Max Planck Institute for Mathematics
California Institute of Technology

Complex Chern-Simons theory

Arbeitstagung, 27 May 2013

Interview with **Sir Michael Atiyah** on **math**, **physics** and **fun**

What makes a mathematics problem fun for you?

The main thing that interests me in mathematics always is the interconnection between different parts of mathematics, the fact that one problem may have half a dozen different ways of being looked at in different subjects, a bit of algebra, a bit of geometry, a bit of topology. It's this interaction and bridges that interest me.

In this talk, I will explain a connection (motivated from physics) between three seemingly unrelated subjects:

- Quantum and homological invariants of knots and links
- Classical geometry of Higgs bundle moduli spaces
- “Quantum symplectic geometry“, Fukaya category, enumerative invariants, …

Chern-Simons gauge theory

$$S = \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

- non-abelian interacting gauge theory (TQFT)
- has a long history …

- has many applications …
  - to condensed matter physics
  - to string theory
  - to low dimensional topology
  - to quantum information

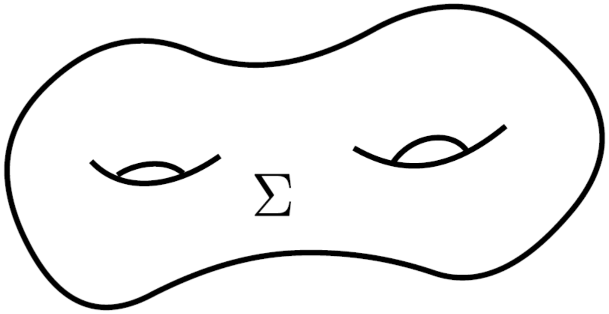
Cutting and Gluing

closed 3-manifold M $\rightsquigarrow$ number $Z(M)$

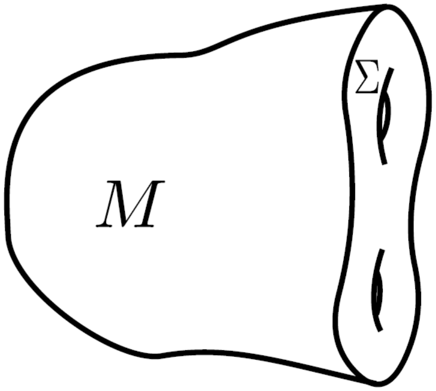
closed 2-manifold Σ $\rightsquigarrow$ vector space $Z(\Sigma)$

closed 1-manifold S^1 $\rightsquigarrow$ category $Z(S^1)$

point p $\rightsquigarrow$ 2-category $Z(p)$.



$\rightsquigarrow \mathcal{H}_\Sigma$



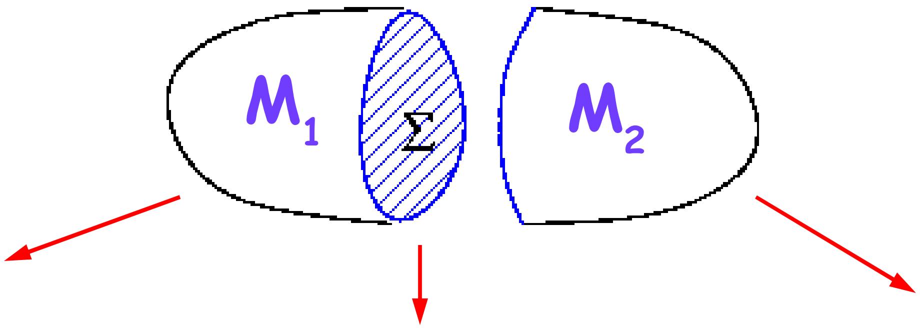
$\rightsquigarrow Z(M) \in \mathcal{H}_\Sigma$

see Kenji Ueno's talk

Cutting and Gluing

M.Atiyah, G.Segal

In three-dimensional TQFT:



vector

$Z_{M_1} \in \mathcal{H}_\Sigma$

vector space

$\mathcal{H}_\Sigma$

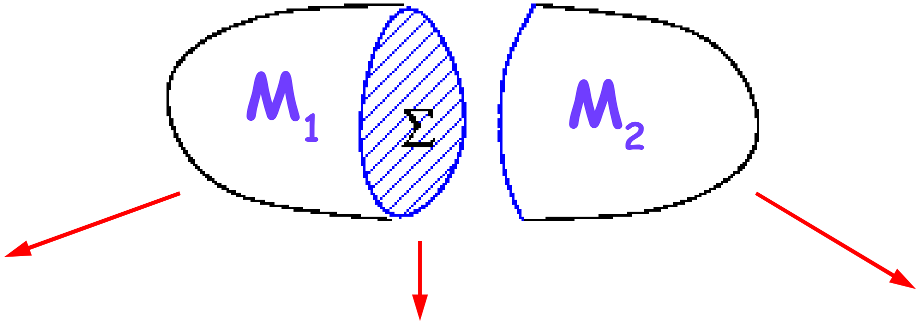
vector

$Z_{M_2} \in \mathcal{H}_\Sigma$

Cutting and Gluing

M.Atiyah, G.Segal

In three-dimensional TQFT:



vector

$Z_{M_1} \in \mathcal{H}_\Sigma$

vector space

$\mathcal{H}_\Sigma$

vector

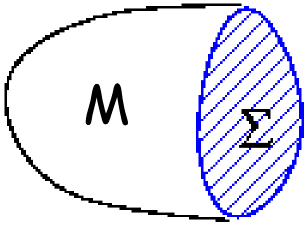
$Z_{M_2} \in \mathcal{H}_\Sigma$

$$M = M_1 \cup M_2 \quad \Rightarrow \quad Z(M) = \langle Z_{M_1} | Z_{M_2} \rangle$$

Chern-Simons gauge theory

$$S = \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

M = 3-manifold (possibly with boundary)



$$Z(M) = \int e^{-S/\hbar} \, \mathcal{D}A$$

"quantum invariant" of **M**

[N.Reshetikhin, V.Turaev]
[E.Witten]

- depends on the choice of the gauge group
- depends on the "coupling constant" $\hbar$

$$\boxed{q = e^{\hbar}}$$

Gauge Group

G = (simple) compact Lie group $SU(2)$

- $\mathcal{H}_\Sigma$ finite-dimensional
- unitary representations *discrete*

$G_\mathbb{C}$ = complexification of G $SL(2,C)$

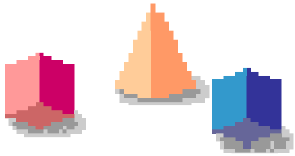
- $\mathcal{H}_\Sigma$ infinite-dimensional
- unitary representations *continuous*

Gauge Group

G = (simple) compact Lie group

- $\mathcal{H}_\Sigma$ finite-dimensional
- unitary representations *discrete*

⇒ state sum model for $Z(M)$



$G_\mathbb{C}$ = complexification of G

- $\mathcal{H}_\Sigma$ infinite-dimensional
- unitary representations *continuous*

⇒ state integral model for $Z(M)$

The role of q

compact G : q = root of 1

complex $G_{\mathbb{C}}$: $q \in \mathbb{C}$

modularity?

(cf. $q = \exp(2\pi i \tau)$ $\tau \to -\frac{1}{\tau}$)

- Surprising hidden symmetry:

$$G \to {}^L G \qquad \hbar \to {}^L\hbar = -\frac{4\pi^2}{\hbar}$$

The role of q

Galois representations of G	↔	automorphic representations of ^{L}G
U(N)		U(N)
SO(2N)		SO(2N)
SO(2N+1)		Sp(2N)
E6		E6/Z_3
E8		E8

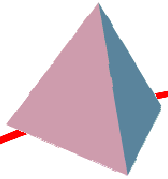
Robert Langlands

- Surprising hidden symmetry:

$$G \rightarrow {}^L G \qquad \hbar \rightarrow {}^L \hbar = -\frac{4\pi^2}{\hbar}$$

Computing $G_{\mathbb{C}}$ partition functions

- Dehn surgeries
- triangulations



quantum dilogarithm

$$Z^{CS}(M;\hbar) = \int_{C_\rho} \prod_{j=1}^{N} \Phi_\hbar(\Delta_j)^{\pm 1} \prod_{i=1}^{N-b_0(\Sigma)} \frac{dp_i}{\sqrt{4\pi\hbar}}$$

choice of contour

$$\overset{\hbar\to 0}{\sim} \exp\left(\frac{1}{\hbar}S_0 + S_1 + \hbar S_2 + \dots\right)$$

Ray-Singer torsion of M

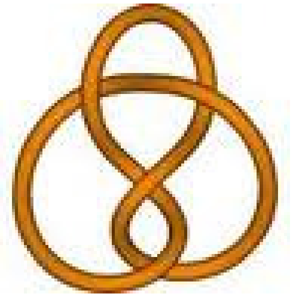
arithmetic of M

For details see e.g. arXiv:0903.2472
with T.Dimofte, J.Lenells, D.Zagier

"Looking back"

knot K

[R. Kashaev, 1996]



$\longrightarrow$

invariant $\langle K \rangle_n \in \mathbb{C}$

labeled by a positive integer n

- defined via R-matrix
- very hard to compute

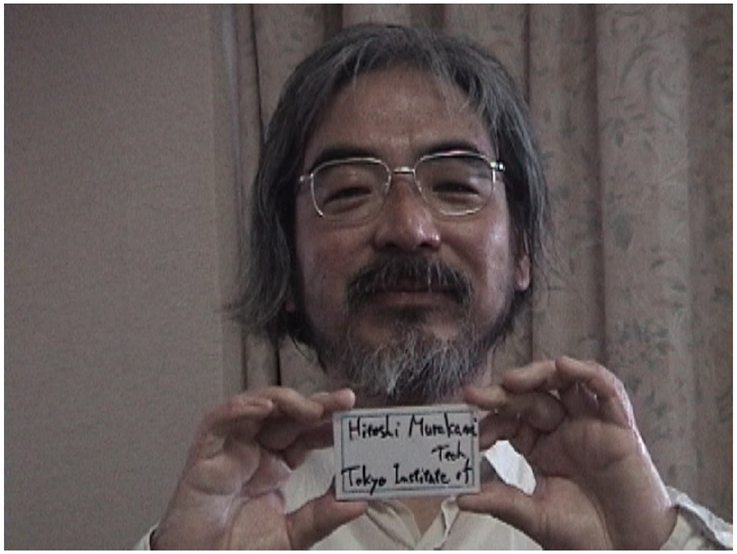
$$\lim_{n\to\infty} \frac{1}{n} \log \langle K \rangle_n = \mathrm{Vol}\,(S^3 \setminus K)$$

("volume conjecture")

A first step to understanding the Volume Conjecture

$\langle K \rangle_n = J_n(q)$ colored Jones polynomial

with $q = \exp(2\pi i/n)$



Hitoshi Murakami

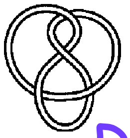
Jun Murakami (1999)

Colored Jones polynomial

$J_2(q) = J(q) =$ Jones polynomial

- In Chern-Simons TQFT [E.Witten, 1989]

Wilson loop operator

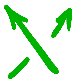
$\langle$  $\rangle_R$ = polynomial in q

R ← 2-dimn'l representation of $SU(2)$

Colored Jones polynomial

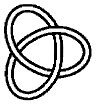
$J_2(q) = J(q) =$ Jones polynomial

- Skein relations:

$q^2 J($  $) - q^{-2} J($  $) = (q^{-1} - q) \cdot J($  $)$

$J($unknot$) = q^{-1} + q$

Example:

$J($  $) = q + q^3 + q^5 - q^9$

Colored Jones polynomial

knot **K** $\longrightarrow$ **n**-colored Jones polynomial:

$J_n(K;q) \in \mathbb{Z}[q, q^{-1}]$

R = **n**-dimn'l representation of **SU(2)**

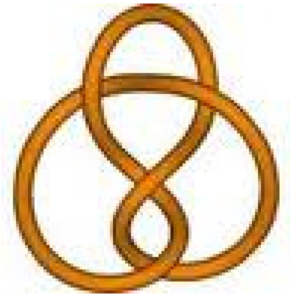
- "Cabling formula":

$$J_{\oplus_i R_i}(K;q) = \sum_i J_{R_i}(K;q)$$

$$J_R(K^n;q) = J_{R^{\otimes n}}(K;q) \, ,$$

Colored Jones polynomial

knot **K** → **n**-colored Jones polynomial:



$J_n(K;q) \in \mathbb{Z}[q, q^{-1}]$

R = **n**-dimn'l representation of **SU(2)**

$$\begin{aligned} & J_1(K;q) = 1\,, \\ & J_2(K;q) = J(K;q)\,, \\ \mathbf{2}^{\otimes 2} = \mathbf{1} \oplus \mathbf{3} \Rightarrow\ & J_3(K;q) = J(K^2;q) - 1\,, \\ \mathbf{2}^{\otimes 3} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{4} \Rightarrow\ & J_4(K;q) = J(K^3;q) - 2J(K;q) \\ & \ldots, \end{aligned}$$

Volume Conjecture

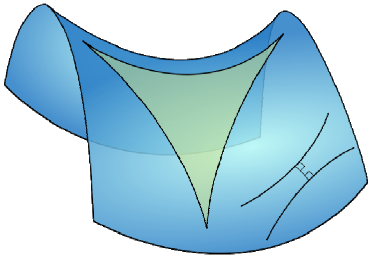
Murakami & Murakami:

cf. Arf(K) = J(i)

$$\langle K\rangle_n = J_n(K; q = e^{2\pi i/n})$$

$$\lim_{n\to\infty} \frac{2\pi \log|J_n(K; q = e^{2\pi i/n})|}{n} = \mathrm{Vol}(M)$$

quantum group invariants (combinatorics, representation theory) ⟷ classical hyperbolic geometry



Interpretation in Chern-Simons theory

- analytic continuation of **SU(2)** is **SL(2,ℂ)**

$$\lim_{n\to\infty} \frac{2\pi \log |J_n(K; q = e^{2\pi i/n})|}{n} = \mathrm{Vol}(M)$$

- constant negative curvature metric on **M** → flat **SL(2,ℂ)** connection on **M = S^3 \ K**

$$\boxed{R_{ij} = -2g_{ij}}$$

$$\boxed{d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0}$$

Large Color Limit

Moral:

$$\lim_{n\to\infty}\left(\text{SU(2) Chern-Simons}\right) \simeq \text{SL(2,C) Chern-Simons}$$

Classical limit **$q = \exp(2\pi i/n) \to 1$**

- leads to many generalizations...

$$q = e^{\hbar} \to 1\,, \qquad n \to \infty\,, \qquad q^n = \boxed{x} \;\; (\text{fixed})$$

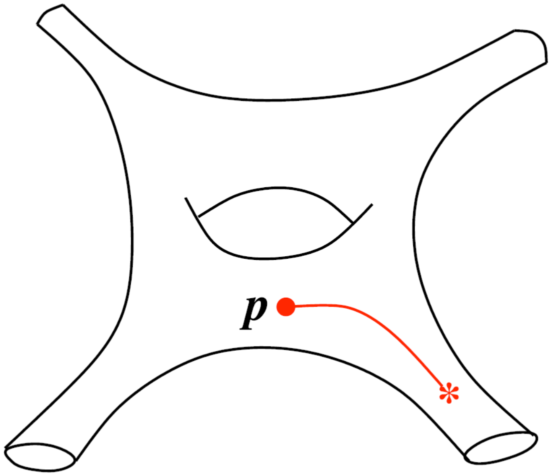
Knots and Algebraic Curves

Generalized Volume Conjecture:

$$\boxed{J_n(K; q = e^{\hbar}) \overset{\substack{n\to\infty \\ \hbar\to 0}}{\sim} \exp\left(\frac{1}{\hbar}\int \log y \frac{dx}{x} + \ldots\right)}$$

where

$$x = q^n = \text{fixed}$$



planar algebraic
curve

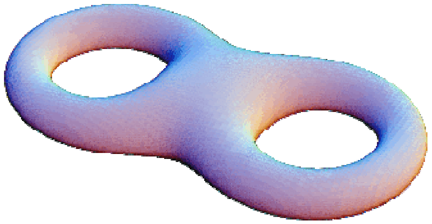
$A(x,y) = 0$

Classical A-polynomial

[D.Cooper, M.Culler, H.Gillet, D.Long, P.B.Shalen]

M = 3-manifold
with a toral boundary,
e.g. a knot complement

planar algebraic curve:



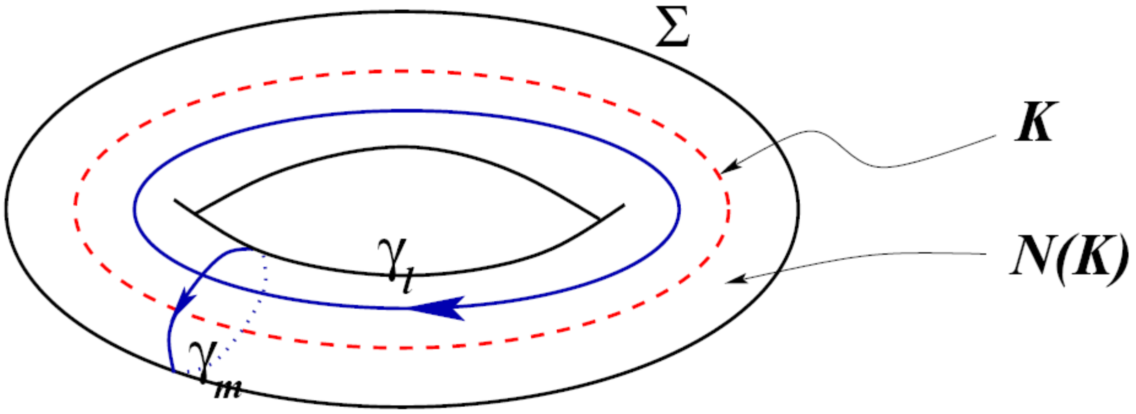
A-polynomial
of a knot K

$$\left\{(x,y)\in\mathbb{C}^*\times\mathbb{C}^*\,\middle|\,\underline{A(x,y)\ =\ 0}\right\}$$

representation
variety:

$$\rho\colon\pi_1(M)\to SL(2,\mathbb{C})$$

Consider, for a example, a knot complement:



$$\rho(\gamma_l) = \begin{pmatrix} y & * \\ 0 & y^{-1} \end{pmatrix} \quad , \quad \rho(\gamma_m) = \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix}$$

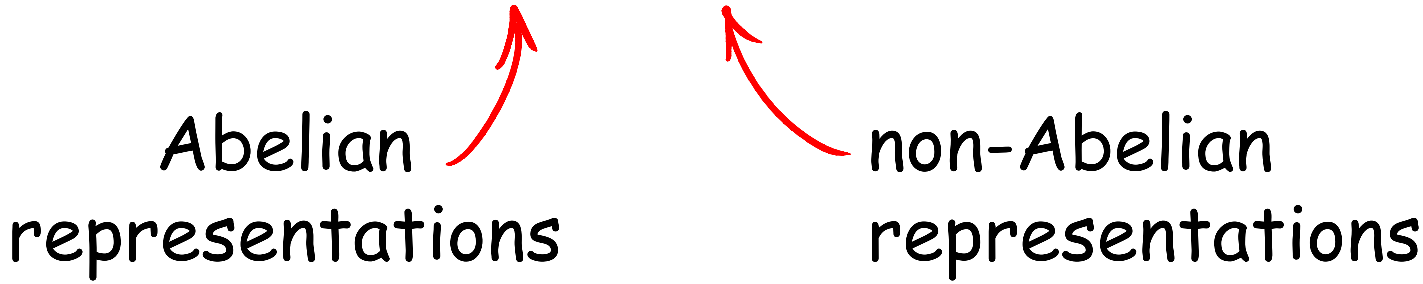
$$\pi_1 = < a, b \mid a\, b\, a = b\, a\, b >$$

$$\begin{cases} m = a \\ \ell = b\, a^2\, b\, a^{-4} \end{cases} \quad \Rightarrow \quad A(x,y) = (y-1)(y+x^3)$$

Properties of the A-polynomial

$H_1(M) \cong \mathbb{Z}$ for any knot complement

⇨ $A(x,y) = (y-1)\,(\;\ldots\;)$



Abelian representations

non-Abelian representations

- If **K** is a hyperbolic knot, then $A(x,y) \neq y-1$.
- If **K** is a knot in a homology sphere, then the A-polynomial involves only even powers of x.

Properties of the A-polynomial

- A-polynomial can distinguish mirror knots:

$$A(x,y) = 0 \xleftrightarrow{\text{parity}} A(x^{-1},y) = 0$$

- If **K** is a hyperbolic knot, then $A(x,y) \neq y-1$.
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Properties of the A-polynomial

- A-polynomial can distinguish mirror knots:

$$A(x,y) = 0 \quad \xleftrightarrow{\text{parity}} \quad A(x^{-1},y) = 0$$

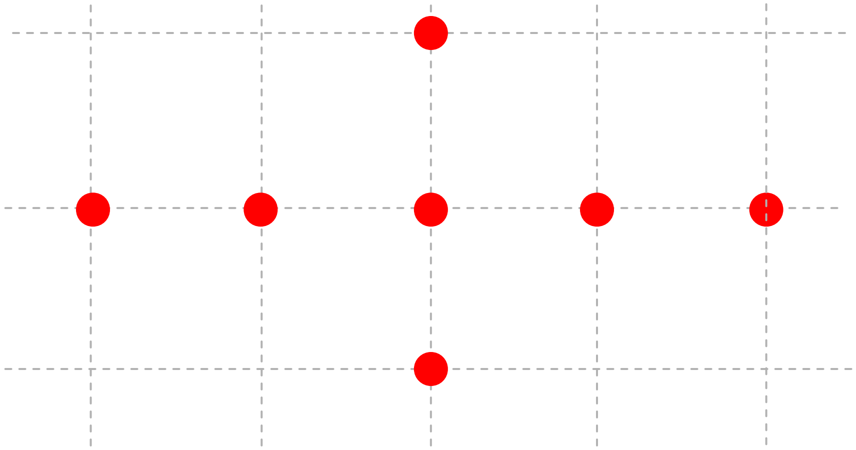
- The A-polynomial is reciprocal:

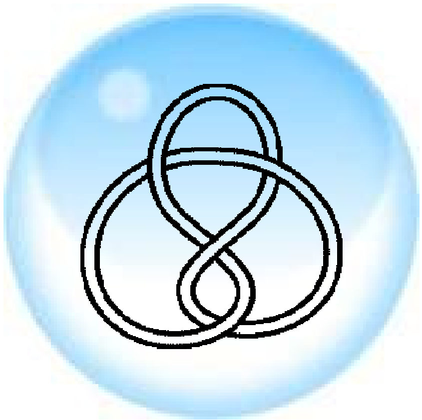
$$A(x,y) \sim A(x^{-1},y^{-1})$$

- The A-polynomial has integer coefficients

Properties of the A-polynomial

- The A-polynomial is tempered, *i.e.* the faces of the Newton polygon of $A(x,y)$ define cyclotomic polynomials in one variable:

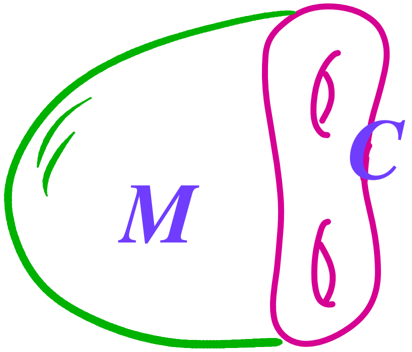




- The slopes of the sides of the Newton polygon of $A(x,y)$ are boundary slopes of incompressible surfaces in **M**.

Branes in Hitchin moduli space

M = 3-manifold with boundary C (= genus-g Riemann surface)

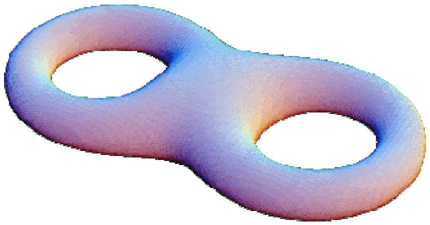


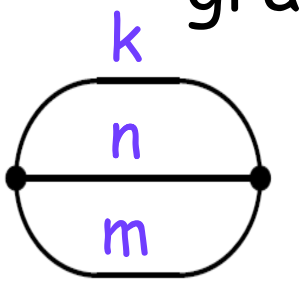
Example: $g=1$

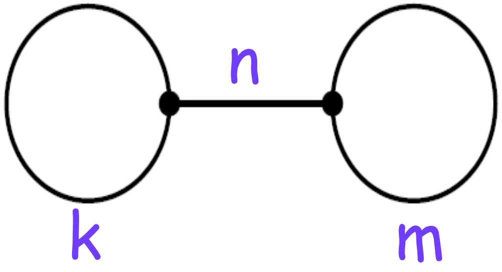
knot complement

Example: $g=2$

graph complement







Branes in Hitchin moduli space

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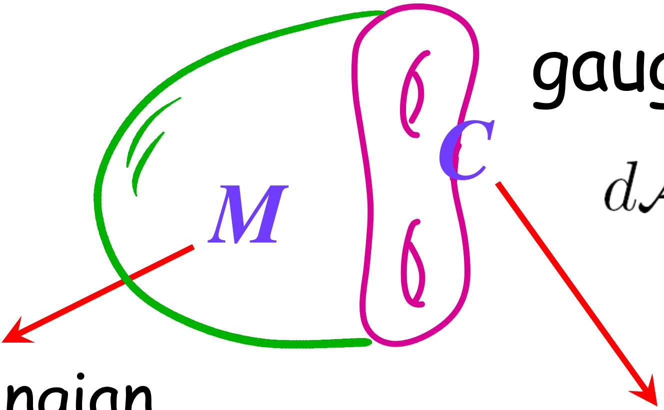


gauge theory on M

$$dA + A \wedge A \; = \; 0$$

Lagrangian submanifold

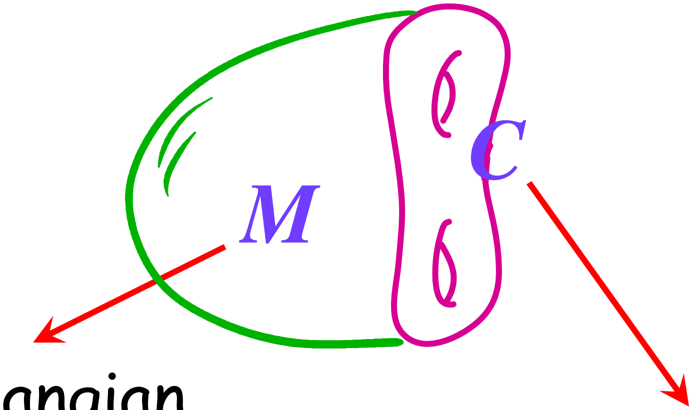
hyper-Kahler manifold

$$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M) \subset \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, C) \cong \mathcal{M}_H(G, C)$$

with respect to $\Omega_J = \omega_K + i\omega_I = \dfrac{1}{4\pi^2\hbar}\displaystyle\int_C \text{Tr}\delta\mathcal{A} \wedge \delta\mathcal{A}$

Branes in Hitchin moduli space

M = 3-manifold with boundary ***C*** (= genus-g Riemann surface)



Lagrangian submanifold

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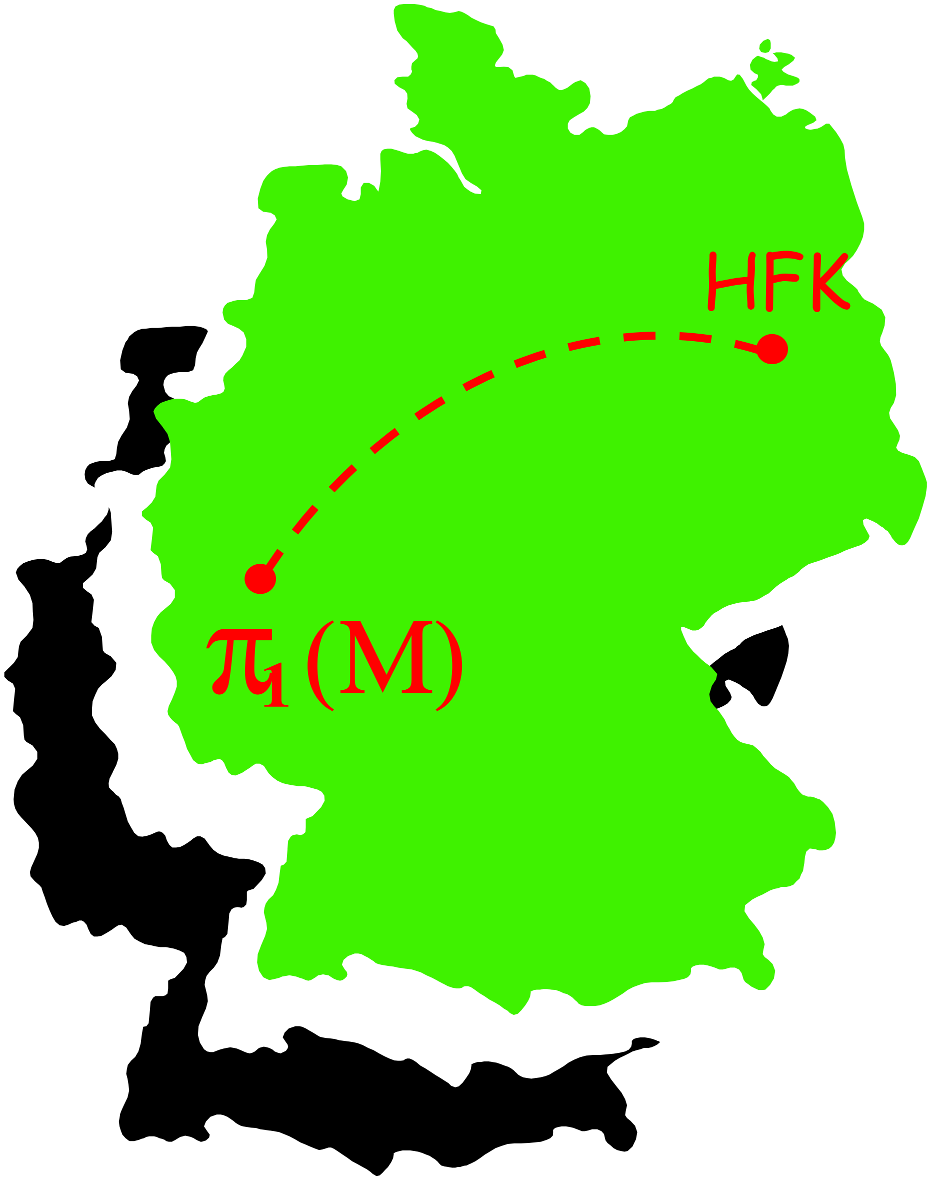
with respect to $\Omega_J = \omega_K + i\omega_I$ ➡ (A,B,A) brane !

Lessons

- A-polynomial as a limit shape (in large color limit)
- the A-polynomial curve should be viewed as a holomorphic Lagrangian submanifold (as opposed to a complex equation) in moduli space of Higgs bundles
- its quantization with symplectic form $\frac{dy}{y} \wedge \frac{dx}{x}$ leads to an interesting wave function
- has all the attributes to be an analog of the Seiberg-Witten curve for knots and 3-manifolds
- Generalizations!

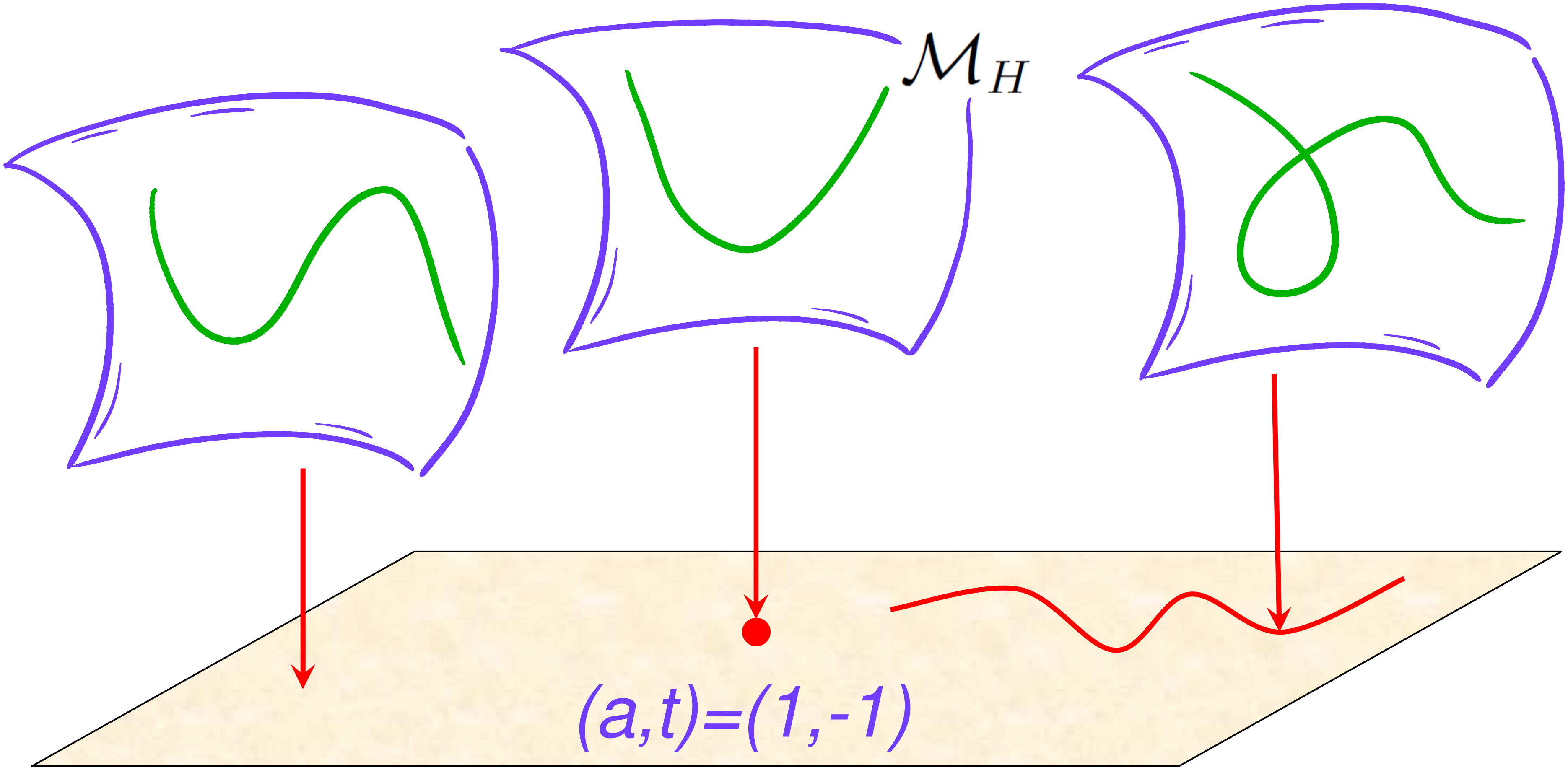
see Zoltan Szabo's talk

From old to new ...



"Looking Forward"

- Two commutative deformations:



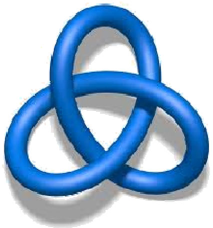
"Looking Forward"

- Two commutative deformations:

$$A(x,y) \longrightarrow A^{\text{super}}(x,y;a,t)$$

Example:

$$A(x,y) = (y-1)(y+x^3)$$



$$\Big\downarrow \ (a,t)\text{ -deformation}$$

$$y^2 - \frac{a(1-t^2x+2t^2(1+at)x^2+at^5x^3+a^2t^6x^4)}{1+at^3x}y + \frac{a^2t^4(x-1)x^3}{1+at^3x}$$

"Looking Forward"

- Two commutative deformations:

$$A(x,y) \longrightarrow A^{\text{super}}(x,y;a,t)$$

- One non-commutative deformation:

$$x\ ,\ y \quad \rightsquigarrow \quad \widehat{x}\ ,\ \widehat{y}$$

$$\Omega_J = \frac{dy}{y} \wedge \frac{dx}{x} \qquad\qquad \widehat{y}\widehat{x} = \enclose{circle}{q}\,\widehat{x}\widehat{y}$$

$$A^{\text{super}}(x,y;a,t) \longrightarrow \hat{A}^{\text{super}}(\hat{x},\hat{y};a,q,t)$$

Deformation and Quantization

using $x = q^n$ and $\widehat{y} P_n = P_{n+1}$

we obtain the following recursion relation:

$$\hat{A}^{\text{super}} P_n(a,q,t) = 0$$

- One non-commutative deformation:

$$x \, , \, y \quad \rightsquigarrow \quad \widehat{x} \, , \, \widehat{y}$$

$$\Omega_J = \frac{dy}{y} \wedge \frac{dx}{x} \qquad \widehat{y}\widehat{x} = \boxed{q}\,\widehat{x}\widehat{y}$$

$$A^{\text{super}}(x,y;a,t) = 0 \longrightarrow \hat{A}^{\text{super}}(\hat{x},\hat{y};a,q,t)\, P = 0$$

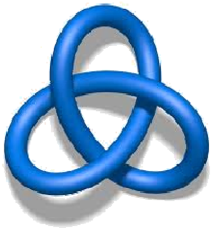
Deformation and Quantization

using $x = q^n$ and $\widehat{y} P_n = P_{n+1}$

we obtain the following recursion relation:

$$\hat{A}^{\text{super}} P_n (a,q,t) = 0$$

Example: $\widehat{A}^{\text{super}}(\widehat{x}, \widehat{y}; a, q, t) = \alpha + \beta \widehat{y} + \gamma \widehat{y}^2$



$$\Rightarrow \quad \alpha P_n + \beta P_{n+1} + \gamma P_{n+2} = 0$$

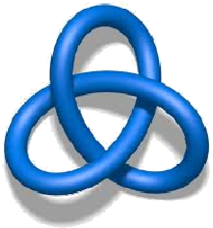
rational functions of a, q, $x=q^n$, and t

Deformation and Quantization

Let's try to solve this recursion relation with

$P_n(a,q,t) = 0$ for $n < 1$ and $P_1(a,q,t) = 1$

Example: $$\widehat{A}^{\text{super}}(\widehat{x},\widehat{y};a,q,t) = \alpha + \beta\widehat{y} + \gamma\widehat{y}^2$$



$$\Rightarrow \quad \underset{\uparrow}{\alpha} P_n + \underset{\uparrow}{\beta} P_{n+1} + \underset{\uparrow}{\gamma} P_{n+2} = 0$$

rational functions of a, q, $x=q^n$, and t

What is $P(a,q,t)$?

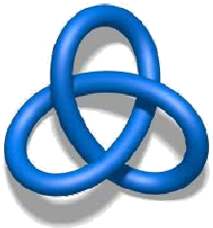
Let's try to solve this recursion relation with

$P_n(a,q,t) = 0$ for $n < 1$ and $P_1(a,q,t) = 1$

n	$P_n(a,q,t)$
1	1
2	$aq^{-1} + aqt^2 + a^2t^3$
3	$a^2q^{-2} + a^2q(1+q)t^2 + a^3(1+q)t^3 + a^2q^4t^4 + a^3q^3(1+q)t^5 + a^4q^3t^6$
4	$a^3q^{-3} + a^3q(1+q+q^2)t^2 + a^4(1+q+q^2)t^3 + a^3q^5(1+q+q^2)t^4 +$ $+ a^4q^4(1+q)(1+q+q^2)t^5 + a^3q^4(a^2+a^2q+a^2q^2+q^5)t^6 +$ $+ a^4q^8(1+q+q^2)t^7 + a^5q^8(1+q+q^2)t^8 + a^6q^9t^9$

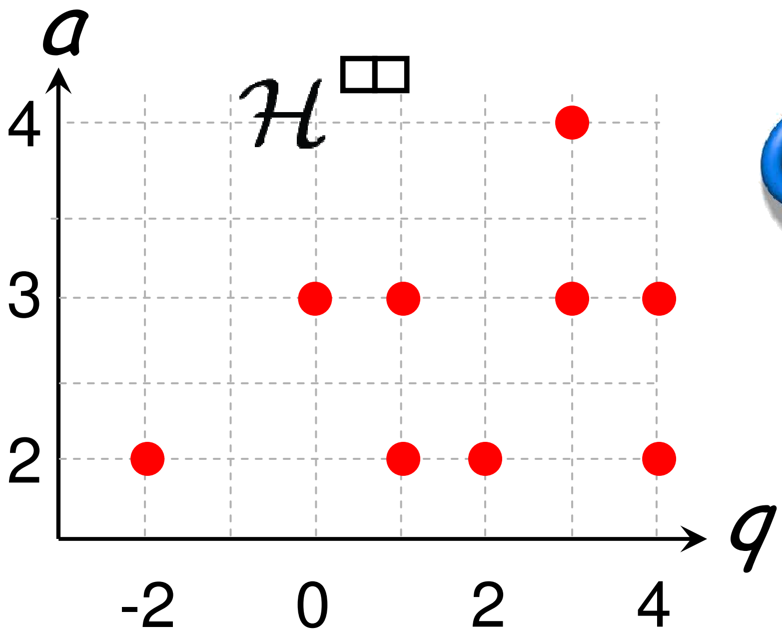
What is $P(a,q,t)$?

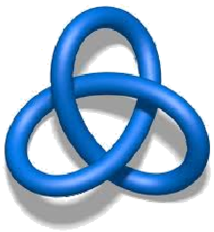
Note, all $P_n(a,q,t)$ involve only positive integer coefficients

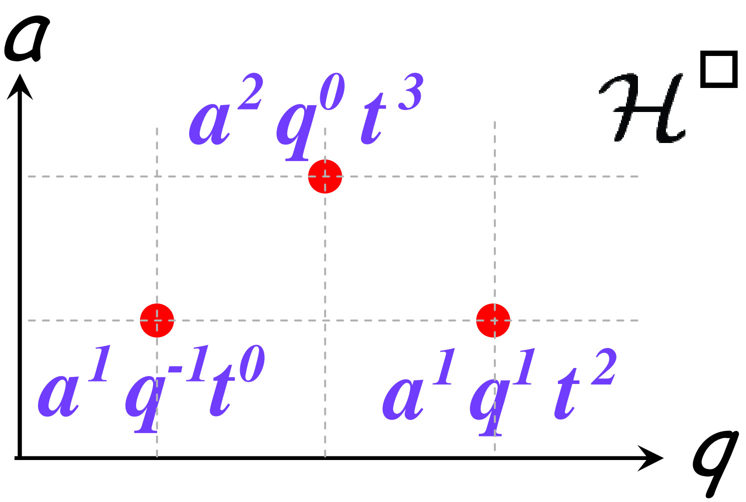


n	$P_n(a,q,t)$
1	1
2	$aq^{-1} + aqt^2 + a^2t^3$
3	$a^2q^{-2} + a^2q(1+q)t^2 + a^3(1+q)t^3 + a^2q^4t^4 + a^3q^3(1+q)t^5 + a^4q^3t^6$
4	$a^3q^{-3} + a^3q(1+q+q^2)t^2 + a^4(1+q+q^2)t^3 + a^3q^5(1+q+q^2)t^4 +$ $+a^4q^4(1+q)(1+q+q^2)t^5 + a^3q^4(a^2+a^2q+a^2q^2+q^5)t^6 +$ $+a^4q^8(1+q+q^2)t^7 + a^5q^8(1+q+q^2)t^8 + a^6q^9t^9$

Colored HOMFLY homology







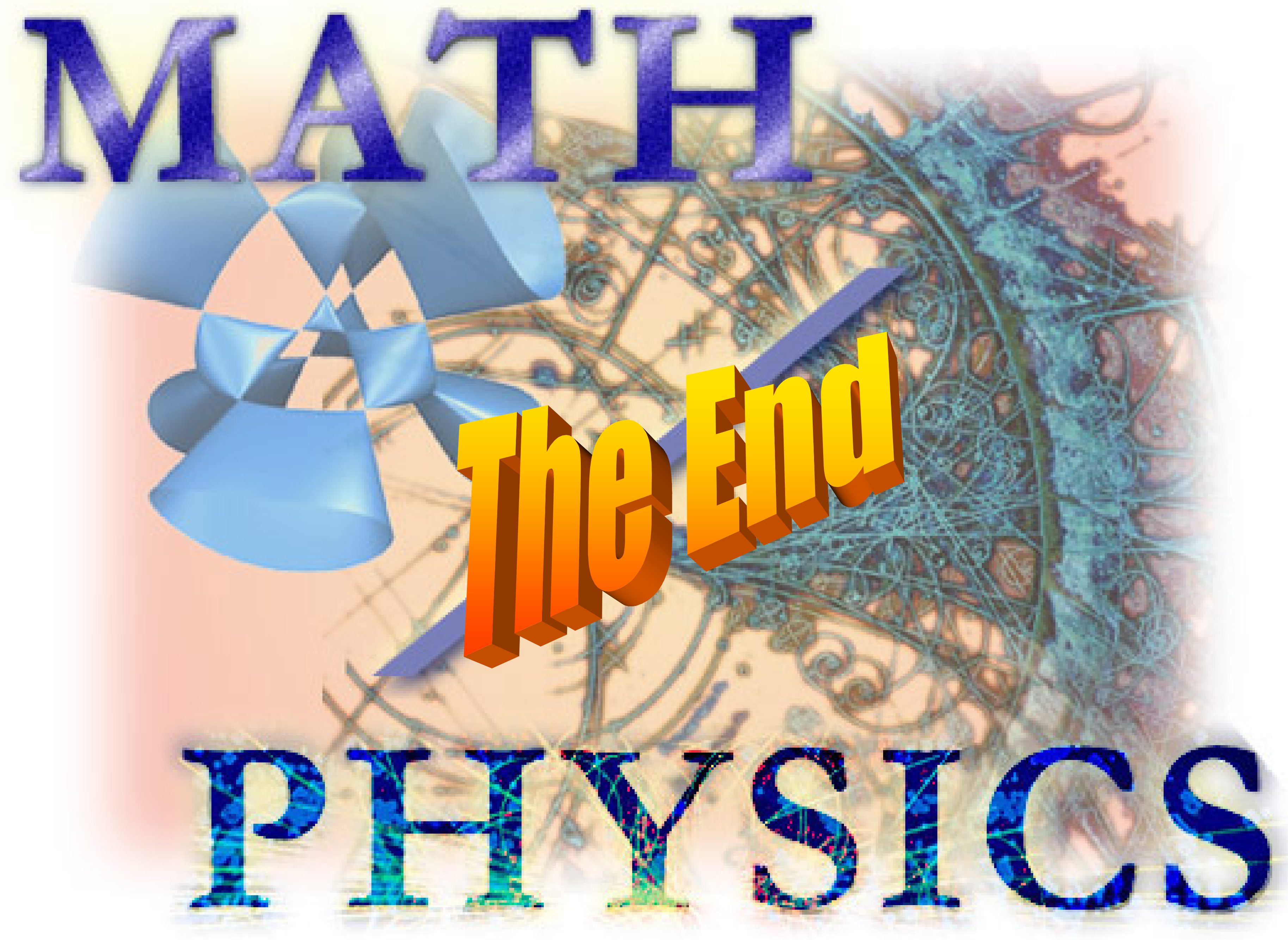
n	$P_n(a,q,t)$
1	1
2	$aq^{-1} + aqt^2 + a^2t^3$
3	$a^2q^{-2} + a^2q(1+q)t^2 + a^3(1+q)t^3 + a^2q^4t^4 + a^3q^3(1+q)t^5 + a^4q^3t^6$

Colored Recursions

- colored Jones polynomial **$J_n(q)$**
  - mathematically well defined for all **n**
- colored sl(2) homology
  - mathematical definitions (!) for all **n**
- colored HOMFLY-PT polynomial $P_n(a,q)$
  - mathematical definition for all **n**
- colored HOMFLY homology

$$P_n(a,q,t) = P\,(\mathcal{H}_{\underbrace{\square\square\square\square\square\square\square\square\square\square}_{n}})$$

- defined for **n=1** / conjectured for all **n**


MATH
The End
PHYSICS