrow \pi_1(M) \cong \mathbb{Z}$.