Celebrating

One Hundred Fifty Years of

Topology

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ARBEITSTAGUNG

Bonn, May 22, 2013

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LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primum Græcè & Latinè editi, atque absolutissimis
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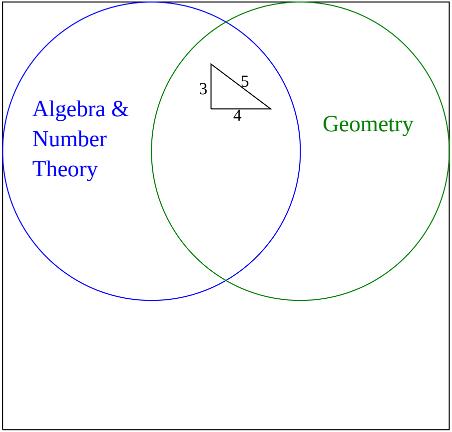
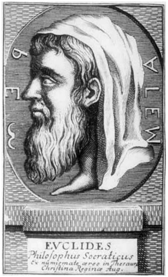
AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V.C.
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via
Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.



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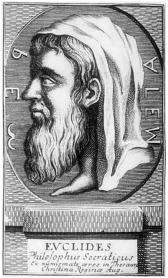

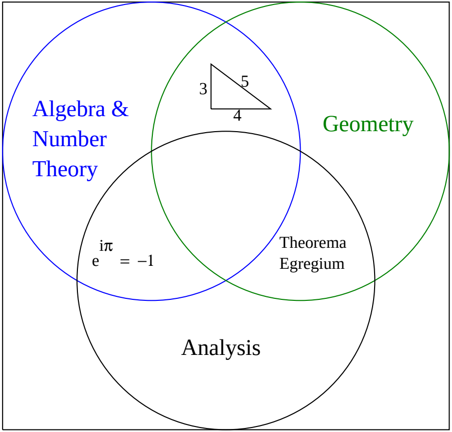

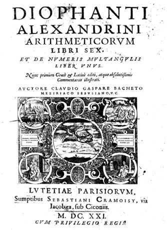

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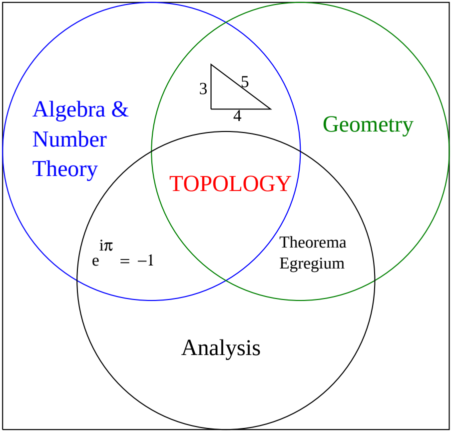
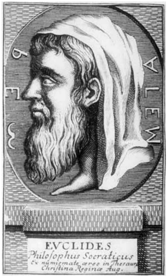
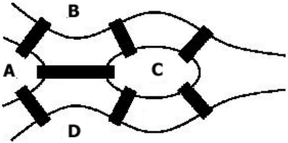




When did topology start?

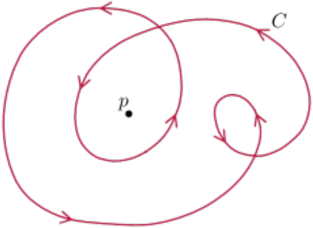


The bridges of Konigsberg

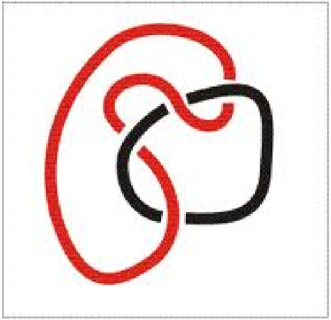
1736

$$V - E + F = 2$$

Cauchy, 1825



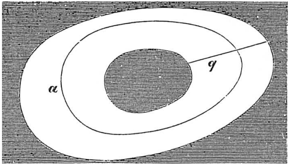
$$W_C(p) = \frac{1}{2\pi i} \oint_C \frac{dz}{z - p}$$



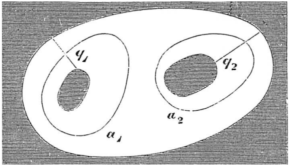
$$L = \frac{1}{4\pi} \int\int_{x,y} \frac{(x - y) \cdot (dx \times dy)}{\|x - y\|^3}$$

1833

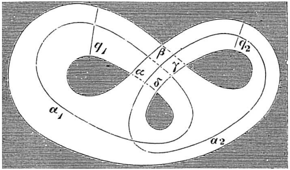
Riemann, 1857



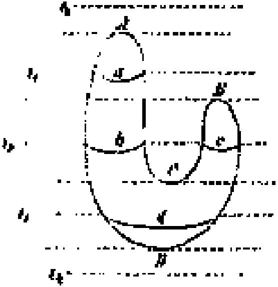
doubly connected

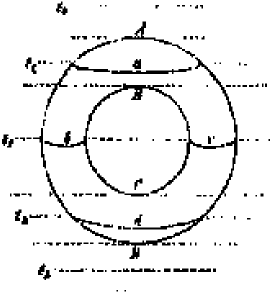


triply connected



triply connected





August Ferdinand Möbius

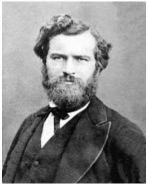
The Möbius Classification of surfaces in $\mathbb{R}^3$: 1863

Definition of the “class” of a surface:

On a closed surface of the ***n-th class*** [= genus $n-1$] , there exist $n-1$ closed curves which do not disconnect the surface.

Theorem. Any two closed surfaces of the same class are elementarily related.

> *Two geometric figures will be called **“elementarily related”** if to any infinitely small element of any dimension in one figure there corresponds an infinitely small element in the other figure, such that two neighboring elements in one figure correspond to two elements in the other which also come together;* $\cdots$



Camille Jordan, 1877,

Jordan curve theorem.

Walther von Dyck, 1888: Topology studies properties invariant under continuous functions with continuous inverse.

$$\chi(M) = \frac{1}{2\pi} \int\int K \, dA \, .$$

Henri Poincaré, 1892–1904,

homology, Betti numbers, duality, homotopy, fundamental group, covering spaces, $\cdots$

20^{th} century

Felix Hausdorff

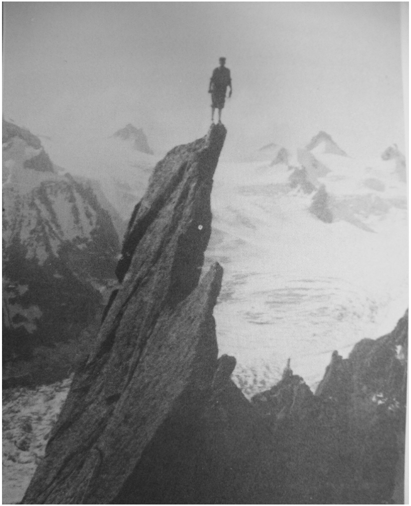
L. E. J.
Brouwer

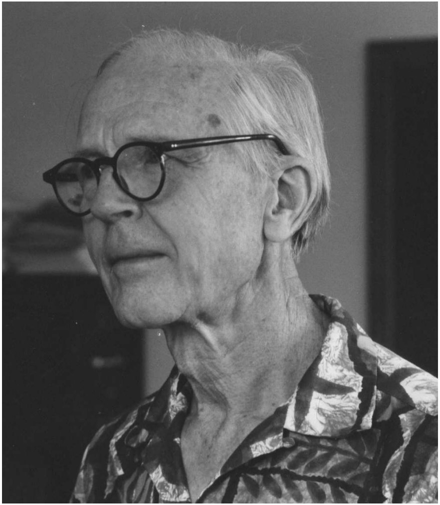
H. Kneser and
H. Hopf

Solomon Lefschetz and James Alexander

The Alexander Chimney in Colorado

Hassler Whitney



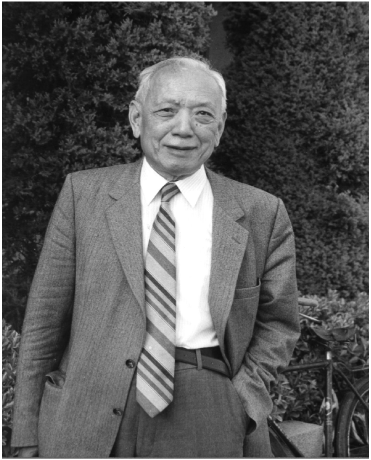


$E \xrightarrow{S^{n-1}} X \quad \mapsto \quad$ characteristic classes $\quad w_i \in H^i(X;\ \mathbb{Z}/2)$.

Lev Pontryagin

ξ real vector bundle over X

$\mapsto \quad p_j(\xi) \in H^{4j}(X;\ \mathbb{Z})$

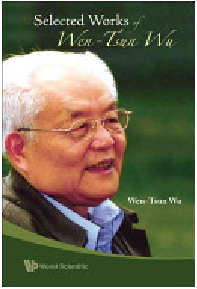


Shiing-Shen Chern

γ complex vector bundle over X

$\mapsto \quad c_j(\gamma) \in H^{2j}(X;\ \mathbb{Z})$

Many people put these into more modern form



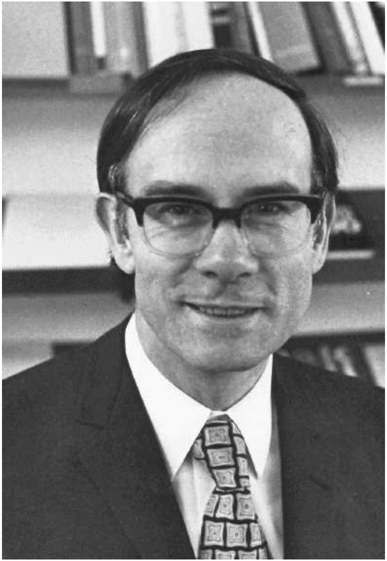
Wu Wen-Tsün

$$H^*(B_{GL(\mathbb{R})};\ \mathbb{Z}/2) \cong (\mathbb{Z}/2)[w_1,\ w_2,\ w_3,\ \cdots]$$

$$H^*(B_{GL(\mathbb{C})};\ \mathbb{Z}) \cong \mathbb{Z}[c_1,\ c_2,\ c_3,\ \cdots]$$

$$H^*(B_{GL(\mathbb{R})};\ \mathbb{Z}) \cong \mathbb{Z}[p_1,\ p_2,\ \cdots] \oplus (2-\text{torsion})$$

Neue topologische Methoden · · ·



Todd genus (≈ arithmetic genus)

Francesco Severi

David Hilbert

J. A. Todd

Lemma. $\exists$ a unique

$$\mathbf{T} \;=\; 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^{\,2} + c_2) + \tfrac{1}{24}c_1 c_2 + \cdots \;\in\; H^{\Pi}(B_{GL(\mathbb{C})};\mathbb{Q}) \;,$$

such that the "genus" $T(V_n) \;=\; \mathbf{T}(\tau_{V_n})[V_n]$ is **multiplicative**:

$$T(V \times V') = T(V)\cdot T(V')\;, \qquad \text{with}\;\; T\big(P_n(\mathbb{C})\big) = +1\;.$$

Theorem.

$$T(V_n) \;=\; \sum_{k=0}^{n}(-1)^k \dim_{\mathbb{C}}\{\text{holomorphic}\;\; k\text{–forms}\}\;.$$

Some notation

If γ is a holomorphic vector bundle over V, let (γ) denote the sheaf of germs of local holomorphic sections. Let **1** denote the trivial line bundle.

Pierre Dolbeault:

$$\text{Theorem}: \; H^k\big(V; (\mathbf{1})\big) \;\cong\; \{\text{holomorphic } k\text{-forms}\}\,.$$

Hence $\quad T(V) \;=\; \sum_k (-1)^k \dim_{\mathbb{C}} H^k\big(V;\; (\mathbf{1})\big)\,.$

Classical Riemann-Roch Theorems

Gustav Roch

Max Noether

Andre Weil

The **Chern character** of a complex n-plane bundle over X

$$\mathbf{ch}(\gamma_n) = n + \frac{c_1}{1\,!} + \frac{c_1^{\,2} - 2c_2}{2\,!} + \frac{c_1^{\,3} - 3c_1 c_2 + 3c_3}{3\,!} + \cdots$$

is an element of $H^{\Pi}(X;\ \mathbb{Q})$ characterized by two properties:

$$\mathbf{ch}(\gamma_1) = e^{c_1(\gamma_1)}\,.$$

$$\mathbf{ch}(\gamma_m \oplus \gamma'_n) = \mathbf{ch}(\gamma_m) + \mathbf{ch}(\gamma'_n)\,,$$

$$\Longrightarrow \quad \mathbf{ch}(\gamma_m \otimes \gamma'_n) = \mathbf{ch}(\gamma_m)\,\mathbf{ch}(\gamma'_n)\,.$$

Hirzebruch's Riemann-Roch Theorem: For any holomorphic vector bundle γ over V,

$$\sum_k (-1)^k \dim_{\mathbb{C}} H^k(V;\ (\gamma)) = \Big(\mathbf{ch}(\gamma)\,\mathbf{T}(\tau_V)\Big)[V]\,.$$

Rene Thom's cobordism theory was based on deep geometric intuition, plus hard algebraic topology. Essential ingredients:

Jean-Pierre Serre
on spectral
sequences

Norman Steenrod
on cohomology
operations

The L-genus of an oriented $4n$-manifold

Hirzebruch showed that there is one and only one sum

$$\mathbf{L} \;=\; 1 + \frac{p_1}{3} + \frac{7p_2 - p_1^2}{45} + \cdots \;\in\; H^{\Pi}(B_{GL(\mathbb{R})};\ \mathbb{Q})\ ,$$

such that the "L-genus", $L(M^{4n}) \;=\; \mathbf{L}(\tau_{M^{4n}})[M^{4n}]$,

is multiplicative $L(M \times M') \;= L(M)\, L(M')$,

with $L\big(\mathbb{P}_{2n}(\mathbb{C})\big) = +1$.

Theorem (Thom, Hirzebruch). The L-genus $L(M^{4k})$ is equal to the signature of the quadratic form

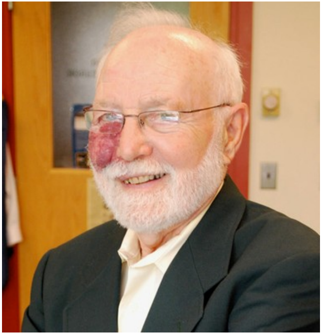
$$\begin{array}{ccc} H^{2k}(M^{4k};\ \mathbb{Q}) & \longrightarrow & \mathbb{Q} \\ x & \mapsto & (x \cup x)[M^{4k}]\ . \end{array}$$

Hirzebruch also defined the $\widehat{A}$-genus $\widehat{A}(M^{4k})$, where

$$\widehat{\mathbf{A}} = 1 - \frac{p_1}{24} + \frac{7p_1^2 - 4p_2}{5760} + \cdots .$$

If M^{4k} is a spin manifold ($w_2 = 0$), then

Michael Atiyah



Is Singer

proved that $\widehat{A}(M^{4k})$ is equal to the index of the associated Dirac operator, and hence is an integer.

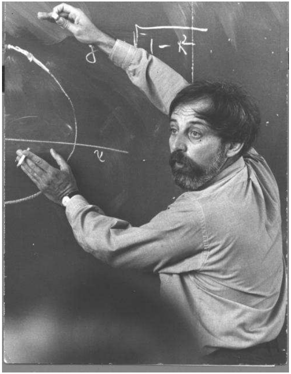
Almost parallelizable manifolds

Suppose that $M^{4k} \setminus (\mathrm{point})$ is parallelizable, so that $p_j(\tau_{M^{4k}})$ is zero for $j < k$. Hirzebruch's formulas then take the form

$$L(M^{4k}) \;=\; \left(2^{2k}(2^{2k-1}-1)\frac{B_k}{(2k)\,!}\right) p_k[M^{4k}]\;,$$

and since $w_2 = 0$,

$$\widehat{A}(M^{4k}) \;=\; \frac{-B_k}{2\,(2k)\,!}\, p_k[M^{4k}]\;.$$



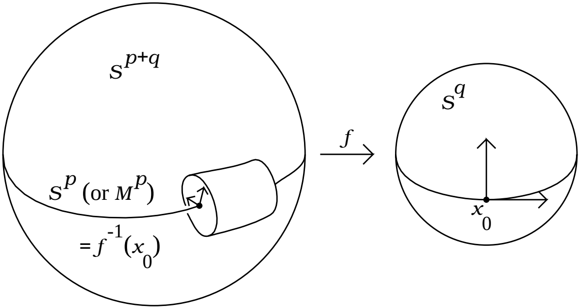
Raoul Bott at the Arbeitstagung in 1969

Bott showed that $\pi_{4k-1}(SO) \cong \mathbb{Z}$. Furthermore, a generator gives rise to a vector bundle ξ over S^{4k} with

$$p_k(\xi)[S^{4k}] = \begin{cases} (2k-1)! & \text{for } k \text{ even} \\ 2(2k-1)! & \text{for } k \text{ odd}. \end{cases}$$

$\Longrightarrow$ the Pontrjagin number $p_k[M^{4k}]$ of an almost parallelizable manifold is always divisible by $(2k-1)!$.

The Pontrjagin-Thom construction.



Any $M^p \subset S^{p+q}$ with framed normal bundle determines a homotopy class in $\pi_{p+q}(S^q)$.

Taking $M^p = S^p$ we obtain the ***J*-homomorphism**

$$J : \pi_p(SO_q) \to \pi_{p+q}(S^q) .$$

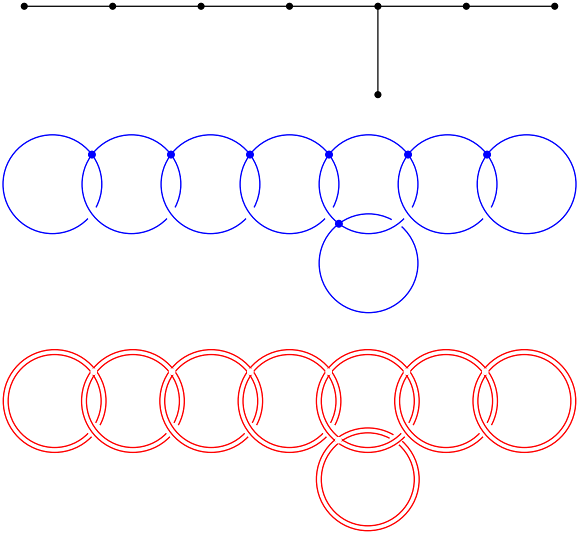
In the stable case $q >> p$, I will write $J : \pi_p(SO) \to \Pi_p$.

The Adams Conjecture

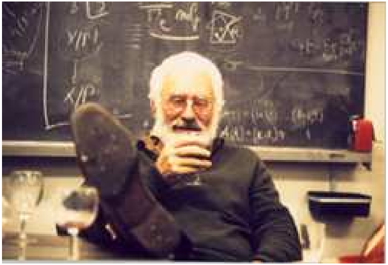
Frank Adams

$$\left| J\big(\pi_{4k-1}(SO)\big) \right| \;=\; \text{denominator}\left(\frac{B_k}{4k}\right) .$$

The E_8 manifold-with-boundary W^{4k}



The boundary ∂W^{4k} is a homotopy $(4k-1)$-sphere **if** $k > 1$.



Michel Kervaire

Theorem The group of homotopy spheres which bound parallelizable manifolds is cyclic of order

$$2^{2k-2}(2^{2k-1}-1)\,\mathrm{numerator}\left(\frac{4B_k}{k}\right),$$

with generator ∂W_{4k-1}.

The last 50 years

Amazing progress in low dimensional topology:

Freedman, Donaldson,
Thurston Geometrization, Perelman

Ever deeper connections with mathematical physics:

gauge theory, Seiberg-Witten theory,
symplectic topology, $\cdots$

TO BE CONTINUED !