Parts I & II : $\mathcal{T}$ tensor-triangulated cat.

$\rightsquigarrow$ t.t.-spectrum $\mathrm{Spc}(\mathcal{T}^c)$: a top. space

- source of 'geom.' for $\mathcal{T}$: objects have a support in $\mathrm{Spc}(\mathcal{T}^c)$
- classifies objects of $\mathcal{T}^c$ up to t.t.-structure
- generalizes chromatic theory in top.

Major problem: Compute Spc for 'motivic' $\mathcal{T}$.

Motivic ($\mathbb{A}^1$-homotopy) theory : Wish list:

Fix a field $\mathbb{F}$, perfect and char$(\mathbb{F}) \neq 2$.

$S^{(\mathrm{Sm}/\mathbb{F})^{op}}$

- Every $X \in \mathrm{Sm}/\mathbb{F}$ gives an object X_+ ('motive' of X)

localization
- Geom on $\mathrm{Sm}/\mathbb{F}$: Groth. top. (NIS, étale)
- Impose $\mathbb{A}^1 \times X \xrightarrow{\sim} X$.

Day convol.
- Tensor st. $X_+ \otimes Y_+ = (X \times_{\mathbb{F}} Y)_+$.

Stabilization (spectra)
- Stabilize : inverting $\mathbb{P}^1$.

Topol. way	Alg. way
Coeff. in $\mathcal{S} = \mathrm{Sp}$	Coeff in $\mathcal{S} = \mathrm{Ch}(R)$ R comm. ring e.g. $\mathbb{Z}$.
$SH_{mot}(\mathbb{F})$	$DM(\mathbb{F}; R)$ [⚠ ↗ also encodes transfers]

I. $\boxed{SH(\mathbb{F})}$

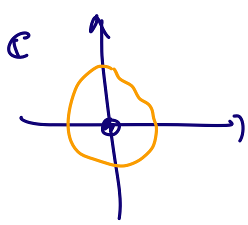
tt-functors:

$$X \longmapsto X(\mathbb{C})_{an}$$

$$\begin{tikzcd} SH \arrow[r, "cst"] \arrow[rrr, bend right, "Id"'] & SH(\mathbb{F}) \arrow[r, "\text{if } \mathbb{F} \subseteq \mathbb{C}"] & SH(\mathbb{C}) \arrow[r, "Re"] & SH \end{tikzcd}$$

$S^n \longmapsto S^n$ top. spheres

Also geom. spheres: $\mathbb{P}^1 \simeq S^1 \otimes \mathbb{G}_m$, $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$

$Re(\mathbb{G}_m) = \mathbb{C}^* =$  $\simeq S^1$.

Hopf maps: • usual $\eta^{top} : \mathbb{1} \to (S^1)^{\otimes -1}$

$$\begin{array}{c} SH \\ \downarrow \\ SH(\mathbb{F}) \end{array} \qquad \begin{array}{ccc} S^3 & \longrightarrow & S^2 \\ \wr\!\! & & \wr\!\! \\ \mathbb{C}^2 \setminus \{0\} & \longrightarrow & \mathbb{CP}^1 \end{array}$$

- algebraic version:

$$\boxed{\begin{array}{ccc} \mathbb{A}^2 \setminus \{0\} & \longrightarrow & \mathbb{P}^1 \\ \wr\!\!| & & \wr\!\!| \\ S^1 \otimes \mathbb{G}_m^{\otimes 2} & & S^1 \otimes \mathbb{G}_m \end{array}}$$ in $SH(\mathbb{F})$

$$\begin{array}{ccc} SH(\mathbb{F}) & \eta^{alg} : \mathbb{1} \longrightarrow \mathbb{G}_m^{\otimes -1} \\ \big\downarrow & \big\downarrow \mathrm{Re} \\ SH & \eta^{top} \end{array}$$

<u>Morel</u>: η^{alg} is <u>never</u> $\otimes$-nilpotent.

<u>Cor</u>: $\mathrm{Spc}(\mathrm{Re}) : \mathrm{Spc}(SH^c) \hookrightarrow \mathrm{Spc}(SH(\mathbb{F})^c)$??

$\mathbb{F} \subseteq \mathbb{C}$ ✓

is not surj.: image $\subseteq$

$\mathrm{supp}(\mathrm{cone}(\eta^{alg})) = \mathrm{supp}(\mathbb{P}^2)$.

Ref: <u>Heller–Ormsby</u> $\rho^{\bullet}_{\mathbb{G}_m}$ is onto. $(u = \mathbb{G}_m)$

$$\begin{array}{cc} & \mathrm{Spc}(\mathcal{T}^c) \\ & \big\downarrow \rho^{\bullet}_u \\ \rho & \mathrm{Spec}^h(\underbrace{\mathrm{Hom}(\mathbb{1}, u^{\otimes \bullet})}) \\ & \big\downarrow \\ & \mathrm{Spec}(\mathrm{End}(\mathbb{1})) \end{array}$$

u $\otimes$-invertible

Milnor–Witt K-theory (Morel).

II. $\boxed{DM(\mathbb{F};R)}$

H-functor $\quad SH(\mathbb{F}) \longrightarrow DM(\mathbb{F};R)$

$$\eta \longmapsto 0$$

In DM: $\quad End(\mathbb{1}, \mathbb{G}_m^{\otimes n}) = \begin{cases} K_n^M(\mathbb{F}) & n \geq 0 \\ 0 & n < 0. \end{cases}$

<u>$R=\mathbb{Q}$</u>: <u>Conj.</u>: $\boxed{Spc(DM(\mathbb{F};\mathbb{Q})^c) = *}$.

$\Downarrow$

'Conservativity conj.'

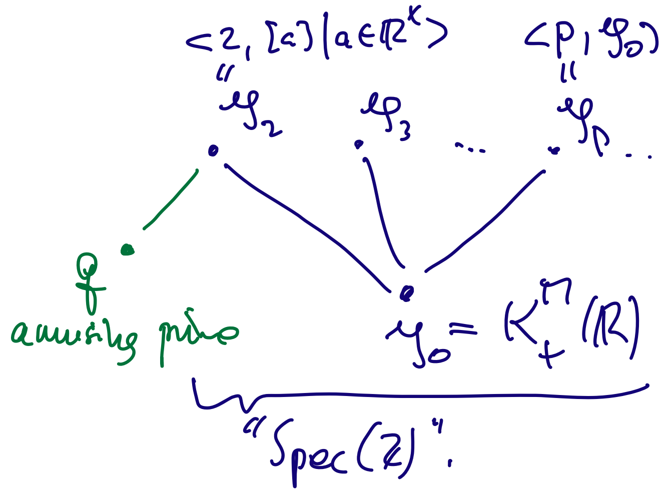
<u>$R=\mathbb{Z}$</u>: $\longleftrightarrow$ $R=\mathbb{Z}/p$

<u>Comp. map</u>: $\rho_u^{\bullet} : Spc(\mathcal{T}^c) \longrightarrow Spec^h(End_u^{\bullet}(\mathbb{1}))$

$u = \mathbb{G}_m \qquad End(\mathbb{1}) = \mathbb{Z} \qquad End^{\bullet}_{\mathbb{G}_m}(\mathbb{1}) = K_{\bullet}^M(\mathbb{F})$.

<u>Ex</u>: $\mathbb{F} = \mathbb{R}$

$Spec^h(K_0^M(\mathbb{R})) =$



$q = \langle 2, [a] \mid a > 0 \rangle$

$\Downarrow$

$Spc(DM(\mathbb{R};\mathbb{Z})^c)$

should have something interesting "at 2".

III. $\boxed{\text{Simpler variants}}$

$$
\begin{tikzcd}
 & & \mathrm{DM} \arrow[rr, two heads] & & \mathrm{DM}_{et} \\
\mathrm{DAM} \arrow[r, phantom, "\subset"] & \mathrm{DATM} \arrow[ur, phantom, "\subset"] & & & \\
\text{Artin} \arrow[u] & \mathrm{DTM} \arrow[u, phantom, "\cup"] & & & \\
 & \text{Tate} \arrow[u] & & &
\end{tikzcd}
$$

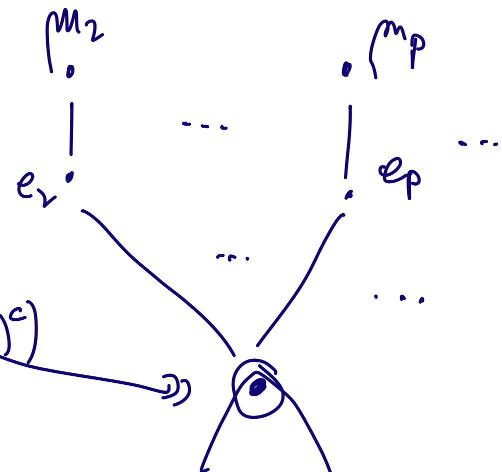
tt-subcat. gen. by: - T : $\mathbb{G}_m^{\otimes n}$.
- A : $\mathbb{E}/\mathbb{F}$ finite field ext.

$\underset{\text{rigid}}{\mathcal{K}} \subset \mathcal{L} \Rightarrow \mathrm{Spc}(\mathcal{L}) \twoheadrightarrow \mathrm{Spc}(\mathcal{K})$

<u>Thm.</u> (Peter '13) $\mathrm{Spc}(\mathrm{DTM}(\bar{\mathbb{Q}};\mathbb{Q})^c) = *$

<u>Thm.</u> (Gallauer '19) $\mathrm{Spc}(\mathrm{DTM}(\bar{\mathbb{F}};\mathbb{Z}/p)^c) = \begin{matrix} \bullet\, m \\ | \\ \bullet\, e \end{matrix}$
$p \in \mathbb{F}^\times$

<u>Cor</u> $\mathrm{Spc}(\mathrm{DTM}(\bar{\mathbb{F}};\mathbb{Z})^c) =$

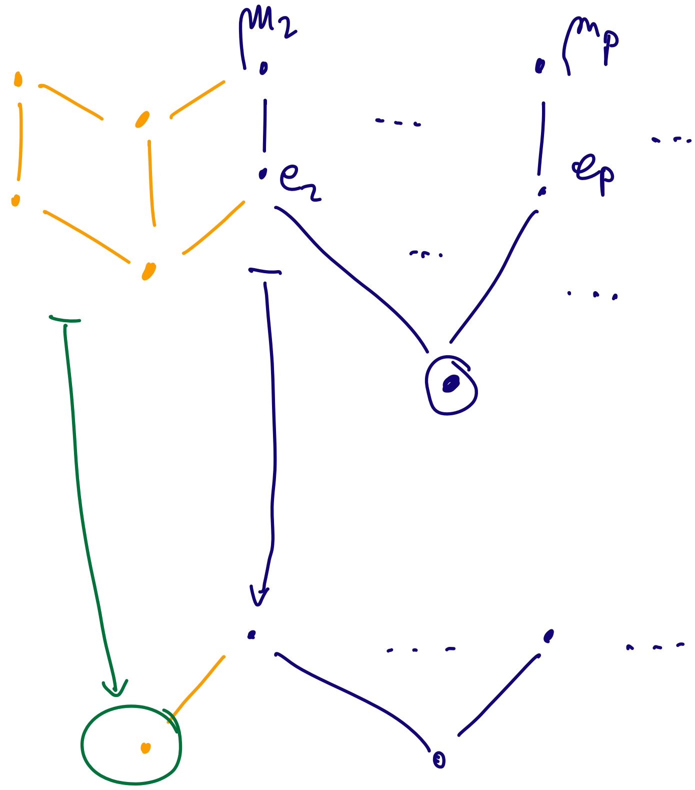


$\mathrm{Spc}(\mathrm{DM}(\mathbb{F};\mathbb{Q})^c) \twoheadrightarrow$

Thm (B. + G.)

$\mathrm{Spc}\left(\mathrm{DATM}(\mathbb{R};\mathbb{Z})^c\right)$

$\mathbb{C}/\mathbb{R}$



Techniques: modular repr. th. of C_2, (filtered) over $\mathbb{F}_2$.

Promo: Tomorrow: Math @ "New directions in group th. & tr. cat".

(Oct. 26: Scott's talk.)