Introduction to The Cobordism Hypothesis

1. What are topological field theories?
2. Symmetric Monoidal Categories & the Cobordism Hypothesis in 1d
3. Symmetric Monoidal Bicategories & the Cobordism Hypothesis in 2d

1. What are Topological Field Theories?

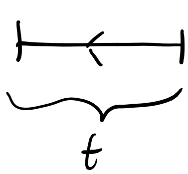
What is quantum mechanics?

- a complex Hilbert space $\mathcal{H}$ of <u>states</u>.
- a self-adjoint operator H on $\mathcal{H}$, the <u>Hamiltonian</u>.
- given a state $\psi \in \mathcal{H}$ at $t=0$,
  the time-evolution is determined by
  the <u>Schrödinger equation</u>

$$i\hbar \frac{d\psi}{dt} = H\psi \implies \psi(t) = e^{-\frac{i}{\hbar}tH}\psi$$

In pictures:

$$\mathrm{pt} \longmapsto \mathcal{H}$$



$$\longmapsto e^{-\frac{it}{\hbar}H} : \mathcal{H} \to \mathcal{H}$$

<u>Idea</u> Quantum mechanics

$$\|$$

1-dimensional quantum field theory

End of 80's:

Witten: [88, '89] There exist interesting QFTs with trivial time propagation

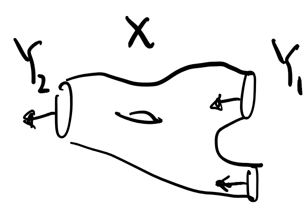
Atiyah: ['89] A d-dimensional topological quantum field theory (TQFT) is an assignment

"space": (d-1)-dim closed oriented manifold

$$Z(Y^{d-1}) \in \mathrm{Vect}_{\mathbb{C}} \quad \text{"state space"}$$

$$Z(X^d : Y_1^{d-1} \rightsquigarrow Y_2^{d-1}) : Z(Y_1) \to Z(Y_2) \quad \mathbb{C}\text{-linear map}$$

"Spacetime": d-dim oriented bordism



with the following properties:

1. $Z(X_1) = Z(X_2)$ if X_1 & X_2 are diffeomorphic relative boundary

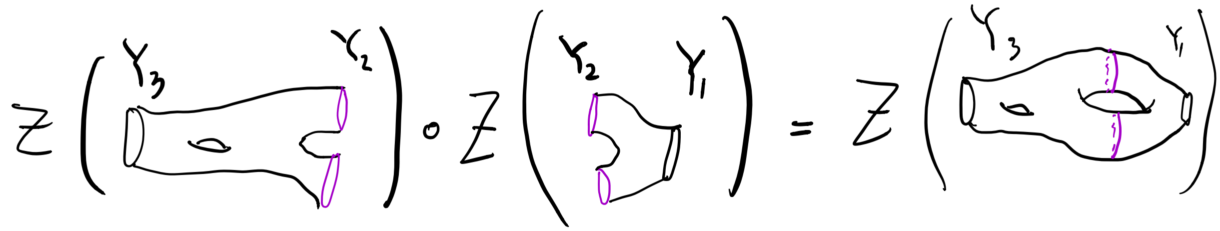
2. $Z(Y \times [0,1]) : Z(Y) \to Z(Y)$ is the identity

3. $Z(Y_1 \sqcup Y_2) \cong Z(Y_1) \otimes Z(Y_2)$

4. $Z(X_1 \sqcup X_2) = Z(X_1) \otimes Z(X_2)$

5. $Z(\phi) \cong \mathbb{C}$

6. Linear maps compose under gluing bordisms



<u>Idea</u> Z is a representation of the bordism category

$\Rightarrow$ TQFTs help us to understand manifolds.

e.g. a closed manifold

$$X^d : \phi \rightsquigarrow \phi$$

is a bordism from the empty manifold to itself.

$\Rightarrow Z(X^d) : \mathbb{C} \to \mathbb{C}$ gives a number $\in \mathbb{C}$

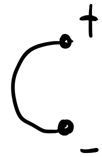
$\Rightarrow$ smooth manifold invariant.

<u>Example</u> given a finite-dimensional vector space V

We can make a one-dimensional TQFT:

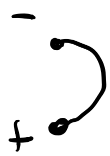
$\bullet^{+} \mapsto V$

$\bullet^{-} \mapsto V^{*}$



$\mapsto \mathrm{ev} : V \otimes V^{*} \to \mathbb{C}$

$v \otimes \xi \mapsto \xi(v)$

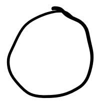


$\mapsto \mathrm{coev} : \mathbb{C} \to \mathrm{End}\, V \cong V^{*} \otimes V$

$1 \mapsto \mathrm{id}_V \mapsto \sum_i \epsilon^i \otimes e_i$

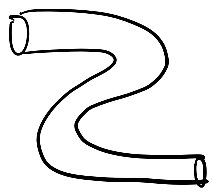
where $\{e_1, \dots, e_n\}$ is a basis of V

with dual basis $\{\epsilon^1, \dots, \epsilon^n\} \subseteq V^*$



$\mapsto \mathrm{tr}\, \mathrm{id}_V = \dim V = n$

<u>Zorro's lemma</u> Let Y^{d-1} oriented closed. Then



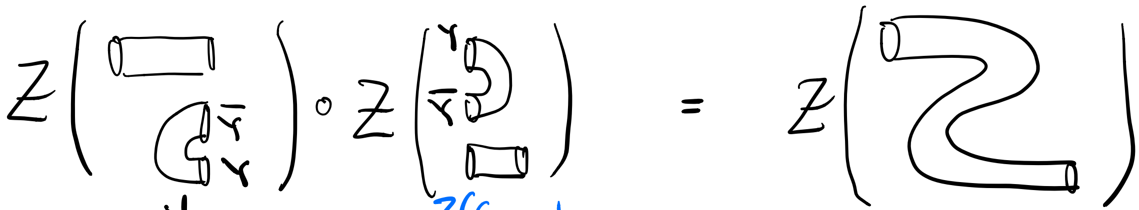
Y

$\cong Y \times [0,1]$

<u>Corollary</u> $Z(Y)$ is finite-dimensional

Proof Let $\bar{Y}$ denote the orientation-reversal.

Chop up the first bordism into two pieces



$$\begin{tikzcd}
Z(Y) & Z(Y) \arrow[l, "\mathrm{id}_Y"'] & \\
& \otimes & \mathbb{C} \arrow[ul, "Z(\mathrm{coev})"'] \arrow[dl] \\
& Z(\bar{Y}) \arrow[dl, "Z(\mathrm{ev})"'] & \\
\mathbb{C} & \otimes & \\
& Z(Y) \arrow[ul] & Z(Y) \arrow[l, "\mathrm{id}_Y"]
\end{tikzcd}
\quad = \quad Z(Y) \xleftarrow{\mathrm{id}_Y} Z(Y)$$

Write

$$Z(\mathrm{coev})(1) = \sum_{i=1}^{n} v_i \otimes w_i \quad \in Z(Y) \otimes Z(\bar{Y})$$

Then

$$\sum_{i=1}^{n} v_i \, Z(\mathrm{ev})(w_i \otimes v) = v$$

So $Z(Y)$ is spanned by $\{v_1, \dots, v_n\}$ $\square$

In Fact $Z(\mathrm{ev})$ is a non-degenerate bilinear pairing giving a canonical isomorphism

$$Z(\bar{Y}) \cong Z(Y)^*$$

with the linear dual.

2. Symmetric Monoidal Categories

Def A <u>monoidal category</u> is a category $\mathcal{C}$ together with

1. a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
2. an <u>associativity</u> natural isomorphism
   $$\alpha \quad (-\otimes -)\otimes - \;\cong\; -\otimes(-\otimes -)$$
   of functors $\mathcal{C}\times\mathcal{C}\times\mathcal{C}\to\mathcal{C}$
3. a <u>unit object</u> $\mathbb{1} \in \mathrm{Obj}\,\mathcal{C}$
4. <u>Unitor</u> natural isomorphisms
   $$\mathbb{1}\otimes(-) \overset{\lambda}{\cong} (-) \overset{\rho}{\cong} (-)\otimes\mathbb{1}$$

such that

a) the <u>triangle axiom</u>

$$\begin{tikzcd}
(c_1\otimes\mathbb{1})\otimes c_2 \arrow[rr, "\alpha_{c_1,\mathbb{1},c_2}"] \arrow[dr, "\rho_{c_1}\otimes \mathrm{id}_{c_2}"'] & & c_1\otimes(\mathbb{1}\otimes c_2) \arrow[dl, "\mathrm{id}_{c_1}\otimes\lambda_{c_2}"] \\
& c_1\otimes c_2 &
\end{tikzcd}$$

b) the <u>pentagon axiom</u>

<u>Def</u> A monoidal category is <u>braided</u> if it comes equipped with a natural isomorphism

$$\beta : \otimes \to \otimes \circ \tau \qquad \text{called a } \underline{\text{braiding}}$$

where $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the flip functor

$$\tau(c_1, c_2) = (c_2, c_1).$$

$\left(\text{So } \beta_{c_1, c_2} : c_1 \otimes c_2 \xrightarrow{\sim} c_2 \otimes c_1 \right)$

such that a hexagon diagram commutes.

A <u>symmetric monoidal category</u> is a braided monoidal category such that

$$\beta_{c_2, c_1} \circ \beta_{c_1, c_2} = id_{c_1 \otimes c_2}$$

<u>Examples</u>

- $\mathcal{C} = (Vect_{\mathbb{C}}, \otimes)$
- $\mathcal{C} = (Rep_{\mathbb{C}}(G), \otimes)$
- $\mathcal{C} = (Set, \times)$
- $\mathcal{C} = (Vect_{\mathbb{C}}, \oplus)$
- $\mathcal{C} = \mathcal{B}ord_{d,d-1}$ with

objects: Y^{d-1} oriented closed manifold

morphisms: Diffeomorphism classes rel ∂ of $X^d: Y_1 \rightsquigarrow Y_2$ oriented bordisms

$\otimes := \amalg$

Def A symmetric monoidal functor

$$F: \mathcal{C}_1 \to \mathcal{C}_2$$

is a functor between symmetric monoidal categories together with

1. an isomorphism $\mathbb{1}_{\mathcal{C}_2} \cong F(\mathbb{1}_{\mathcal{C}_1})$
2. a natural isomorphism

$$F(-) \otimes F(-) \cong F(- \otimes -)$$

such that F is unital, "associative" & intertwines the braidings.

Note A TQFT $\Longleftrightarrow$ a symmetric monoidal functor
$$Z : \mathrm{Bord}_{d,d-1} \to \mathrm{Vect}_{\mathbb{C}}$$

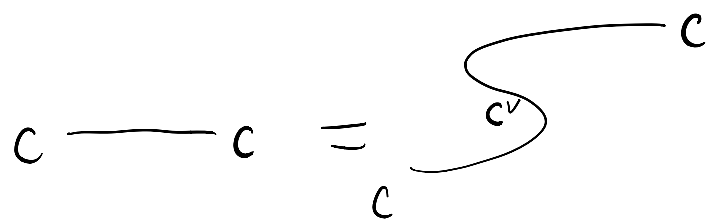
We can replace the target category $\mathrm{Vect}_{\mathbb{C}}$ by any symmetric monoidal category.
What will the finite-dimensionality condition be replaced with?

Zorro's diagram

gave us

$$
\begin{tikzcd}
C^\vee & C^\vee \arrow[l] & \\
\otimes & \otimes & \mathbb{1} \arrow[ul, bend right] \arrow[dl, bend left, "coev_C"'] \\
 & C & \otimes \\
\mathbb{1} & \otimes \arrow[l, bend left, "ev_C"'] & \\
 & C^\vee & C^\vee \arrow[l]
\end{tikzcd}
\quad = \quad id_C
$$

the mirrored diagram (Superman's lemma?)



can give more information in general.

<u>Def</u> A <u>dual</u> of an object $c \in \mathcal{C}$ in a symmetric monoidal category is an object $c^\vee \in \mathcal{C}$ together with morphisms

$$ev_c : c \otimes c^\vee \to \mathbb{1}, \quad coev_c : \mathbb{1} \to c^\vee \otimes c$$

such that

$$c^\vee \cong \mathbb{1} \otimes c^\vee \xrightarrow{coev_c \otimes id_{c^\vee}} c^\vee \otimes c \otimes c^\vee \xrightarrow{id_{c^\vee} \otimes ev_c} c^\vee \otimes \mathbb{1} \cong c^\vee$$

$$c \cong c \otimes \mathbb{1} \xrightarrow{id_c \otimes coev_c} c \otimes c^\vee \otimes c \xrightarrow{ev_c \otimes id_c} \mathbb{1} \otimes c \cong c$$

are identities.

<u>Remarks</u> 1. If $c \in \mathcal{C}$ has a dual it is unique up to isomorphism.

2. In a symmetric monoidal category, c is a dual of $c^\vee$

3. We didn't use the braiding to define duals. In a general monoidal category one gets a different notion of dual (right v.s left dual) by replacing c & $c^\vee$.

<u>Cobordism hypothesis for 1-categories</u>:

The category of 1d TQFTs

$$Z: \mathrm{Bord}_{1,0} \to \mathcal{C}$$

is equivalent to

$$(\mathcal{C}^{\text{dualizable}})^\sim$$

the core (the largest subgroupoid) of the full subcategory of dualizable objects of $\mathcal{C}$.

3. Symmetric Monoidal Bicategories

"**Def**" A bicategory $\mathcal{B}$ is a gadget with

objects $c \in \mathcal{B}$

1-morphisms $c_1 \xrightarrow{f} c_2$ composing as usual

2-morphisms
$$\begin{tikzcd} c_1 \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\varphi"] & c_2 \end{tikzcd}$$
between 1-morphisms

which can be composed vertically

$$\begin{tikzcd}[row sep=large, column sep=large] c_1 \arrow[r, bend left=60, "f_1"] \arrow[r, "f_2" description] \arrow[r, bend right=60, "f_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\varphi_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow\varphi_2"] & c_2 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd}[column sep=large] c_1 \arrow[r, bend left=60, "f_1"] \arrow[r, bend right=60, "f_3"'] \arrow[r, phantom, "\Downarrow\varphi_2 \circ_v \varphi_1"] & c_2 \end{tikzcd}$$

but also horizontally

$$\begin{tikzcd} c_1 \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\varphi"] & c_2 \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\psi"] & c_3 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd}[column sep=large] c_1 \arrow[r, bend left, "g_1 \circ f_1"] \arrow[r, bend right, "g_2 \circ f_2"'] \arrow[r, phantom, "\Downarrow\psi \circ_h \varphi"] & c_3 \end{tikzcd}$$

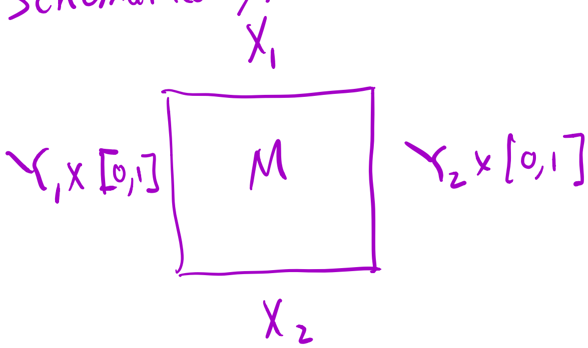
Example The bordism bicategory $\mathrm{Bord}_{d,d-1,d-2}$ has

Objects: Y^{d-2} closed oriented manifolds

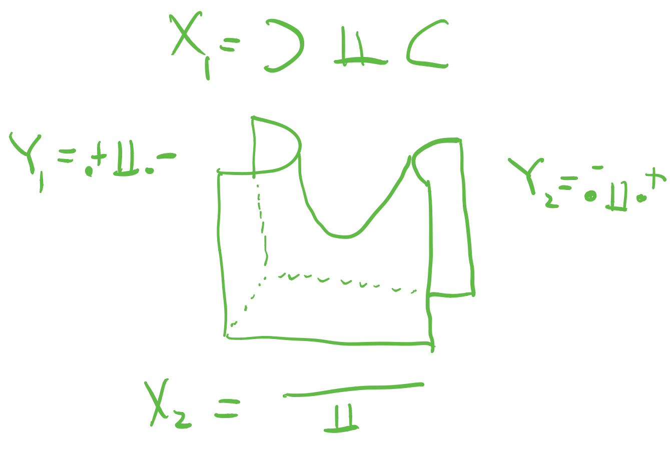
1-morphisms: $X^{d-1} : Y_1^{d-2} \rightsquigarrow Y_2^{d-2}$ oriented bordisms

2-morphisms:
$$\begin{tikzcd} Y_1 \arrow[r, bend left, squiggly, "X_1"] \arrow[r, bend right, squiggly, "X_2"'] \arrow[r, phantom, "\Downarrow M^d"] & Y_2 \end{tikzcd}$$
oriented bordisms with corners

Schematically:



Example for $d=2$:



Example 2. Categories form a bicategory $\mathcal{B} = \mathcal{C}at$:

objects: categories $\mathcal{C}$

1-morphisms: functors $\mathcal{C}_1 \xrightarrow{F} \mathcal{C}_2$

2-morphisms: natural transformations
$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathcal{C}_2 \end{tikzcd}$$

3. Algebras over a field k form a bicategory $\mathcal{B} = \mathrm{Alg}_k$:

<u>objects</u>: algebras A

<u>1-morphisms</u>: (A_2, A_1)-bimodules $A_1 \xrightarrow{M} A_2$

<u>2-morphisms</u>: intertwiners

$$\begin{tikzcd} A_1 \arrow[r, bend left, "M_1"] \arrow[r, bend right, "M_2"'] \arrow[r, phantom, "\Downarrow \varphi"] & A_2 \end{tikzcd}$$

<u>Recall</u> if $\mathcal{C}_1, \mathcal{C}_2$ are categories and $\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "F"] & \mathcal{C}_2 \arrow[l, bend left, "G"] \end{tikzcd}$ functors,

we say F is left adjoint to G if there are natural transformations

$$\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}_2} \quad \eta : \mathrm{id}_{\mathcal{C}_1} \Rightarrow GF$$

such that

$$F = F \circ \mathrm{id}_{\mathcal{C}_1} \xRightarrow{\mathrm{id}_F \circ_h \eta} FGF \xRightarrow{\varepsilon \circ_h \mathrm{id}_F} \mathrm{id}_{\mathcal{C}_2} \circ F = F$$

$$G = \mathrm{id}_{\mathcal{C}_1} \circ G \xRightarrow{\eta \circ_h \mathrm{id}_G} GFG \xRightarrow{\mathrm{id}_G \circ_h \varepsilon} G \circ \mathrm{id}_{\mathcal{C}_2} = G$$

are identities.

<u>Def</u> A <u>left dual</u> (or <u>adjoint</u>) to a 1-morphism

$g: C_1 \to C_2$ is a 1-morphism $f: C_2 \to C_1$ together with 2-morphisms

$$\varepsilon: f\circ g \Rightarrow id_{C_2} \qquad \eta: id_{C_1} \Rightarrow g\circ f$$

such that

$$f = f\circ id_{C_1} \xRightarrow{id_f \circ_h \eta} fgf \xRightarrow{\varepsilon \circ_h id_f} id_{C_2} \circ f$$

$$g = id_{C_1} \circ g \xRightarrow{\eta \circ_h id_g} gfg \xRightarrow{id_g \circ_h \varepsilon} g \circ id_{C_2}$$

are identities.

<u>Example</u> Let $A, B \in Alg_k$ and M a (B,A)-bimodule that is finitely-generated projective as a right A-module. Then $Hom_A(M,A)$ is an (A,B)-bimodule which is right dual to M.

<u>Example</u> For $\mathcal{C}$ a monoidal category, consider the bicategory $B\mathcal{C}$ with

<u>objects</u> $\{*\}$

<u>1-morphisms</u> $\mathcal{C}$ with composition $\otimes$

2-morphisms $\mathrm{Mor}\,\mathcal{C}$

Then a left dual of a 1-morphism of $\mathcal{BC}$ is the same as a left dual of an object in $(\mathcal{C}, \otimes)$.

Def An object c in a symmetric monoidal bicategory $\mathcal{B}$ is fully dualizable if

1-dualizable: ev_c, $coev_c$ exist as before,

2-dualizable: ev_c, $coev_c$ admit both left & right dual.

Example $A \in \mathrm{Alg}_{\mathbb{C}}$ is fully dualizable if and only if it is finite-dimensional & semi-simple.

Cobordism hypothesis for bicategories

The bicategory of 2d framed TFTs

$$\mathrm{Bord}^{fr}_{2,1,0} \to \mathcal{B}$$

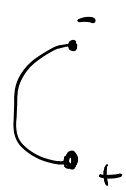
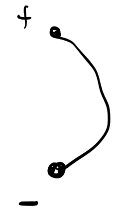
is equivalent to

$$(\mathcal{B}^{f.d.})^{\sim}$$

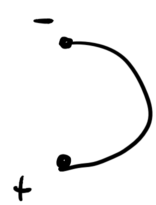
the core of the fully dualizable objects in $\mathcal{B}$.

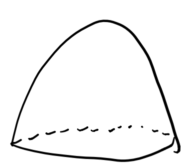
Why is $\bullet^+ \in \mathrm{Bord}^{fr}_{2,1,0}$ fully dualizable?

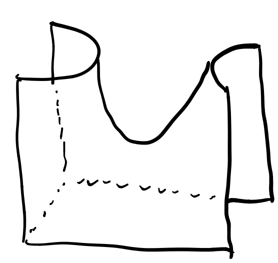
It is 1-dualizable as before with

$ev =$  $coev =$ 

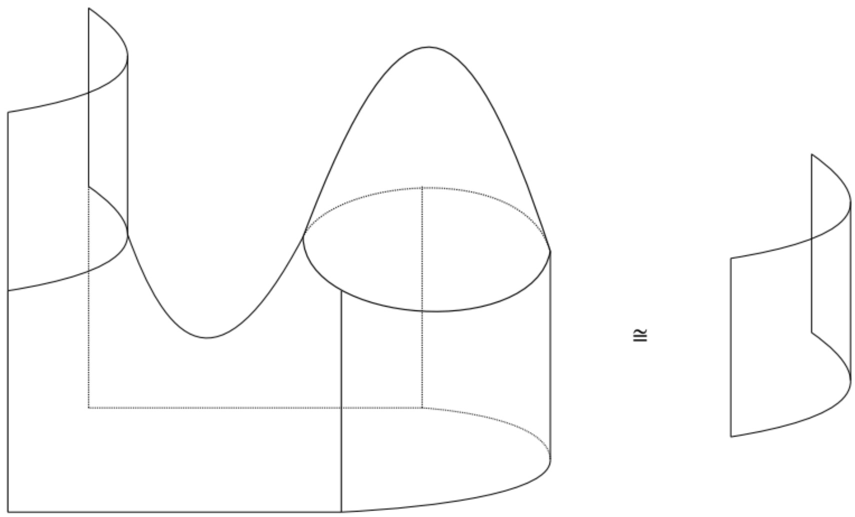
We now have to show the existence of the 4 left/right adjoints of ev/coev.

<u>Claim:</u> $coev^\vee :=$  is a right dual of coev.

$\eta:$  $\mathrm{id}_\emptyset \to coev^\vee \circ coev$

$\varepsilon:$  $coev^\vee \circ coev \to \mathrm{id}_{\bullet_- \sqcup \bullet_+}$

The right dualizability condition is



Hiro Lee Tanaka

"lectures on factorization homology, ∞-categories and topological field theories"