Motivic TT-Geometry

II

Recall from last week: "tt-classification problem"

$(\mathcal{T}, \otimes, \mathbb{1})$ a $\otimes$-Δ-category

classify tt-ideals in $\mathcal{T}$

$\mathcal{K} \subseteq \mathcal{T}$ full, replete

thick:

* $0 \in \mathcal{K}$
* $X \in \mathcal{K} \Leftrightarrow \Sigma X \in \mathcal{K}$
* $X \to Y \to Z \to \Sigma X$ distinguished Δ
  If 2-o-o-3 of $\{X, Y, Z\}$ are in $\mathcal{K}$, so is the third
* $X \oplus Y \in \mathcal{K} \Rightarrow X, Y \in \mathcal{K}$

tensor ideal:

* $X \in \mathcal{K}$, $T \in \mathcal{T}$, $T \otimes X \in \mathcal{K}$

Helpful analogy: $\otimes$-Δ-categories $\Leftrightarrow$ commutative rings

Hypothesis

① T is essentially small

② T is rigid $\rightsquigarrow$ no need to worry about radical

$\Rightarrow X^{\otimes n} \in K$

$\Rightarrow X \in K$

* $D(R)^{\omega}$, $D(X)^{\omega}$, SH^{ω} + more

Def^{n}: A tt-ideal $P \subsetneq T$ is prime if

$X \otimes Y \in P \Rightarrow X$ or Y in P

* The spectrum of T is the set

$\operatorname{Spc} T := \{ P \subseteq T \mid P \text{ prime tt-ideal} \}$

* $X \in T$, the support of X is the subset

$\operatorname{Supp}(X) := \{ P \in \operatorname{Spc} T \mid X \notin P \}$

Q: why "supported"? If $P \in \operatorname{Supp}(X)$, we can form a Verdier quotient

$$\pi : T \to T/P$$

P thick $\Rightarrow \ker(\pi) = P$

As $X \notin P$, $\pi(X) \neq 0$ in the quotient.

<u>Prop</u> $X \mapsto \mathrm{supp}(X)$ satisfies

1) $\mathrm{supp}(\mathbb{1}) = \mathrm{Spc}\, T$ $\quad \mathrm{supp}(0) = \emptyset$

2) $\mathrm{supp}(X \oplus Y) = \mathrm{supp}(X) \cup \mathrm{supp}(Y)$

3) $\mathrm{supp}(\Sigma X) = \mathrm{supp}(X)$

4) $X \to Y \to Z \to \Sigma X$, $\quad \mathrm{supp}(Y) \subseteq (\mathrm{supp}(X) \cup \mathrm{supp}(Z))$

5) $\mathrm{supp}(X \otimes Y) = \mathrm{supp}(X) \cap \mathrm{supp}(Y)$

6) $\mathrm{supp}\, X = \emptyset \Leftrightarrow X \cong 0$

This satisfies a nice universal property.

<u>Def^n</u> The Zariski topology on $\mathrm{Spc}\, T$ has a basis of closed sets $\{\mathrm{supp}\, X \mid X \in T\}$

$\Rightarrow$ Arbitrary closed set is of the form

$$Z(S) = \bigcap_{X \in S} \mathrm{supp}(X)$$

$(\operatorname{Spc} T, \tau) \rightsquigarrow$ Balmer spectrum

Thm $\operatorname{Spc} T$ is a spectral topological space $\rightsquigarrow$ is of the form $\operatorname{Spec}(R)$

$R \in$ Comm.

Q: So what?

Def^n X a spectral space. A subspace $W \subseteq X$ is Thomason if W is of the form $\bigcup_I W_i$, W_i closed w/ q-c complement

$\rightsquigarrow \operatorname{Thom}(X)$

Aside If X is a Noetherian spectral space

$$\operatorname{Thom}(X) \cong \left\{ \begin{array}{c} \text{specialization closed} \\ \text{subsets} \end{array} \right\}$$

Intitoition:

$\Rightarrow$ For any $X \in \mathcal{T}$, $\mathrm{supp}(X)$ is Thomason, as is any union of supports

$\Leftarrow$ Any Thomason subset can be written as a union of supports.

Main Theorm (Balmer) The assignments

$$\sigma : \mathrm{Thick}^{\otimes}(\mathcal{T}) \to \mathrm{Thom}(\mathrm{Spc}\,\mathcal{T})$$
$$\sigma(\mathcal{K}) = \bigcup_{X \in \mathcal{K}} \mathrm{supp}(X)$$

$$\tau : \mathrm{Thom}(\mathrm{Spc}\,\mathcal{T}) \to \mathrm{Thick}^{\otimes}(\mathcal{T})$$
$$\tau(W) = \{X \in \mathcal{T} \mid \mathrm{supp}(X) \subseteq W\}$$

is an isomorphism of lattices

$$\mathrm{Thick}^{\otimes}(\mathcal{T}) \cong \mathrm{Thom}(\mathrm{Spc}\,\mathcal{T})$$

__Ex__ $R \in \mathrm{Comm}$

$$\mathrm{Spc}(D(R)^{\omega}) \cong \mathrm{Spec}(R)$$

$$\mathrm{Thick}^{\otimes}(D(R)^{\omega}) \cong \mathrm{Thom}(\mathrm{Spec}\, R)$$

Last week: R Noetherian

$$\mathrm{Thick}^{\otimes}(D(R)^{\omega}) \cong \text{specialization closed subsets of } \mathrm{Spec}(R)$$

__Ex__ $\mathrm{Spc}(SH_{(p)}^{\omega})$

$$\circ \rightsquigarrow \bullet \rightsquigarrow \bullet \rightsquigarrow \bullet \rightsquigarrow \cdots \rightsquigarrow \bullet$$

__Intuition__ we have a $\otimes$-Δ-category $\mathcal{T}$ that we really care about. Probe $\mathcal{T}$ with categories that we do know.

Prop If $F: S \to T$ is a $\otimes$-Δ-functor, we get a continuous map

$$\mathrm{Spc}(F): \mathrm{Spc}(T) \to \mathrm{Spc}(S)$$
$$Q \mapsto F^{-1}(Q)$$

$$(\mathrm{Spc}(F))^{-1}(\mathrm{supp}_S(X)) = \mathrm{supp}_T(F(X))$$

Ex $\pi: T \to T/K$ a Verdier loc^n.

$$\mathrm{Spc}(\pi): \mathrm{Spc}(T/K) \to \mathrm{Spc}(T)$$

$$\mathrm{Spc}(T/K) \cong \{P \in \mathrm{Spc}(T) \mid K \subseteq P\}$$

$\nexists$! we do hypothesis.

Thm (Balmer) $F: S \to T$ detects $\otimes$-nilpotence of morphisms (i.e every $f: X \to Y$ in S with $F(f) = 0$, satisfies $f^{\otimes n} = 0$ for some $n \geq 1$)

Then

$$\mathrm{Spc}(F): \mathrm{Spc}(T) \to \mathrm{Spc}(S)$$

is surjective.

(this is an iff if $F: S \to T$ admits a right adjoint)

<u>Thm</u> $F: S \to T$, TFAE

* $F: S \to T$ is conservative (ie detects isos)
* $\mathrm{Spc}(F)$ surjective on closed points

Fix a $\otimes$-Δ-category T. Associated to T is a natural ring $\mathrm{End}(\mathbb{1})$

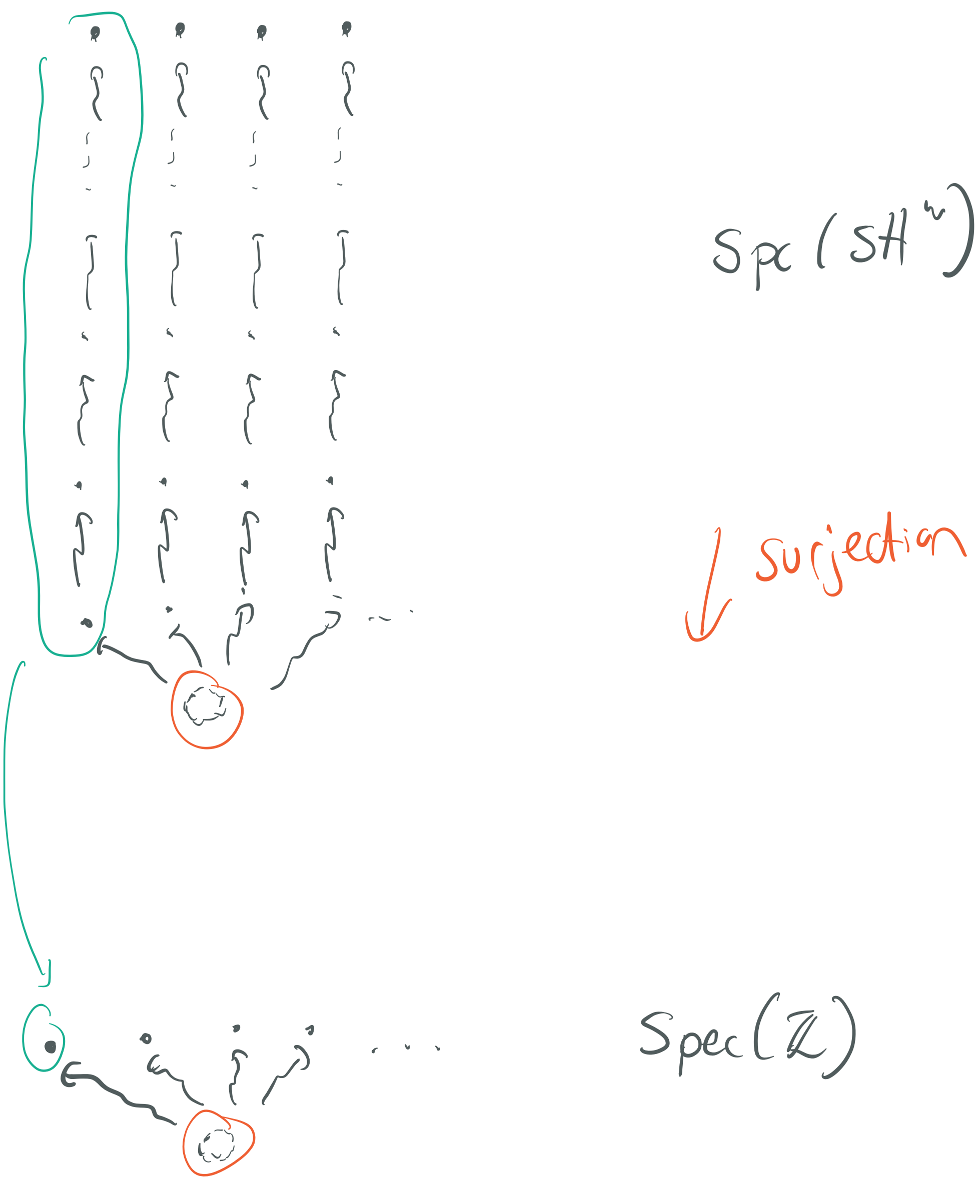
<u>Thm</u> (Balmer) There is a continuous morphism

$$\rho_T : \mathrm{Spc}(T) \to \mathrm{Spec}(\mathrm{End}(\mathbb{1}))$$

If T is "connective" $\Rightarrow \mathrm{Hom}(\Sigma^i \mathbb{1}, \mathbb{1}) \cong 0$ for all $i < 0$.

$\Rightarrow \rho_T$ is surjective.

Ex $\mathcal{T} = SH^{\omega}$ $\quad \mathrm{End}(\mathbb{S}) = \mathbb{Z}$.



Also a graded ring

$$\mathrm{End}^{\circ}_{\mathcal{T}}(\mathbb{1}) = \mathrm{Hom}(\mathbb{1}, \Sigma^{\circ}(\mathbb{1}))$$

Have a continuous morphism

$$\rho^\bullet_T : \mathrm{Spc}(T) \to \mathrm{Spec}^h(\mathrm{End}^\bullet_T(\mathbb{1}))$$

<u>Thm</u> If $\mathrm{End}^\bullet_T(\mathbb{1})$ is a coherent ring (e.g. Noetherian) then both of

ρ_T <u>and</u> $\rho^\bullet_T$

are surjective.