Sheared Witt vectors and Barsotti-Tate groups

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Plan

Once and for all, fix a prime p.
"Barsotti-Tate group"="p-divisble group"="BT group".

$W :=$ the ring scheme of p-typical Witt vectors.

1. I will define a certain ring space ${}^sW$ equipped with an epimorphism ${}^sW \twoheadrightarrow W$. ("Space"="something like an algebraic space".)
${}^sW$ is called the space of sheared Witt vectors because it is closely related to the theory of sheared prismatization (Bhatt, Kanaev, Mathew, Vologodsky, Zhang).
In some sense, ${}^sW$ goes back to a 2001 work of Th. Zink.

2. Then I will discuss a conjectural description of the p-adic completion of the stack of BT groups. It is in the spirit of Dieudonné and Th. Zink, but the role of W is played by ${}^sW$.

Slogan behind the conjecture:
Dieudonné theory becomes better if you replace W with ${}^sW$.

Meaning of “space”

A ring R is said to be *p-nilpotent* if the element $p \in R$ is nilpotent. (Natural class of test rings for doing integral p-adic geometry.)

Let $\mathrm{p\text{-}Nilp}$ denote the category of p-nilpotent rings, so $\mathrm{p\text{-}Nilp}^{\mathrm{op}}$ is the category formed by spectra of p-nilpotent rings. We equip $\mathrm{p\text{-}Nilp}^{\mathrm{op}}$ with the flat (=fpqc) topology. A *space* is a sheaf of sets on $\mathrm{p\text{-}Nilp}^{\mathrm{op}}$. A *stack* is a stack on $\mathrm{p\text{-}Nilp}^{\mathrm{op}}$.

The category of p-adic formal schemes is a full subcategory of the category of spaces. Namely, a functor $\mathrm{p\text{-}Nilp}^{\mathrm{op}} \to \{\text{Sets}\}$ is said to be a p-adic formal scheme if its restriction to the category of $\mathbb{Z}/p^n$-algebras is representable by a scheme for each $n \in \mathbb{N}$. (Such a functor is an fpqc-sheaf by the representability condition.) Equivalently, a p-adic formal scheme X is a collection of schemes X_n over $\mathbb{Z}/p^n$ (for all $n \in \mathbb{N}$) with isomorphisms $X_n \xrightarrow{\sim} X_{n+1} \otimes \mathbb{Z}/p^n$.

I will usually say “scheme” instead of “p-adic formal scheme” (we live in the p-adic world. . .).

The ring scheme W^{perf}

By W I mean the functor $R \mapsto W(R)$, where $R \in \text{p-Nilp}$. This functor is a ring scheme (where "scheme"= "p-adic formal scheme).

Let $F : W \to W$ be the Witt vector Frobenius. F is a faithfully flat morphism of schemes, so it is surjective as a morphism of sheaves.

Let W^{perf} be the "perfection" of W, i.e., the projective limit of the diagram

$$\ldots \xrightarrow{F} \!\!\!\!\rightarrow W \xrightarrow{F}\!\!\!\!\rightarrow W.$$

W^{perf} is a ring scheme equipped with an epimorphism $W^{\mathrm{perf}} \twoheadrightarrow W$. What is its kernel? For $n \in \mathbb{N}$ let $W^{(F^n)} := \mathrm{Ker}(F^n : W \twoheadrightarrow W)$. The schemes $W^{(F^n)}$ form a projective system:

$$\ldots \xrightarrow{F}\!\!\!\!\rightarrow W^{(F^2)} \xrightarrow{F}\!\!\!\!\rightarrow W^{(F)}.$$

Let $T_F(W)$ be the projective limit (the "F-Tate module" of W). Then

$$\mathrm{Ker}(W^{\mathrm{perf}} \twoheadrightarrow W) = T_F(W).$$

Definition of ${}^sW$

Format: ${}^sW := W^{\mathrm{perf}}/T_F(\hat{W})$, where $T_F(\hat{W}) \subset T_F(W)$ is a certain smaller ideal, which is an ind-scheme. So one has ${}^sW \twoheadrightarrow W$.

$\hat{W} \subset W$ is the following ind-subscheme: let $R \in \text{p-Nilp}$, then $\hat{W}(R)$ is the set of Witt vectors x over R such that all components of x are nilpotent and almost all of them equal 0.

It is easy to show that $\hat{W} \subset W$ is an ideal and $F(\hat{W}) \subset \hat{W}$. It is known that $F : \hat{W} \to \hat{W}$ is a surjective map of sheaves.

Finally, $T_F(\hat{W})$ is the "F-Tate module" of $\hat{W}$, i.e.,

$$T_F(\hat{W}) = \varprojlim(\ldots \xrightarrow{F}\!\!\!\!\rightarrow \hat{W}^{(F^2)} \xrightarrow{F}\!\!\!\!\rightarrow \hat{W}^{(F)}).$$

Remark. By definition, ${}^sW$ is the fpqc sheafification of the presheaf $R \mapsto W^{\mathrm{perf}}(R)/(T_F(\hat{W}))(R)$. So it is hard to describe ${}^sW(R)$ for a *general* $R \in \text{p-Nilp}$ (sheafification is hard to control). But there is a direct description of ${}^sW(R)$ if R_{red} is perfect. If one also assumes that R is Artinian then ${}^sW(R) = W(R_{\mathrm{red}}) \oplus \hat{W}(R) \subset W(R)$ (as in Zink-2001).

Pieces of structure on ${}^sW$

$F : W \to W$ induces $F : W^{\mathrm{perf}} \to W^{\mathrm{perf}}$.
${}^sW$ is a quotient of W^{perf} by an F-stable ideal.
So we have a ring homomorphism $F : {}^sW \to {}^sW$.

One also has a map $\tilde{V} : {}^sW \to {}^sW$ satisfying the identities similar to those satisfied by $V : W \to W$ (if $p = 2$ the identities are slightly different). This is less obvious since there is no $V : W^{\mathrm{perf}} \to W^{\mathrm{perf}}$.
Difficulty: $VF \neq FV$. But this difficulty doesn't appear in characteristic p (i.e., if the test ring R is an $\mathbb{F}_p$-algebra).
So the map $\tilde{V} : {}^sW_{\mathbb{F}_p} \to {}^sW_{\mathbb{F}_p}$ is clear.
Here ${}^sW_{\mathbb{F}_p}$ is the reduction of ${}^sW$ modulo p, i.e., the restriction of the functor ${}^sW : \mathrm{p\text{-}Nilp} \to$ Rings to the category of $\mathbb{F}_p$-algebras.

Barsotti-Tate groups (recollections)

Definition. An n-truncated BT group (abbreviation: BT_n group) over a ring R is a finite locally free commutative group R-scheme H killed by p^n and satisfying a certain condition. If $n > 1$ the condition is as follows: the fpqc sheaf of $\mathbb{Z}/p^n$-modules on $\operatorname{Spec} R$ corresponding to H is flat. (I will not discuss the case $n = 1$.)

Known: the order of H equals p^{nh}, where $h : \operatorname{Spec} R \to \mathbb{Z}_{\geq 0}$ is locally constant. h is called the height of H.

Fix $h \in \mathbb{Z}_{\geq 0}$. Notation: $\mathrm{BT}_n^h(R)$ is the groupoid of n-truncated BT groups over R of height h.
BT_n^h is clearly an algebraic stack of finite type over $\mathbb{Z}$.

Theorem (Grothendieck). BT_n^h is *smooth* over $\mathbb{Z}$.

BT_n^h has dimension 0 over $\mathbb{Z}$: indeed, $\mathrm{BT}_n^h \otimes \mathbb{Q}$ is just the classifying stack of $GL(h, \mathbb{Z}/p^n\mathbb{Z})$.

Barsotti-Tate groups (continued)

Problem: describe the p-adic completion of BT_n^h, i.e., describe $\mathrm{BT}_n^h(R)$ for $R \in \text{p-Nilp}$.

From now on, I assume that $R \in \text{p-Nilp}$, so BT_n^h will be a stack on $\text{p-Nilp}^{\mathrm{op}}$.

Let $H \in \mathrm{BT}_n(R)$. Let $\bar{H} \in \mathrm{BT}_n(R/pR)$ be the base change of H.
Known: $\mathrm{Lie}(\bar{H})$ is a locally free module of rank $\leq h$. This rank is called the dimension of H.

Fix $d \in \mathbb{Z}$, $0 \leq d \leq h$. Have a substack $\mathrm{BT}_n^{h,d} \subset \mathrm{BT}_n^h$; it is both open and closed.

We have a morphism $\mathrm{BT}_{n+1}^{h,d} \to \mathrm{BT}_n^{h,d}$, namely $H \mapsto \mathrm{Ker}(H \xrightarrow{p^n} H)$.
It is smooth and surjective (Grothendieck).

Let $\mathrm{BT}^{h,d} = \mathrm{BT}_\infty^{h,d}$ be the projective limit of $\mathrm{BT}_n^{h,d}$. This is the stack of Barsotti-Tate groups (a.k.a. p-divisible groups). So an object of $\mathrm{BT}^{h,d}(R)$ is a diagram $H_1 \hookrightarrow H_2 \hookrightarrow \ldots$, where $H_n \in \mathrm{BT}_n(R)$ and $H_n = \mathrm{Ker}(H_{n+1} \xrightarrow{p^n} H_{n+1})$.

I will mostly assume $n = \infty$ (formulations become easier).

Th. Zink's theory of displays (improved by E. Lau)

Usual Dieudonné theory describes $\mathrm{BT}^{h,d}(k)$ (where k is a perfect field of characteristic p) in terms of semilinear algebra over $W(k)$.

Zink's theory of displays is a version of Dieudonné theory for arbitrary p-nilpotent rings:
(i) he defined a stack $\mathrm{Disp}^{h,d}$ (the stack of displays); for any $R \in \mathrm{p\text{-}Nilp}$ the definition of $\mathrm{Disp}^{h,d}(R)$ is given in terms of F-linear algebra over $W(R)$;
(ii) he defined a morphism $\pi : \mathrm{BT}^{h,d} \to \mathrm{Disp}^{h,d}$ such that if k is a perfect field of characteristic p then the map $\mathrm{BT}^{h,d}(k) \to \mathrm{Disp}^{h,d}(k)$ is an isomorphism;
(iii) E. Lau proved that π is a gerbe banded by a flat commutative group scheme with generic fiber $\mathbb{Z}_p(1)^N$, $N = d(h-d)$.
(iv) Lau also proved that the reduced part of each of the fibers of the group scheme is zero (this "explains" why π induces an isomorphism $\mathrm{BT}^{h,d}(k) \xrightarrow{\sim} \mathrm{Disp}^{h,d}(k)$ if k is a perfect field).

The stack of displays (Th. Zink, 2001)

$\mathrm{Disp}^{h,d} := GL(h,W)/\{\text{action of } \mathcal{A}\}$ (quotient in the sense of sheaves). Here $GL(h,W)$ is the functor $R \mapsto GL(h,W(R))$ and $\mathcal{A} \subset GL(h,W)$ a certain subgroup acting on $GL(h,W)$ (viewed as a *scheme*).

Definition of $\mathcal{A}$ and the action. $\mathcal{A}$ is formed by block-matrices

$$A = \begin{pmatrix} A_{11} & V(A_{12}) \\ A_{21} & A_{22} \end{pmatrix} \text{ where } A_{11} \text{ has size } d, \ A_{22} \text{ has size } h-d.$$

Here $V(A_{12})$ is the matrix obtained by applying V to each element of A_{12}. The action of $A \in \mathcal{A}$ on $GL(h,W)$ is given by

$$U \mapsto AU\Phi(A)^{-1}, \quad \text{where } \Phi(A) := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} F(A) \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}.$$

$\Phi(A)$ makes sense: $p^{-1}F(V(A_{12})) = A_{12}$.

Agreement with classical Dieudonné theory

Claim. If k is a perfect field of characteristic p then

$$\mathrm{BT}^{h,d}(k) \simeq GL(h, W(k))/\,\{\text{action of } \mathcal{A}(k)\}.$$

Indeed, by (covariant) Dieudonné theory, $\mathrm{BT}^{h,d}(k)$ "is" the groupoid of pairs $(M, \varphi : M \to M)$, where $M \simeq W(k)^h$, φ is F-linear, $\varphi(M) \supset pM$, and $\dim_k(M/\varphi(M)) = h - d$. This groupoid identifies with $GL(h, W(k))/\,\{\text{action of } \mathcal{A}(k)\}$: to $U \in GL(h, W(k))$ one associates (M, φ), where $M = W(k)^h$ and $\varphi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$.

Conjectural description of $\mathrm{BT}^{h,d}$

We have a map $\mathrm{BT}^{h,d} \to \mathrm{Disp}^{h,d}$.
I will define a stack $\mathrm{BT}^{h,d,?}$ and a map $\mathrm{BT}^{h,d,?} \to \mathrm{Disp}^{h,d}$.
Conjecture 1. $\exists$ an isomorphism $\mathrm{BT}^{h,d} \xrightarrow{\sim} \mathrm{BT}^{h,d,?}$ over $\mathrm{Disp}^{h,d}$.

Definition of $\mathrm{BT}^{h,d,?}$. Recall: $\mathrm{Disp}^{h,d} := GL(h,W)/\{\text{action of } \mathcal{A}\}$, where the subgroup $\mathcal{A} \subset GL(h,W)$ and its action on the scheme $GL(h,W)$ are defined using the maps $F, V : W \to W$.
Definition: $\mathrm{BT}^{h,d,?} := GL(h, {}^sW)/\ \{\text{action of } {}^s\mathcal{A}\}$, where the subgroup ${}^s\mathcal{A} \subset GL(h, {}^sW)$ and its action on $GL(h, {}^sW)$ are defined similarly using the maps $F, \tilde{V} : {}^sW \to {}^sW$ (not quite similarly if $p = 2$).

Variant of the conjecture

The above conjecture says that $\mathrm{BT}^{h,d} = GL(h, {}^sW)/\left\{\text{action of } {}^s\mathcal{A}\right\}$. If so then for any $R \in \text{p-Nilp}$ we have an inclusion

$$\mathrm{BT}^{h,d}(R) \supset GL(h, {}^sW(R))/\left\{\text{action of } {}^s\mathcal{A}(R)\right\}$$

Conjecture 2. This inclusion is an equality if $R \in \text{p-Nilp}$ satisfies the following assumptions:
(i) R_{red} is perfect,
(ii) all finitely generated projective R-modules are free.

(Easy lemma: Spectra of rings $R \in \text{p-Nilp}$ satisfying (i)-(ii) form a base for the fpqc topology on $\text{p-Nilp}^{\mathrm{op}}$.)

Th. Zink (2001) and E. Lau (2014) proved Conjecture 2 under extra assumptions for R.

n-truncated version of the conjecture

Recall: conjecturally, $\mathrm{BT}^{h,d} = GL(h, {}^sW)/\{\text{action of } {}^s\mathcal{A}\}$, where ${}^s\mathcal{A}$ is a certain group space acting on $GL(h, {}^sW)$ viewed as a space.

Conjectural description of $\mathrm{BT}_n^{h,d}$: ${}^sW$ is replaced by ${}^s\mathscr{R}_n$, where ${}^s\mathscr{R}_n := \mathrm{Cone}({}^sW \xrightarrow{p^n} {}^sW)$ is a ring stack; the group space ${}^s\mathcal{A}$ is replaced by a certain group stack ${}^s\mathcal{A}_n$ acting on the stack $GL(h, {}^s\mathscr{R}_n)$.
Conjecture. $\mathrm{BT}_n^{h,d} = \mathrm{BT}_n^{h,d,?}$, where
$\mathrm{BT}_n^{h,d,?} := GL(h, {}^s\mathscr{R}_n)/\{\text{action of } {}^s\mathcal{A}_n\}$.
A priori, $\mathrm{BT}_n^{h,d,?}$ is a 2-stack (but in fact, a 1-stack).

Does the conjecture give a presentation $\mathrm{BT}_n^{h,d} = S/\Gamma$, where S is a scheme and Γ is a flat groupoid acting on S?
Yes, and S is just $GL(h, W_n)$ (I am not explaining why).
But Γ is not very explicit. The scheme of morphisms of Γ is an affine scheme of finite type, which is described not directly but as a quotient of an ind-scheme by an action of a group ind-scheme.

Shimurian analogs of $\mathrm{BT}^{h,d}$ and $\mathrm{Disp}^{h,d}$

Define $\mathrm{Disp}^{G,\mu}$ similarly to $\mathrm{BT}^{h,d}$ but replacing $GL(h)$ with G and $\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}$ with $\mu(p)$; here G is a smooth group scheme over $\mathbb{Z}_p$ and $\mu \in \mathrm{Hom}(\mathbb{G}_m, G)$. (Details worked out by Bültel-Pappas and Lau.)

One also has a stack $\mathrm{BT}^{G,\mu}$ and a morphism $\mathrm{BT}^{h,d} \to \mathrm{Disp}^{h,d}$; the definition is more delicate, it is based on the prismatic theory.

Gardner and Madapusi proved that $\mathrm{BT}^{G,\mu}$ is "reasonable" if μ is *1-bounded*, i.e., if all weights of $\mathbb{G}_m$ acting on $\mathrm{Lie}(G)$ via μ are ≤ 1. If G is reductive and almost simple then "1-bounded" means "minuscule or zero".

Conjecturally, the stacks $\mathrm{BT}^{G,\mu}$ are related to Shimura varieties. (This would be similar to the construction which starts with a principally polarized abelian scheme and produces a p-divisible group equipped with a symplectic self-duality.)

If μ is 1-bounded then there is a conjectural description of $\mathrm{BT}_n^{G,\mu}$, $n \in \mathbb{N} \cup \infty$, similar to the case $G = GL(h)$.