HIGGS BUNDLES AND CHARACTERISTIC CLASSES

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- Kähler manifold M, $w_2 = 0$

- spin structure: $K^{1/2}$

- Dirac operator D

$$D = \frac{1}{\sqrt{2}}(\bar{\partial} + \bar{\partial}^*) : \Omega^{0,ev}(M, K^{1/2}) \to \Omega^{0,od}(M, K^{1/2})$$

The polynomials T_k are called TODD polynomials. The identity

$$\frac{x}{1-e^{-x}} = \exp\left(\frac{1}{2}x\right)\frac{\frac{1}{2}x}{\sinh\frac{1}{2}x} \quad \text{(we write } \exp(a) = e^a\text{)}$$

is useful for the calculation of the first few TODD polynomials. It implies, using Lemma 1.3.1, formula (6_m) in the (c_i, x, γ_i) formulation and the relations (7), that

$$T_k(c_1, \ldots, c_k) = \Sigma \frac{1}{2^{4s} r!}\left(\frac{1}{2}c_1\right)^r A_s(p_1, \ldots, p_s) \tag{12}$$

where the sum is over all non-negative integers r, s with $r + 2s = k$.

F.Hirzebruch, *Topological Methods in Algebraic Geometry,* Springer Verlag, Berlin-Heidelberg-New York, (1966)

certain divisibility properties modulo a power of 2 in the case w_2 vanishes. The Riemann-Roch theorem $(M_{n,1})$ of algebraic geometry[4] seems to indicate that a generalization of Rohlin's theorem to higher dimensions must have something to do with the A-genus. It can be proved[4] for an *algebraic variety* V_{2k} that the vanishing of the second Stiefel-Whitney class w_2 implies: $A(V_{2k}) = 0 \pmod{2^{4k}}$, and that for k *odd*, the vanishing of w_2 implies: $A(V_{2k}) = 0 \pmod{2^{4k+1}}$. It is easy to calculate that for the non-singular hypersurface $V_{2k}^{(2r)}$ of degree $2r$ imbedded in the complex-projective space $\mathbf{P}_{2k+1}$ the Stiefel-Whitney class w_2 vanishes and that

$$A(V_{2k}^{(2r)}) = 2^{4k+1}\binom{r+k}{2k+1}.$$

F.Hirzebruch, *Problems on Differentiable and Complex Manifolds*, Ann.of Math. **60** (1954) 213–236.

- compact 2-manifold Σ, real Lie group G^r

- character variety $\mathsf{Hom}(\pi_1(\Sigma), G^r)/G^r$

- compact 2-manifold Σ, real Lie group G^r

- character variety $\mathrm{Hom}(\pi_1(\Sigma), G^r)/G^r$

- flat G^r-bundle, maximal compact K

- characteristic classes in

  $H^1(\Sigma, \pi_0(K)), H^2(\Sigma, \pi_1(K))$

  ... help to distinguish components of the character variety

EXAMPLE $G^r = SL(2,\mathbf{R})$

- maximal compact $SO(2)$, Chern class $c \in H^2(\Sigma, \mathbf{Z}) = \mathbf{Z}$
- $|c| \leq 2g-2$ (Milnor-Wood)
- $|c| < 2g-2$ determines a connected component
- $|c| = 2g-2 \Rightarrow 2^{2g}$ connected components

PLAN

- Higgs bundles and real forms

- Characteristic class w_2 for $SL(n, \mathbf{R})$

- (B,A,A)-branes and (B,B,B)-branes

HIGGS BUNDLES

- compact Riemann surface Σ
- principal G-bundle P
- $\mathcal{A}$ affine space of connections on P
- infinite-dimensional flat Kähler manifold

- $\mathcal{G}$ group of gauge transformations

- $(A, \Phi) \in T^*\mathcal{A} = \mathcal{A} \times \Omega^{1,0}(\Sigma, \mathfrak{g})$ flat hyperkähler manifold

- moment map $(F_A + [\Phi, \Phi^*], \bar{\partial}_A \Phi)$

- quotient moduli space of Higgs bundles

- $\bar{\partial}_A \Phi = 0 \qquad F_A + [\Phi, \Phi^*] = 0$

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- $\bar{\partial}_A \Phi = 0 \qquad F_A + [\Phi, \Phi^*] = 0$

- complex structure I: moduli space of (stable) pairs (A, Φ)

  $G = U(n)$ vector bundle V, $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

- complex structure J: flat G^c-connection

  $\nabla_A + \Phi + \Phi^*$ (representations $\pi_1(\Sigma) \to G^c$)

- complex structure K: flat G^c-connection

  $\nabla_A + i\Phi - i\Phi^*$

REAL FORM G^r

- $K \subset G^r$ maximal compact
- principal K^c-bundle
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$
- Higgs field $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$
- holonomy of $\nabla + \Phi + \Phi^* \in G^r$

EXAMPLE $G^r = SL(n, \mathbf{R})$

- orthogonal vector bundle V

- $\Lambda^n V \cong \mathcal{O}$

- $\Phi = \Phi^T \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

ABELIANIZATION

THE FIBRATION

- hyperkähler moduli space $\mathcal{M}(G)$

  $\dim_{\mathbf{R}} = 4(g-1)\dim G$

- principal G^c-bundle, $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K)$

- invariant polynomials $p_1, \ldots, p_\ell$ on $\mathfrak{g}$

  $p_m(\Phi) \in H^0(\Sigma, K^{d_m})$

- fibration $\mathcal{M}^{2k}(G) \to \mathbf{C}^k$

- integrable system
- generic fibre abelian variety A
- $G^c = GL(n, \mathbf{C})$ $\det(x - \Phi) = 0$ spectral curve S
- fibre $=$ Jac(S)

- spectral curve S: $\det(x - \Phi) = 0$
- curve in the cotangent bundle $\pi : K \to \Sigma$
- $\pi : S \to \Sigma$ n-fold cover

- $V = \pi_* L$
- $\Phi = \pi_*(L \xrightarrow{\eta} L \otimes \pi^* K)$
- $\eta \in \pi^* K$ canonical section on total space

- L trivial bundle

- $\pi_* L = \mathcal{O} \oplus K^{-1} \oplus \ldots \oplus K^{-n-1}$

- $\Phi =$ companion matrix

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_{n-1} & \ldots & \ldots & a_1 \end{pmatrix}$$

- section of $\mathcal{M}^{2k}(G) \to \mathbf{C}^k$

- $G^c = SL(n, \mathbf{C})$

- $L = U \otimes \pi^* K^{(n-1)/2}$, $\deg U = 0$

- $\mathsf{Nm} : \mathsf{Pic}^0(S) \to \mathsf{Pic}^0(\Sigma)$

- $U \in \mathsf{P}(S, \Sigma)$ = Prym variety = kernel

- $L = \pi^* K^{(n-1)/2}$

- $\pi_* L = K^{(n-1)/2} \oplus K^{(n-3)/2} \oplus \ldots \oplus K^{-(n-1)/2}$

- Φ = companion matrix

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_{n-1} & \ldots & \ldots & 0 \end{pmatrix}$$

- section of $\mathcal{M}^{2k}(SL(n, \mathbf{C})) \to \mathbf{C}^k$

REAL FORM G^r

- real form/compact $\sim$ involution $\sigma : \mathfrak{g} \to \mathfrak{g}$

- ± 1 eigenspaces $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$

- Higgs field $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$: $\sigma^* \Phi = -\Phi$

- G^r character variety $\sim$ fixed points $\mathcal{M}^\sigma$

- σ respects fibration $\mathcal{M}^\sigma \to (\mathbf{C}^k)^\sigma$

- fibre over $p \in (\mathbf{C}^k)^\sigma =$

- $=$ fixed points A^σ in an abelian variety A

- $0 \to A_0^\sigma \to A^\sigma \to \mathbf{Z}_2^{2N} \to 0$

THE CASE $G^r = SL(n, \mathbf{R})$

- involution $\sigma(a) = -a^T$
- $\sigma^*(\Phi) = -\Phi^T$

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- $\sigma^*(\Phi) = -\Phi^T$
- $\det(x - \Phi^T) = \det(x - \Phi)$
- $\Rightarrow$ action on $\mathbf{C}^k$ trivial
- $\Rightarrow$ fibre $\sim \mathbf{Z}_2^{2k}$

- fixed points in fibre $\sim \{U \in \mathsf{P}(S, \Sigma) : U^2 = \mathcal{O}\}$

- $L = U \otimes \pi^* K^{(n-1)/2}$

- vector bundle $V = \pi_* L$ where $L^2 \cong \pi^* K^{n-1}$

- V is an orthogonal bundle

- $\Lambda^n V \cong \mathcal{O} \Rightarrow w_1(V) = 0$

- What is $w_2(V)$?

- V is an orthogonal bundle

- $\Lambda^n V \cong \mathcal{O} \Rightarrow w_1(V) = 0$

- What is $w_2(V)$?

.... as a function of $\mathbf{Z}_2^{2k}$

ORTHOGONAL STRUCTURE

- spectral curve $S \subset$ canonical bundle of Σ

- $K_S \cong \pi^* K^n$

- $d\pi \in H^0(S, K_S \otimes \pi^* K^*) = H^0(S, \pi^* K^{n-1}) \cong H^0(S, L^2)$

- inner product on $V = \pi_* L$ at $y \in S$

$$(s, s)_y = \sum_{\pi(x)=y} \frac{s_x^2}{d\pi_x}$$

KO THEORY

- $L^2 \cong K_S \otimes \pi^* K^* =$ KO-orientation of $\pi : S \to \Sigma$

- $\pi_! : KO(S) \to KO(\Sigma)$

- $\pi_!(1) = [V]$

- $KO(\Sigma) \cong \mathbf{Z} \oplus H^1(\Sigma, \mathbf{Z}_2) \oplus H^2(\Sigma, \mathbf{Z}_2)$

- $w_1(V) = 0 \Rightarrow V = [n] + cu$

- $u = [L \oplus L^*] - [2]$, L degree one complex line bundle

- $KO(\Sigma) \cong \mathbf{Z} \oplus H^1(\Sigma, \mathbf{Z}_2) \oplus H^2(\Sigma, \mathbf{Z}_2)$

- $w_1(V) = 0 \Rightarrow V = [n] + cu$

- $u = [L \oplus L^*] - [2]$, L degree one complex line bundle

- $w_2(u) = 1 \in \mathbf{Z}_2 \cong H^2(\Sigma, \mathbf{Z}_2) \Rightarrow w_2(V) = c$

M.F.Atiyah, *Riemann surfaces and spin structures,* Ann.Sci. ENS **4** (1971) 47–62.

THE MOD 2 INDEX

- M^{8k+2} spin manifold, E real vector bundle
- Dirac operator $D : E \otimes S^{+} \to E \otimes S^{-}$
- $\dim \ker D$ mod 2 is a KO-theory characteristic number
- $p : M \to pt.$, $\varphi(E) = p_{!}(E) \in \mathbf{Z}_2$

- L holomorphic degree one line bundle on Σ

- Dirac operator $\sqrt{2}D = \bar{\partial} : (L+L^*)\otimes K^{1/2} \to (L+L^*)\otimes K^{1/2}\bar{K}$

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- Dirac operator $\sqrt{2}D = \bar{\partial} : (L+L^*)\otimes K^{1/2} \to (L+L^*)\otimes K^{1/2}\bar{K}$

- $\dim H^0(\Sigma, LK^{1/2}) + \dim H^0(\Sigma, L^*K^{1/2}) =$

$= \dim H^0(\Sigma, LK^{1/2}) + \dim H^1(\Sigma, LK^{1/2})$

$= (\dim H^0(\Sigma, LK^{1/2}) - \dim H^1(\Sigma, LK^{1/2}))$ mod $2 = 1$

(Riemann-Roch)

- $1 = \varphi(L + L^*) = \varphi[2] + \varphi(u) = \varphi(u) = w_2(u)$

- spin structure $K^{1/2}$, $L^2 \cong K_S \pi^* K^{-1}$ take $K_S^{1/2} = L\pi^* K^{1/2}$

- $\varphi_S(1) = \varphi(\pi_!(1)) = \varphi(V) = \varphi([n]+w_2(V)u) = n\varphi(1)+w_2(V)$

- n even $w_2(V) = \varphi_S(1)$

- n odd $w_2(V) = \varphi_S(1) + \varphi(1)$

EXAMPLE $SL(2,\mathbf{R})$

- Higgs bundle $V = M \oplus M^*$ $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$

  $w_2(V) = c_1(M) \bmod 2$

- spectral curve $0 = \det(x - \Phi) = x^2 - q$ involution $\sigma(x) = -x$

- Prym variety $\mathrm{P}(S, \Sigma) = \{U : \sigma^* U \cong U^*\}$

- $U^2 \cong \mathcal{O} \Rightarrow \sigma^* U \cong U$

- $L = U\pi^* K^{1/2}$

- $\pi_* L = M^+ \oplus M^-$: invariant/anti-invariant local sections

- $c_1(M^+) = g - 1 - k$ where $2k =$ no of fixed points where action of σ on L is -1

- $w_2(V) = (g - 1 - k) \bmod 2$

- How many elements of order 2 give $w_2(V) = 0$?

- How many elements of order 2 give $w_2(V) = 0$?

- fixed points $x = 0 \sim 4g-4$ zeros of quadratic differential q

- $0 = w_2(V) = g-1-k$: g odd $\Rightarrow k$ even

$$\sum_k \binom{4g-4}{4k} =$$

$$= \frac{1}{4}\Big((1+1)^{4g-4} + (1-1)^{4g-4} + (1+i)^{4g-4} + (1-i)^{4g-4}\Big)$$

$$= 2^{4g-6} + 2^{2g-4}((e^{i\pi/4})^{4g-4} + (e^{-i\pi/4})^{4g-4}) = 2^{4g-6} + 2^{2g-3}$$

- No: $= 2^{4g-7} + 2^{2g-4}$

THE CASE $G^r = SU(m,m)$

- $SL(2,\mathbf{R}) \cong SU(1,1)$

- maximal compact $S(U(m) \times U(m))$

- bundle $V = V_+ \oplus V_-$ Higgs field $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$

- $\Lambda^m V_+ \cong \Lambda^m V_-^*$,

- characteristic class $c_1(V_+) \in H^2(\mathbf{\Sigma}, \mathbf{Z})$

L.Schaposnik, *Spectral data for G-Higgs bundles,* arXiv:1301.1981

- $\det(x - \Phi) = x^{2m} + a_1 x^{2m-2} + \ldots + a_m$

- involution $\sigma(x) = -x$ on S

- $L = U\pi^* K^{(2m-1)/2}$, $U \in \mathsf{P}(S, \Sigma)$

- $\sigma^* U \cong U$

- $\det(x - \Phi) = x^{2m} + a_1 x^{2m-2} + \ldots + a_m$
- involution $\sigma(x) = -x$ on S
- $L = U\pi^* K^{(2m-1)/2}$, $U \in \mathsf{P}(S, \Sigma)$
- $\sigma^* U \cong U$
- $c_1(V_+) = m(g-1) - k$

  $2k$ = no of fixed points where action is -1

BRANES

BRANES

- symplectic geometry
- A-brane = Lagrangian submanifold ....+ flat bundle

BRANES

- symplectic geometry

- A-brane = Lagrangian submanifold ....+ flat bundle

- complex geometry

- B-brane = complex submanifold...+ holomorphic bundle

HYPERKÄHLER MANIFOLDS

- complex structures I, J, K
- symplectic structures $\omega_1, \omega_2, \omega_3$
- (B, A, A)-brane: cx wrt I, totally real wrt J, K
- (B, B, B)-brane: HK submanifold + hyperholomorphic bundle

- Mirror symmetry:
- the mirror of a (B, A, A)-brane....
- .... is a (B, B, B)-brane
- What is the mirror of the G^r-character variety?

SYZ MIRROR SYMMETRY

- mirror = fibration by dual abelian varieties

- mirror of $\mathcal{M}(G) = \mathcal{M}(^L G)$

- $SU(m,m) \subset SL(2m, \mathbf{C})$

- $^L SL(2m, \mathbf{C}) = PSL(2m, \mathbf{C})$

- Need a hyperkähler submanifold of $\mathcal{M}(PSL(2m, \mathbf{C}))$

- spectral curve S: $x^{2m}+a_1x^{2m-2}+\ldots+a_m=0$

  quotient curve $\bar{S}=S/\sigma$

- connected component of fibre $\mathsf{P}(\bar{S},\Sigma)\subset\mathsf{P}(S,\Sigma)$

- spectral curve S: $x^{2m} + a_1 x^{2m-2} + \ldots + a_m = 0$

  quotient curve $\bar{S} = S/\sigma$

- connected component of fibre $\mathsf{P}(\bar{S}, \Sigma) \subset \mathsf{P}(S, \Sigma)$

- "annihilator" in dual of $\mathsf{P}(S, \Sigma)$

  $=$ line bundles on $\mathsf{P}(S, \Sigma)$ trivial on $\mathsf{P}(\bar{S}, \Sigma)$

- connected component of fibre $\mathsf{P}(\bar{S},\Sigma) \subset \mathsf{P}(S,\Sigma)$

- dual of $\mathsf{P}(S,\Sigma) = \mathsf{Jac}(S)/\,\mathsf{Jac}(\Sigma)$

- $\mathsf{P}(S,\bar{S})/\mathsf{P}(S,\bar{S}) \cap \mathsf{Jac}(\Sigma) \to \mathsf{Jac}(S)/\,\mathsf{Jac}(\Sigma) \to \mathsf{Jac}(\bar{S})/\,\mathsf{Jac}(\Sigma)$

- $\mathsf{P}(S,\bar{S}) \cap \mathsf{Jac}(\Sigma) = \{U \in \mathsf{Jac}(\Sigma) : \sigma^*\pi^*U \cong \pi^*U^*\} = H^1(\Sigma, \mathbf{Z}_2)$

- spectral curve S: $x^{2m} + a_1 x^{2m-2} + \ldots + a_m = 0$

- Prym variety $\mathrm{P}(S, \bar{S})$

- $=$ moduli space of $Sp(2m, \mathbf{C})$ Higgs bundles

- modulo $H^1(\Sigma, \mathbf{Z}_2) = PSp(2m, \mathbf{C})$ bundles

- hyperkähler submanifold of $PSL(2m, \mathbf{C})$ Higgs bundles

HYPERHOLOMORPHIC BUNDLE

- Higgs bundle equations: dimensional reduction of ASDYM

- ASD connection $A_1 dx_1 + A_2 dx_2 + \phi_1 dx_3 + \phi_2 dx_4$

- $$D^* = \nabla_1 + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \nabla_2 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi_1 + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \phi_2$$

- Dirac operator $$D^* = \begin{pmatrix} \bar{\partial}_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix}$$

$$D^* = \begin{pmatrix} \bar{\partial}_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix} : \begin{pmatrix} V \otimes K \\ V \otimes \bar{K} \end{pmatrix} \to \begin{pmatrix} V \otimes K\bar{K} \\ V \otimes K\bar{K} \end{pmatrix}$$

- $D^*D \sim -\nabla_1^2 - \nabla_2^2 - \phi_1^2 - \phi_2^2 \Rightarrow \ker D = 0$
- index theorem $\Rightarrow \dim \ker D^* = (2g-2)\,\mathrm{rk}\, V$
- $\mathcal{L}^2$ connection is hyperholomorphic

- complex structure I

- $$\begin{array}{ccc} \Omega^{0,p}(V) & \stackrel{\Phi}{\to} & \Omega^{0,p}(V \otimes K) \\ \bar\partial \downarrow & & \bar\partial \downarrow \\ \Omega^{0,p+1}(V) & \stackrel{\Phi}{\to} & \Omega^{0,p+1}(V \otimes K) \end{array}$$

- total differential $\bar\partial \pm \Phi$

- Hodge theory: $\ker D^* \cong$ hypercohomology $\mathbf{H}^1$

HYPERCOHOMOLOGY

- $\mathcal{O}(V) \xrightarrow{\Phi} \mathcal{O}(V \otimes K)$
- ker $D^* \cong$ hypercohomology $\mathbf{H}^1(V)$
- $\pi_*[\mathcal{O}(L) \xrightarrow{\eta} \mathcal{O}(L\pi^*K)]$

$$\mathbf{H}^1(V) \cong \bigoplus_{\{x:\eta(x)=0\}} (L\pi^*K)_x$$

- connected components of $SU(m,m)$-moduli space

  $\sim$ $2k$-element subsets of $\{x : \eta(x) = 0\}$

$$\Lambda^{2k}\mathbf{H}^1(V) \cong \bigoplus_{\{x_i : \eta(x_i)=0\}} (L\pi^*K)_{x_1}(L\pi^*K)_{x_2}\ldots(L\pi^*K)_{x_{2k}}$$

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- **CLAIM**: This is the required hyperholomorphic bundle.

- What about $SL(n, \mathbf{R})$?

- general spectral curve $\Rightarrow$ HK manifold is the full $PSL(2m, \mathbf{C})$-moduli space

- components $w_2 = 0, w_2 = 1$

- What is the hyperholomorphic bundle?

NONABELIANIZATION

- other real forms $U^*(m), Sp(2m, 2m), SO^*(4m)$
- $\det(x - \Phi) = p(x)^2$
- $V = \pi_* U$, U rank 2
- What is the mirror here?