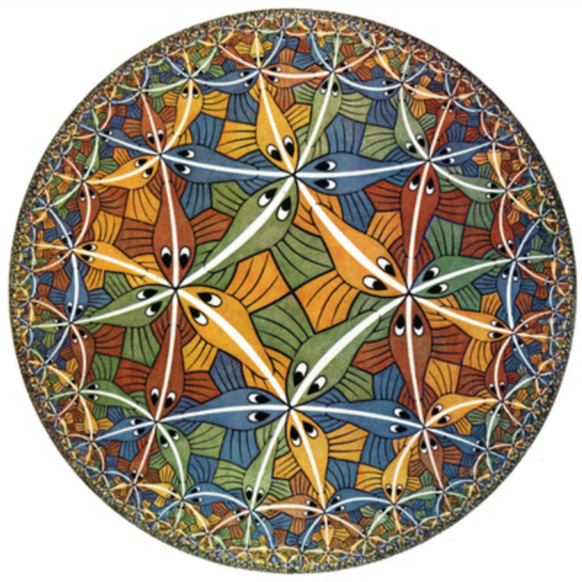

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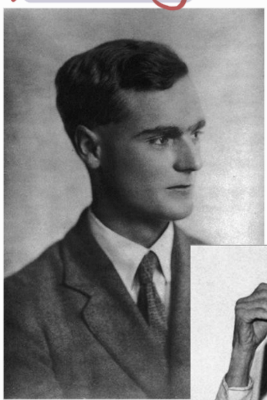
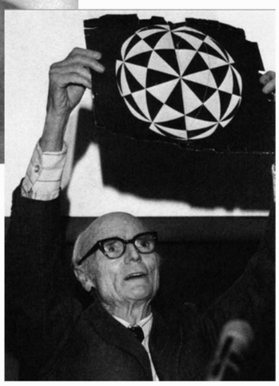
Circle Limit III

Coxeter and Art... in groups

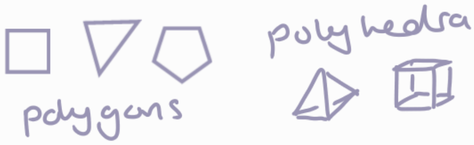
History





Harold Scott Macdonald Coxeter (1907-2003)

- Son of a sculptor and a painter
- prize essay on "dimensional analogy"
- defined regular polytopes



"Thus the chief reason for studying regular polyhedra is still the same as in the time of the Pythagoreans, namely, that their symmetrical shapes appeal to one's artistic sense."

Coxeter, Regular Polytopes

Coxeter Groups

Given
- finite generating set S
- $\forall\ s,t \in S$ $\quad m_{st} = m_{ts}$ an integer $\geqslant 2$ or $m_{st} = \infty$ $\quad s \neq t$

then $$W = \left\langle S \;\middle|\; \begin{matrix} (st)^{m_{st}} = e & \forall\, s \neq t \in S \\ s^2 = e & \forall\, s \in S \end{matrix} \right\rangle$$ **Coxeter group**,

and (W,S) called a Coxeter system.

Can package (W,S) information into a labelled graph Γ with
- vertex set S
- edges

$m_{st} = 2$: $(st)^2 = e$, $st = ts$

$m_{st} = 3$: $(st)^3 = e$, $sts = tst$

$m_{st} > 3$ or ∞: $\underbrace{ststs\ldots}_{m_{st}} = \underbrace{tstst\ldots}_{m_{st}}$

Given such a Γ denote Coxeter group by W_Γ

Remark: Full subgraphs <~> subgroups

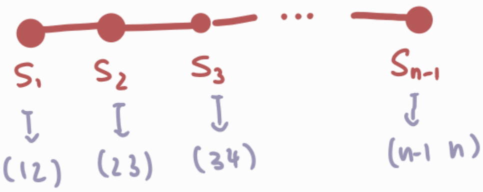
Γ_T $\quad T \subseteq S$ and edges spanned $\qquad W_{\Gamma_T}$ subgp of W_Γ - special subgroup

Examples

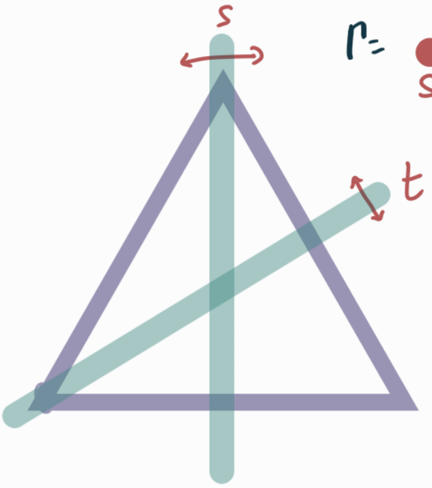
Γ: 

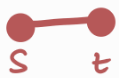
$W_\Gamma = \langle s,t,u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (st)^2 = e \rangle$

$W_\Gamma \simeq S_4$ via $s \mapsto (12)$ $t \mapsto (23)$ $u \mapsto (34)$

In general, $\Gamma =$ 

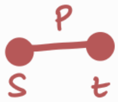
gives $W_\Gamma \cong S_n$



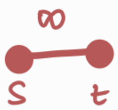
$\Gamma =$ 

$m_{st} = 3$ $W_\Gamma = \langle s,t \mid s^2 = t^2 = (st)^3 = e \rangle$

$W_\Gamma \simeq D_3$

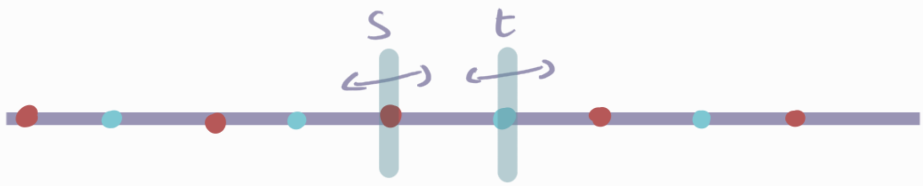
In general $\Gamma =$ 

gives $W \cong D_p$

Γ: 

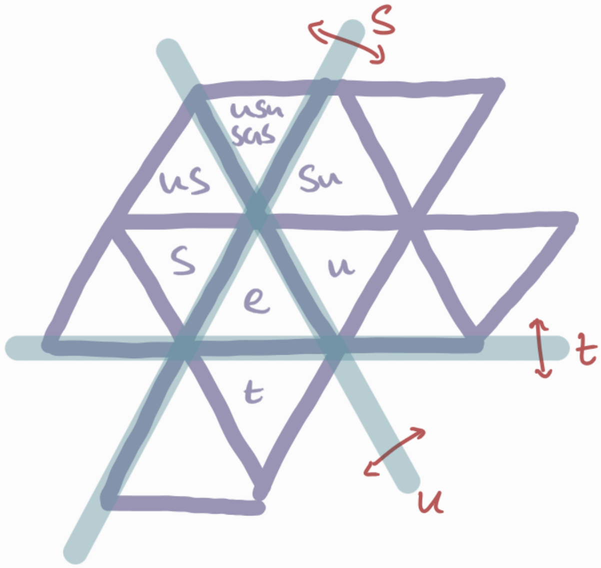
$m_{st} = \infty$

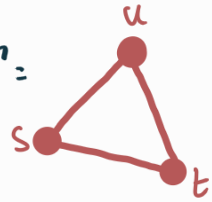
$W_\Gamma \cong D_\infty$



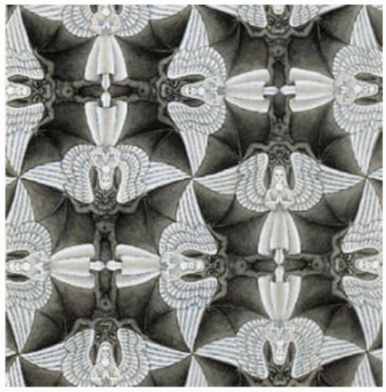
st = translation by 1 to the right

Euclidean Tilings



$\Gamma =$ 

$$W_\Gamma = \langle s, t, u \mid s^2 = t^2 = u^2 = (su)^3 = (st)^3 = (tu)^3 = e \rangle$$



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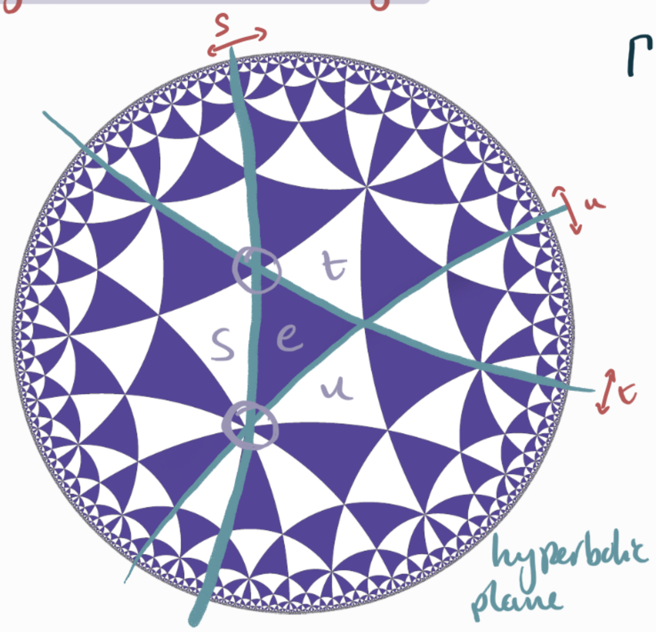
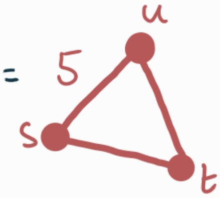
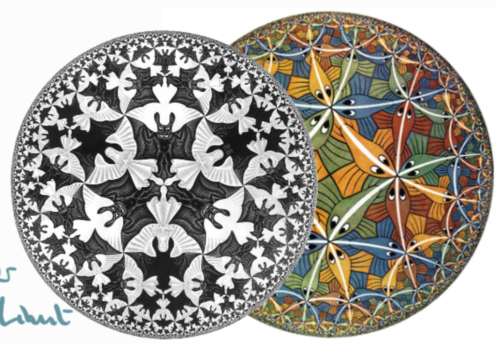
Hyperbolic Tilings



$\Gamma =$ 

$m_{st} = m_{tu} = 3 \quad m_{su} = 5$

$$W_\Gamma = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^5 = e \rangle$$



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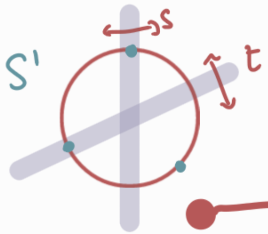
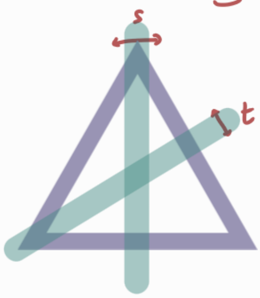
Circle Limit

Spherical Tilings

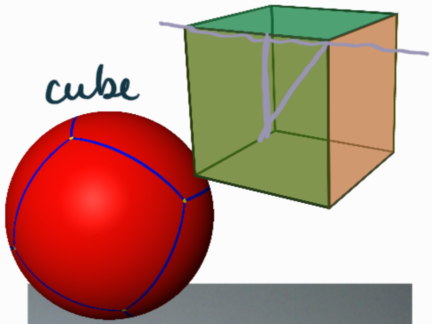
Two-dimensional

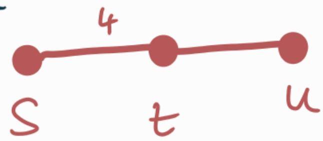




Three-dimensional

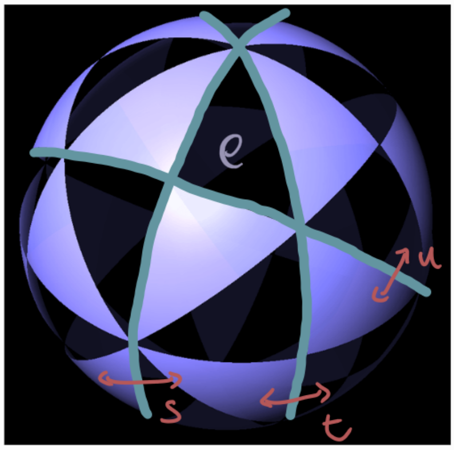
cube



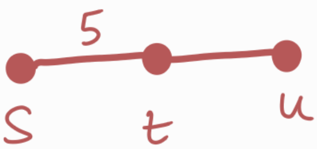


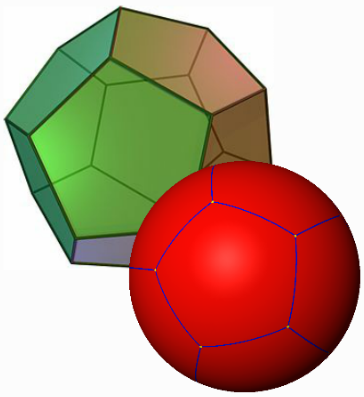
Sym gp of 3cube

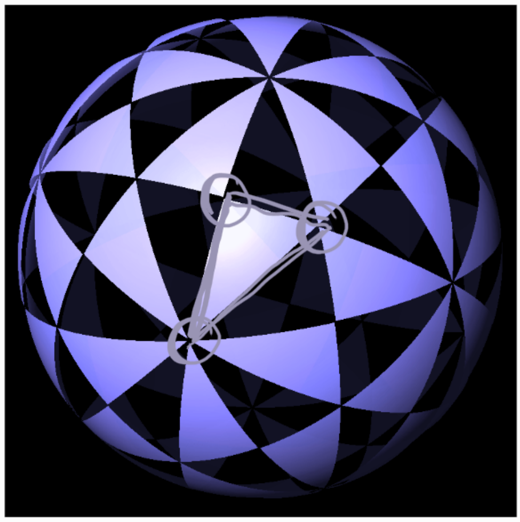
Escher



Dodecahedron

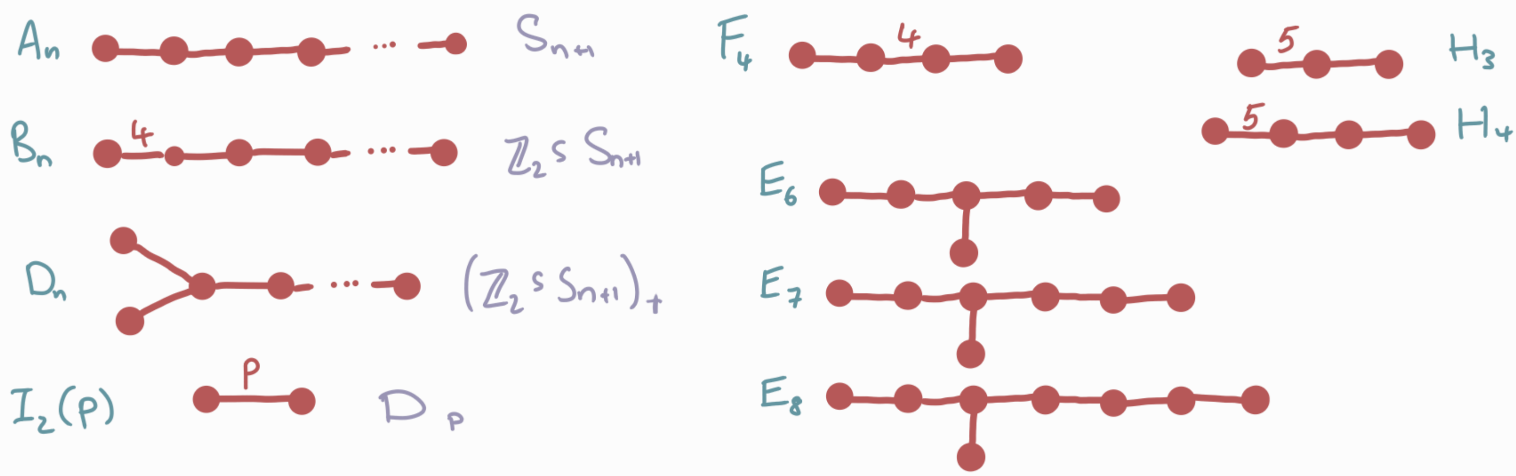




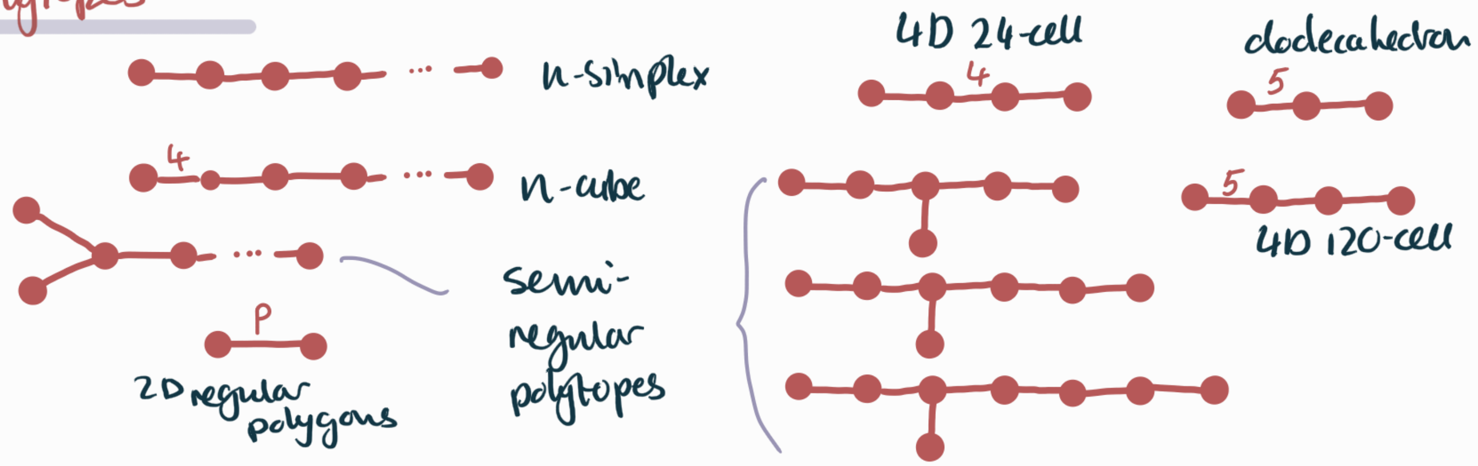


Classification of finite Coxeter groups (Coxeter 1935)

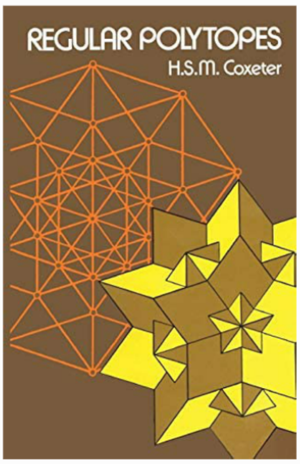
W_Γ is finite $\iff$ Γ is a disjoint union of one or more of the following:



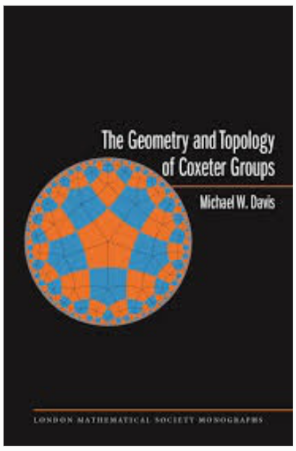
Polytopes



Alicia Boole Stott collaborated with Coxeter to visualise 4 dimensional polytopes



← for more information →



Artin Groups

$$W_\Gamma = \left\langle S \;\middle|\; \begin{array}{ll} (st)^{m_{st}} = e & \forall s \neq t \in S \\ s^2 = e & \forall s \in S \end{array} \right\rangle$$

$$W_\Gamma = \left\langle S \;\middle|\; \begin{array}{ll} \underbrace{stst\ldots}_{m_{st}} = \underbrace{tsts\ldots}_{m_{st}} & \forall s \neq t \in S \\ s^2 = e & \forall s \in S \end{array} \right\rangle$$

Emil Artin (1898 - 1962)

$$A_\Gamma = \left\langle S \;\middle|\; \underbrace{stst\ldots}_{m_{st}} = \underbrace{tst\ldots}_{m_{st}} \quad \forall s \neq t \in S \right\rangle$$

A_Γ is Artin group of Γ introduced by Brieskorn 1971

Example

$\Gamma =$ 

defines $A_\Gamma \cong Br_n$

$s_i \longmapsto \sigma_i$ 

$$Br_n = \left\langle \sigma_i \;\; 1 \leq i \leq n-1 \;\middle|\; \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & 1 \leq i \leq n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| \geq 2 \end{array} \right\rangle$$

"It is a tribute to Artin's extraordinary insight as a mathematician that the definition he proposed in 1925 for equivalence of geometric braids could ultimately be broadened and generalised in many different directions without destroying the essential features of the theory."

Joan Birman, Braids, links and Mapping Class groups

From A_Γ to W_Γ

For any Γ there exists a SES

$$0 \longrightarrow PA_\Gamma \hookrightarrow A_\Gamma \xrightarrow{s^2 \mapsto e} W_\Gamma \longrightarrow 0$$

$\uparrow$ Pure Artin group

$\uparrow$ $\exists !$ element of longest length if W_Γ finite

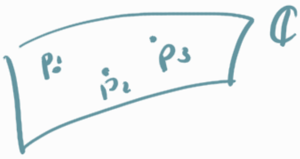
If W_Γ is finite then A_Γ is called finite type

– Garside element (1960s)

From W_Γ to A_Γ

$W_\Gamma \cong S_n \qquad A_\Gamma \cong Br_n$

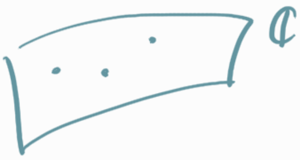
$\Gamma =$ 



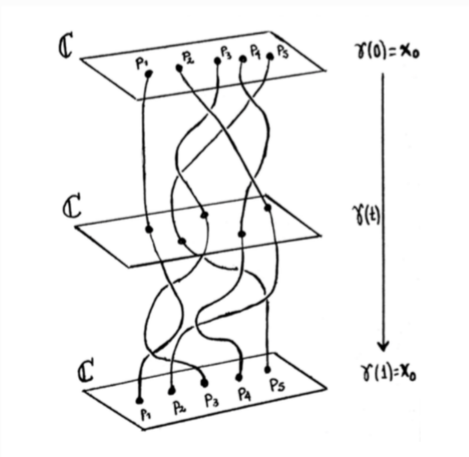
$W_\Gamma = S_n$ acts on $\mathrm{Conf}_n(\mathbb{C})$

$$= \mathbb{C}^n \setminus \bigcup_{1\le i<j\le n} H_{ij} \qquad H_{ij} = \{(z_1, \dots, z_n) \mid z_i = z_j\}$$

$$\mathrm{Conf}_n(\mathbb{C}) / S_n = U\mathrm{Conf}_n(\mathbb{C})$$



$$A_\Gamma = Br_n \cong \pi_1(U\mathrm{Conf}_n(\mathbb{C}))$$



In general, $A_\Gamma = \pi_1(\mathcal{M}_\Gamma / W_\Gamma)$

'hyperplane complement'

For general Γ, $\mathcal{M}_\Gamma$ built from 'Tits cone' for (W,S)

see: Paris "$K(\pi,1)$ conjecture for Artin groups"

$K(\pi,1)$ conjecture: $\mathcal{M}_\Gamma / W_\Gamma$ is a $K(A_\Gamma, 1)$ or BA_Γ.

Arnol'd, Brieskorn, Thom, Pham

Families of Artin Groups

Many open questions for Artin groups in general:

	word problem?	irreducible center?	torsion free?	nice $K(\pi,1)$?
Solution:	Yes	finite type $\mathbb{Z}$ infinite type $\{1\}$	Yes	Yes eg. Sal_Γ

These questions are answered for the following families:

- finite type
- large type – all $m_{st} \geq 3$
- FC type
- Right angled Artin groups (RAAGs) – all m_{st} either 2 OR ∞

RAAGs Introduced by Baudisch 1981 and studied by Droms in late 80s – called 'graph groups'

RAAGs

- All relations are commuting or free

⚠ Given unlabelled graph Γ, define associated RAAG ⚠

$$A_\Gamma = \langle s \in V(\Gamma) \mid st = ts \text{ if } (s,t) \in E(\Gamma) \rangle$$

Edges are $m_{s,t} = 2$!

Eg.

$A_\Gamma = \mathbb{Z}^5$

complete graph $\leftrightsquigarrow \mathbb{Z}^n$

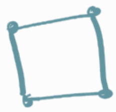
$A_\Gamma = F_5$

no edges $\leftrightsquigarrow F_n$

$A_\Gamma = \mathbb{Z}^2 * \mathbb{Z}^3$

L =

$A_\Gamma = \pi_1(\mathbb{R}^3 \setminus L)$



$A_\Gamma = F_2 \times F_2$

The Salvetti Complex

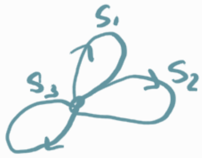
Introduced by Salvetti for finite type in 1987

finite dimensional cw complex $\rightarrow$ $Sal_\Gamma \simeq M_\Gamma / W_\Gamma$

$K(\pi,1)$ conj : $\widetilde{Sal_\Gamma} \simeq *$

For RAAGs build Sal_Γ by...

1. Wedge of circles labeled by generators
2. Attach two-torus for each edge
3. Attach 3-torus for each triangle
4. Attach k-torus for each k-clique



$\partial = sts^{-1}t^{-1}$

The $K(\pi,1)$ Conjecture for RAAGs

$CAT(0)$: triangles are at least as thin as comparisons in $\mathbb{E}^2$

Theorem (Charney - Davis 95) Γ graph A_Γ RAAG

$\widetilde{Sal}_\Gamma$ is a $CAT(0)$ cube complex.

"non - +ve" curvature cubes 'glued together' + cubical metric

Corollary Sal_Γ is a $K(A_\Gamma,1)$. $K(\pi,1)$ conjecture for RAAGs

Main idea in proof

Theorem (Gromov 87) Cube complex K

K is $CAT(0)$ $\Leftrightarrow$ the link of every vertex is flag.

References

Introduction to RAAGs by Ruth Charney
http://people.brandeis.edu/~charney/papers/RAAGfinal.pdf

The K(\pi,1) conjecture for Artin groups by Luis Paris
https://arxiv.org/abs/1211.7339

Problems related to Artin groups by Ruth Charney
http://people.brandeis.edu/~charney/papers/Artin_probs.pdf

Make hyperbolic tilings yourself!
http://www.malinc.se/m/ImageTiling.php

Escher and Coxeter article by Sarah Hart (also lecture on youtube)
https://brewminate.com/escher-and-coxeter-a-mathematical-conversation/