Discontinuous Groups on pseudo-Riemannian Spaces

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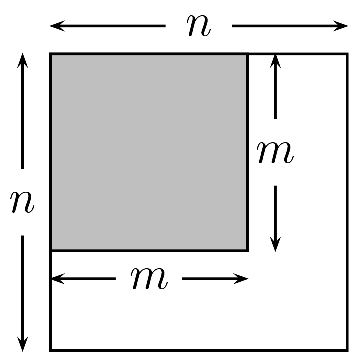
Compact quotients $\Gamma\backslash SL(n)/SL(m)$

Problem (Existence problem for uniform lattice):

Does there exist compact Hausdorff quotients of

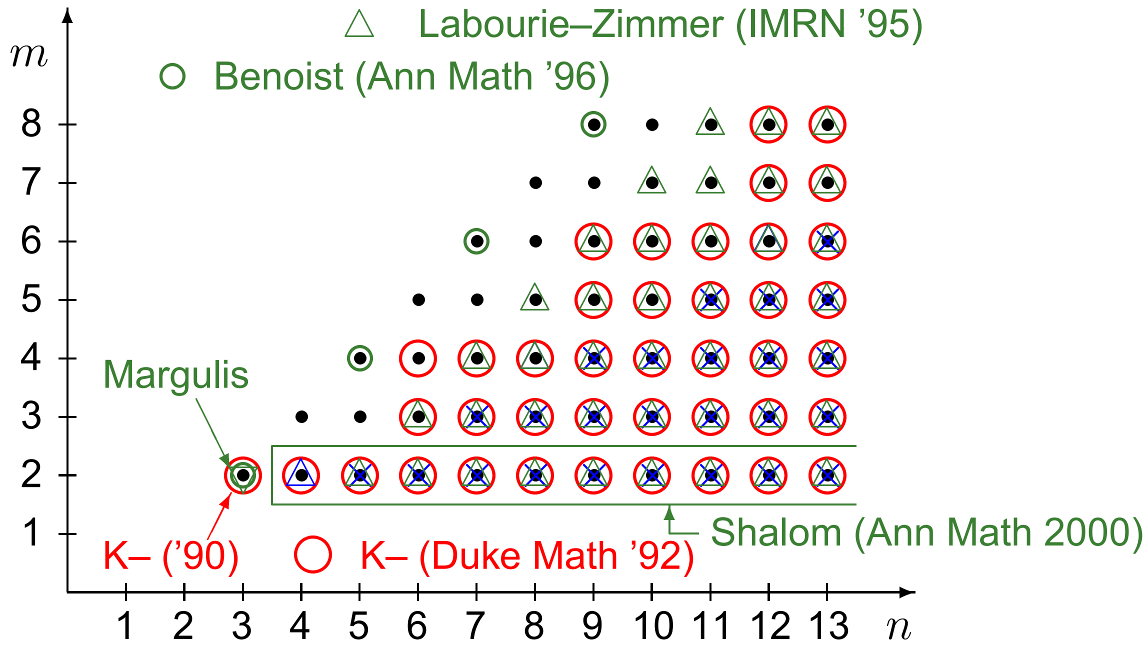
$$SL(n,\mathbb{F})/SL(m,\mathbb{F}) \quad (n > m,\ \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$$

by discrete subgps Γ of $SL(n,\mathbb{F})$?



Compact quotients for $SL(n)/SL(m)$

Uniform lattice does not exist for the following (n, m):



$SL(n)/SL(m)$ case

Conjecture For any $n > m > 1$, there does not exist uniform lattice for $SL(n)/SL(m)$.

Affirmative results:

K–	criterion of proper actions	$\frac{n}{3} > [\frac{m+1}{2}]$
Zimmer	orbit closure thm (Ratner)	$n > 2m$
Labourier–Mozes–Zimmer		
	ergodic action	$n \geq 2m$
Benoist	criterion of proper actions	$n = m + 1,\ m$ even
Margulis	unitary representation	$(n \geq 5, m = 2)$
Shalom	unitary representation	$n \geq 4, m = 2$

Non-Riemannian homo. spaces

$\boxed{\text{Discrete subgp } \overset{\not\Rightarrow}{\Leftarrow} \text{ Discontinuous gp}}$

for non-Riemannian homo. spaces

General Problem

How does a local geometric structure affect the global nature of manifolds?

New phenomena & methods?

2. Complex symmetric structure

G/K: Riemannian symmetric space

$\Downarrow$ complexification

$G_{\mathbb{C}}/K_{\mathbb{C}}$: complex symmetric space

Fact (Borel 1963) Compact quotients exist for $^{\forall}$Riemannian symm sp. G/K.

Conj. Compact quotients exist for $G_{\mathbb{C}}/K_{\mathbb{C}}$
$\Longleftrightarrow G_{\mathbb{C}}/K_{\mathbb{C}} \approx S^7_{\mathbb{C}}$ or complex group mfd

$\Leftarrow$ proved by K–Yoshino 05,
$\Rightarrow$ remaining case $S^{4k-1}_{\mathbb{C}}$, $k \geq 3$ (Benoist, K–)

Space forms (examples)

Space form $\cdots$ $\begin{cases} \text{Signature } (p,q) \text{ of pseudo-Riemannian metric } g \\ \text{Curvature } \kappa \in \{+,0,-\} \end{cases}$

E.g. $q = 0$ (Riemannian mfd)

sphere S^n	$\mathbb{R}^n$	hyperbolic sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

E.g. $q = 1$ (Lorentz mfd)

de Sitter sp	Minkowski sp	anti-de Sitter sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

Space form problem

Space form problem for pseudo-Riemannian mfds

Local Assumption

signature (p,q), curvature $\kappa \in \{+,0,-\}$

$$\Downarrow$$

Global Results

- Do compact forms exist?
- What groups can arise as their fundamental groups?

Compact space forms

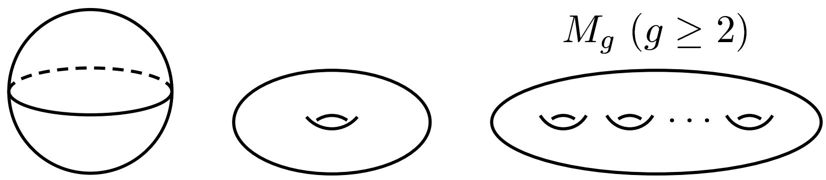
(p, q): signature of metric, curvature $\kappa \in \{+, 0, -\}$

Assume $p \geq q$ (without loss of generality).

- $\kappa > 0$: Calabi–Markus phenomenon
  (Calabi, Markus, Wolf, Wallach, Kulkarni, K–)
- $\kappa = 0$: Auslander conjecture
  (Bieberbach, Auslander, Milnor, Margulis, Goldman, Abels, Soifer, . . .)
- $\kappa < 0$: Existence problem of compact forms

2-dim'l compact space forms

Riemannian case ($\Longleftrightarrow$ signature $(2, 0)$)



curvature $\kappa > 0$ $\kappa = 0$ $\kappa < 0$

Lorentz case ($\Longleftrightarrow$ signature $(1, 1)$)
compact forms do NOT exist
for $\kappa > 0$ and $\kappa < 0$

Compact space forms ($\kappa < 0$)

- Riemannian case $\cdots$ hyperbolic space

Compact quotients

$\iff$ Cocompact discont. gp for $O(n,1)/O(n) \times O(1)$

$\iff$ Cocompact discrete subgp of $O(n,1)$

(uniform lattice)

Exist by $\underbrace{\text{Siegel, Borel–Harish-Chandra, Mostow–Tamagawa}}_{\text{arithmetic}}$,

$\underbrace{\text{Vinberg, Gromov–Piateski-Shapiro}}_{\text{non-arithmetic}} \cdots$

Existence of compact forms

- For pseudo-Riemannian mfd of signature (p,q)

Thm ~~Conjecture~~ Compact space forms of $\kappa < 0$ exist

$\Longleftarrow$
$\Longrightarrow$

① q any, $p = 0$ ($\leftrightarrow \kappa > 0$)

② $q = 0$, p any (hyperbolic sp)

③ $q = 1$, $p \equiv 0 \mod 2$

④ $q = 3$, $p \equiv 0 \mod 4$ (pseudo-Riemannian)

⑤ $q = 7$, $p = 8$

$\Longleftarrow$ True (Proved (1950–2005))

(①② (Riemmanian) ; ③④⑤ (pseudo-Riemannian) Kulkarni, K–)

$\Longrightarrow$ Partial answers:

$q = 1$, $p \leq q$, or pq is odd

Hirzebruch's proportionality principle (K–Ono)

Infinitesimal approximation

$$G = K \exp \mathfrak{p} \implies G_\theta = K \ltimes \mathfrak{p} \quad \text{(Cartan motion gp)}$$

$$\boxed{G/H = O(p, q+1)/O(p,q) \implies G_\theta/H_\theta}$$

<u>Thm</u> (K–Yoshino, 2005)
Compact forms of G_θ/H_θ exist $\iff p \equiv 0 \mod 2^{\varphi(q)}$

Here, $\varphi(q) = \left[\frac{q}{2}\right] + \begin{cases} 0 & (q \equiv 0, 6, 7 \mod 8) \\ 1 & (q \equiv 1, 2, 3, 4, 5 \mod 8) \end{cases}$

<u>E.g.</u>			
	$q = 0$		p any
	$q = 1$	$\varphi(1) = 1$	$p \equiv 0 \mod 2$
	$q = 3$	$\varphi(3) = 2$	$p \equiv 0 \mod 4$
	$q = 7$	$\varphi(7) = 3$	$p \equiv 0 \mod 8$

Radon–Hurwitz number (1922)

<u>Def.</u> (Radon–Hurwitz number)
$\rho(p) := 8\alpha + 2^\beta$
if $p = u \cdot 2^{4\alpha+\beta}$ (u: odd, $0 \le \beta \le 3$)

$$p \equiv 0 \mod 2^{\varphi(q)} \iff q < \rho(p)$$

Radon–Hurwitz number (1922)
$\Downarrow$
Adams: vector fields on sphere (1962)
$\Downarrow$
Uniform lattice for G_θ/H_θ (2005)

General idea: Compact-like actions

Non-compact Lie groups

occasionally behave nicely
when embedded in ∞-dim groups
as if they were

compact groups
(very nice behaviours)

Compact-like linear/non-linear actions

$L \curvearrowright \mathcal{H}$ (linear)

Unitarizability
$=$ existence of L-invariant inner product

Discrete decomposability
$=$ no continuous spectrum
in the L-irreducible decomposition

$L \curvearrowright M$ (non-linear)

Proper acions/properly discontinuous actions
$=$ The action map $\begin{array}{c} L \times M \to M \times M \\ (g, x) \mapsto (x, g \cdot x) \end{array}$ is proper.

Compact-like linear/non-linear actions

$\mathcal{H}$: Hilbert space, unitary reprn.
$L \curvearrowright \mathcal{H}$ discrete decomposability

$\cdots$ L behaves nicely in $U(\mathcal{H})$ (unitary operators)
as if it were a compact group

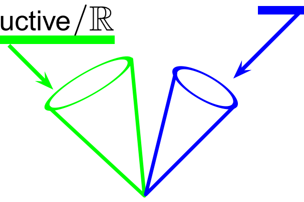
M: topological space
$L \curvearrowright M$ proper actions

$\cdots$ L behaves nicely in $\mathrm{Homeo}(M)$
as if it were a compact group

Criterion of admissible restriction

Theorem A (Criterion) (K– Ann Math '98, Progr Math '05)

Let $G' \underset{\text{reductive}/\mathbb{R}}{\subset} G$ and $\pi \in \widehat{G}$. If



$(\star)$ $\mu(T^*(K/K')) \cap \mathrm{AS}_K(\pi) = \{0\}$ in $\overset{\mathbb{R}^n}{\overset{\|}{\sqrt{-1}\mathfrak{t}^*}}$,

$\Longleftrightarrow$ $\pi|_{K'}$ is K'-admissible.

In particular, the restriction $\pi|_{G'}$ is G'-admissible.
(discretely decomposable & of finite multiplicities)

Proof uses micro-local analysis.

$\pitchfork$ and $\sim$ (definition)

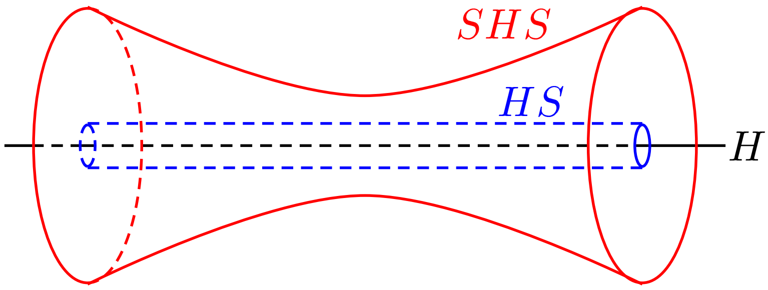
$$L \quad \subset \quad G \quad \supset \quad H$$

Idea: forget even that L and H are group

Def. (K–)

1) $L \pitchfork H \iff \overline{L \cap SHS}$ is compact for $^{\forall}$ compact $S \subset G$

2) $L \sim H \iff {}^{\exists}$ compact $S \subset G$ s.t. $L \subset SHS$ and $H \subset SLS$.



$\pitchfork$ and $\sim$

$$L \quad \subset \quad G \quad \supset \quad H$$

Forget even that L and H are group

1) $L \pitchfork H \iff$ generalization of proper actions

2) $L \sim H \iff$ economy in considering

Meaning of $\pitchfork$:

$$\boxed{L \pitchfork H \iff L \curvearrowright G/H \text{ proper action}}$$

for closed subgroups L and H

$\sim$ provides economies in considering $\pitchfork$

$$H \sim H' \implies \boxed{H \pitchfork L \iff H' \pitchfork L}$$

Criterion for $\pitchfork$ and $\sim$

G: real reductive Lie group
$G = K\exp(\mathfrak{a})K$: Cartan decomposition
ν: $G \to \mathfrak{a}$: Cartan projection (up to Weyl gp.)

<u>Thm B</u> (K– , Benoist)
1) $L \sim H$ in G $\iff$ $\nu(L) \sim \nu(H)$ in $\mathfrak{a}$.
2) $L \pitchfork H$ in G $\iff$ $\nu(L) \pitchfork \nu(H)$ in $\mathfrak{a}$.

abelian

Special cases include

(1)'s $\Rightarrow$: <u>Uniform</u> bounds on errors in eigenvalues when a matrix is perturbed.
(2)'s $\Leftrightarrow$: Criterion for <u>properly discont. actions</u>.

Criterion for compact-like actions

$$\begin{array}{lll} G: & \text{reductive Lie group} & \supset K \\ \cup & & \cup \\ G': & \text{reductive subgp} & \supset K' \end{array}$$

μ : $T^*(K/K') \to \sqrt{-1}\mathfrak{k}^*$ momentum map

<u>Thm A</u> $\pi \in \widehat{G}$, $G' \subset G$
$\mu(T^*(K/K')) \cap \mathrm{AS}_K(\pi) = \{0\}$
$\Longrightarrow \pi|_{G'}$ is discrete decomposable.

G : reductive Lie gp, $G \supset L, H$ (subsets)
$\nu : G \to \mathfrak{a}$ (Cartan projection)

<u>Thm B</u> (proper action)
$L \pitchfork H$ in $G \iff \nu(L) \pitchfork \nu(H)$ in $\mathfrak{a}$

Compact-like linear/non-linear actions

$\mathcal{H}$: Hilbert space
$L \curvearrowright \mathcal{H}$ discrete decomposability
$\cdots$ L behaves nicely in $U(\mathcal{H})$ (unitary operators)
as if it were a compact group

M: topological space
$L \curvearrowright M$ proper actions
$\cdots$ L behaves nicely in $\mathrm{Homeo}(M)$
as if it were a compact group

Local $\Longrightarrow$ Global

$G \supset H$ reductive Lie groups
$\Longrightarrow$ G/H pseudo-Riemannian homo. sp

Cor (Criterion for the Calabi–Markus phenomenon)
Any discont. gp for G/H is finite
$\Longleftrightarrow \mathrm{rank}_{\mathbb{R}}\, G = \mathrm{rank}_{\mathbb{R}}\, H$

Application (space form of signature (p, q), $\kappa < 0$)
Exists a space form M s.t. $|\pi_1(M)| = \infty$
$\Longleftrightarrow p > q$ or $(p, q) = (1, 1)$
(Calabi, Markus, Wolf, Kulkarni, Wallach)
$p > q + 1 \Longrightarrow {}^{\exists}M$ with free non-commutative $\pi_1(M)$

Rigidity, stability, and deformation

$$\begin{tikzcd}
\Gamma \arrow[d, phantom, sloped, "\subset"] \arrow[drr, bend left, "\text{properly discont.}"] & & \\
G & \curvearrowright & X \\
\Gamma' \arrow[u, phantom, sloped, "\supset"] \arrow[urr, dashed, bend right] & &
\end{tikzcd}$$

Suppose Γ' is 'close to' Γ

(R) (local rigidity)	$\Gamma' = g\Gamma g^{-1}$ $({}^{\exists}g \in G)$
(S) (stability)	$\Gamma' \curvearrowright X$ properly discont.

In general,

- (R) $\Rightarrow$ (S).
- (S) may fail (so does (R)).

Local rigidity and deformation

$\Gamma \subset G \curvearrowright X = G/H$ cocompact, discontinuous gp

General Problem

1. When does local rigidity (R) fail?
2. Does stability (S) still hold?

Point: for non-compact H

1. (good aspect) There may be large room for deformation of Γ in G.
2. (bad aspect) Properly discontinuity may fail under deformation.

Rigidity Theorem

$$① \quad \Gamma \curvearrowright G/\{e\} \iff (\Gamma \times 1) \curvearrowright (G \times G)/\Delta G \quad ②$$

$\Gamma \subset G$ simple Lie gp

Fact (Selberg–Weil's local rigidigy, 1964)
$^\exists$uniform lattice Γ admitting continuous deformations for ①
$\iff G \approx SL(2, \mathbb{R})$ (loc. isom).

Thm (K–)
$^\exists$uniform lattice Γ admitting continuous deformations for ②
$\iff G \approx SO(n+1, 1)$ or $SU(n, 1)$ $(n = 1, 2, 3, ...)$.

Local rigidity (R) may fail. Stability (S) still holds.

$\cdots$ Solution to Goldman's stability conjecture (1985), 3-dim case

Compact-like linear/non-linear actions

$\mathcal{H} = L^2(G/H), L^2(G/\Gamma)$: Hilbert space
$L \curvearrowright \mathcal{H}$ discrete decomposability
$\cdots$ L behaves nicely in $U(\mathcal{H})$ (unitary operators)
as if it were a compact group

$M = G/H$: topological space
$L \curvearrowright M$ proper actions
$\cdots$ L behaves nicely in $\mathrm{Homeo}(M)$
as if it were a compact group

Interacting example

$$(G, L, H) = (SO(4,2),\ SO(4,1),\ U(2,1))$$

$\boxed{\text{Tessellation of pseudo-Riemannian mfd } X}$

$$X = SO(4,2)/U(2,1) \quad (\underset{\text{open}}{\subset} \mathbb{P}^3\mathbb{C})$$

discont. gp

$\Gamma \qquad X$

lattice $\cap \qquad G$

$L \qquad \pi \subset L^2(X)$

discrete series

admissible
restriction

π: discrete series of G with GK-dim 5

(quarternionic discrete series)

$\Longrightarrow \pi|_L$ is L-admissible

References

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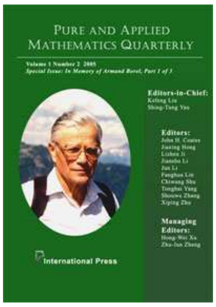

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