Mathematical Perspectives on Music-theoretical Knowledge

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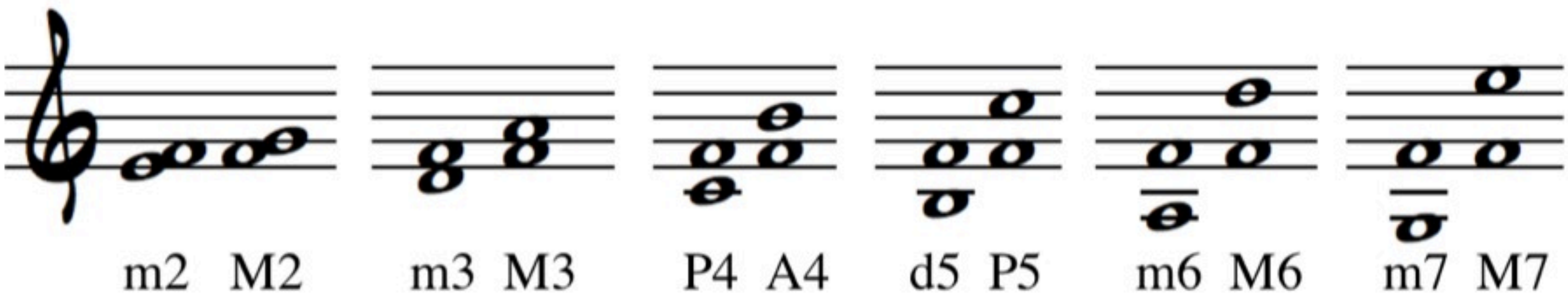
Dear Colleagues,
it is a great pleasure to be invited to this excellent institute and I'm very glad that you are interested in the application of mathematics to music theory. My talk will focus on very basic music-theoretical concepts which are inspired by some properties of the traditional notation system. The links which I will draw to actual music are mostly concerned with the bass line. This is somewhat complementary to the focus of the joint project which Tom Fiore and I are currently working on. That one deals with voice leadings in harmonic progressions.
The mathematical trajectory of my talk will take us from linear maps on Zn to linear maps on the free Z-module of rank 2 and turther on to associated automorphisms of the free group on two generators. All this is closely linked the the field of algebraic combinatorics on words.

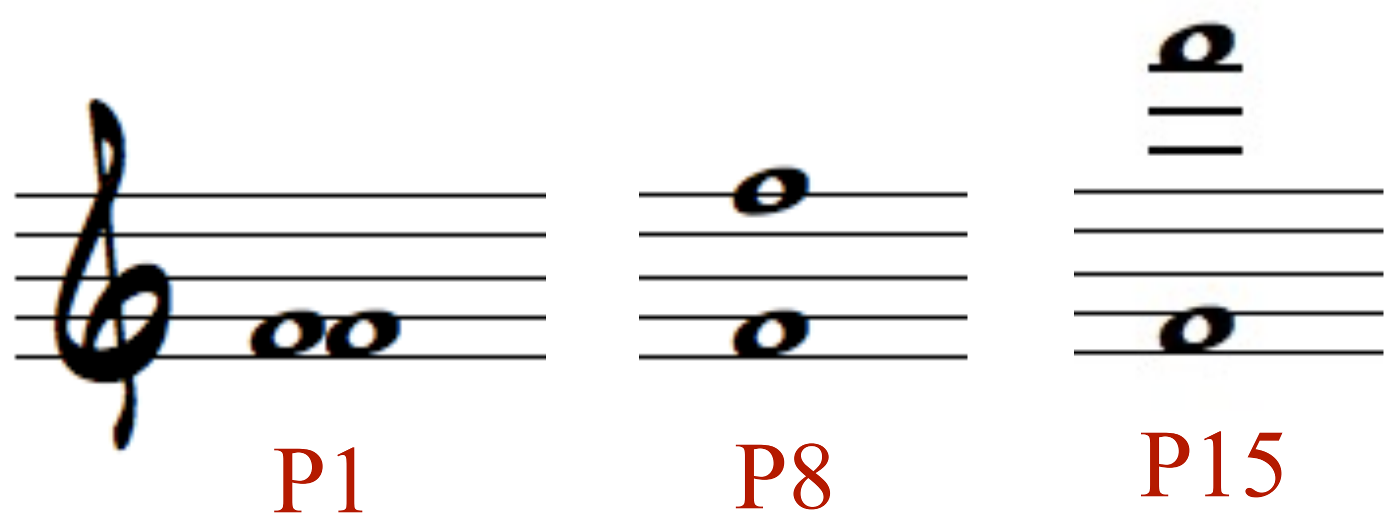
Elementary Observations about Musical Notation and the Diatonic System



(Almost) every generic interval comes in two species.

"Myhill's Property"

Exceptions:



In traditional musical notation note heads are placed on parallel horizontal staff lines and occasionally accidentals are attached to them. This system is well-adopted to the notation of music, which - in the broadest sense - can be called diatonic. Before going into sophisticated aspects of this concept, let us start from a simplified perspective: Diatonic tone relations are those which can be notated with no use sharps or flats.

There is a generic way to measure intervals, namely to count the height differences between two notes on the staff. The traditional interval names prime, second, third, etc. are ordinal numbers and are therefore one higher than the actual generic height. Besides that musicians attach the attributes "perfect", "major", "minor", "augmented", "diminished", to the generic interval names in order to further specify them.

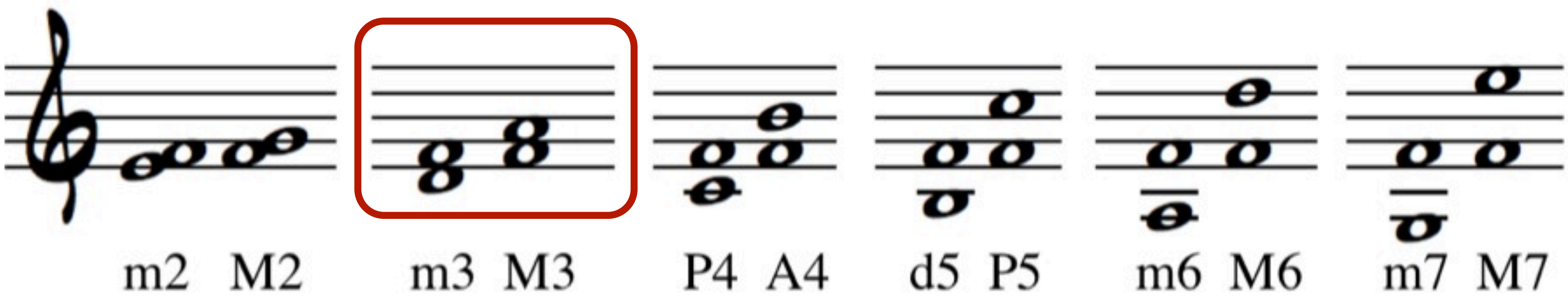
And here is a first elementary mathematical observation about the diatonic intervals (no sharps, no flats): With one type of exception every generic interval occurs in precisely two species: minor second (m2), major second (M2), minor third (m3), major third (M3), perfect fourth (P4), augmented fourth (A4), perfect fifth (P5) diminished fifth (d5), etc.

Music Theorists call this property "Myhill property" (named after the Logician John Myhill). I will come back to this property in a minute.

You may wonder about the different nomenclature for the fifths and fourths. Despite of the traditional motivation of the attribute "perfect" in counterpoint we will see in a minute that these intervals are also structurally distinguished.

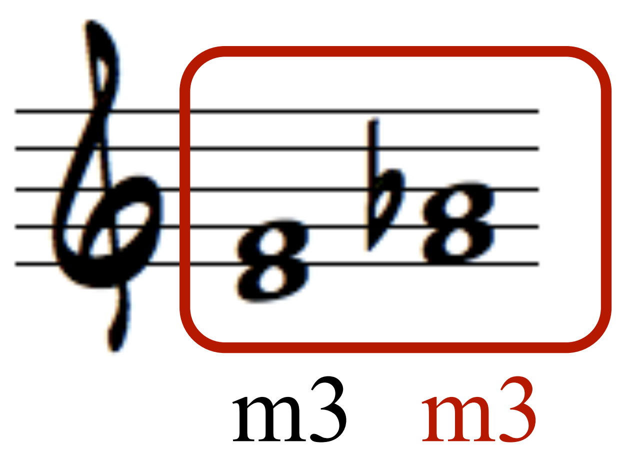
Notice that the generic prime (the zero-interval), the generic octave and all multiples of the octave appear only in one species. If we also take into account our (silent) knowledge about the concrete locations of the minor steps, we see in addition that the octave is the period of the specific diatonic step patterns.

Elementary Observations about Musical Notation and the Diatonic System



(Almost) every generic interval comes in two species.
"Myhill's Property"

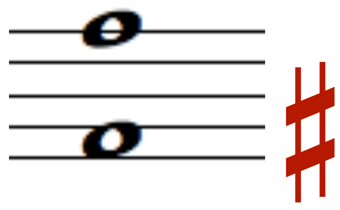
The two species always differ by an alteration (# or b):



Let us now look at the sharps and flats. They extend the domain of notated intervals. Virtually we may attach any number of sharps or flats to a note. But the sharp - designating the interval of an augmented prime - is tightly related to our previous observation (i.e. Myhill property). It is not only that all the non-octave intervals come in two species. In each case the difference between those to species is the same, namely the augmented prime. So for example, by flattening the note A in the major third F-A we turn it into a minor third.

Addition Law for Note Intervals

$$E_{(1,2)}: \; 0 \to \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}\times\{0,1,\ldots,6\} \to \mathbb{Z}_7 \to 0$$



P8

M2

Examples:

$$\left.\begin{array}{c} M2 = (0,0,1) \\ M9 = (1,0,1) \\ m9 = (1,-1,1) \end{array}\right\} \longmapsto 1$$

$$d8 = (1,-1) \mapsto d8 = (1,-1,0) \longmapsto 0$$

In preparation to the study of linear maps on Zn and the free commutative group of rank Z^2, which I initially anounced, we need a group structure on the underlying sets of musical intervals. Actually the notated musical intervals are not intervals in a mathematical sense. We describe them as elements of the free commutative group of rank 2. In view of Myhill's property we may not add the staff heights and sharps independently. In order to build up the interval space we first form the free commutative group generated by the octave and the augmented prime. Here we take advantage from the fact that the octave is a period. In addition we choose the first 7 multiples (from 0 to 6) of the major second M2 in order to represent the 7 staff height positions modulo octave. Every notated interval can be written as a triple of this form. Here are some examples: Major second, major ninth (differing from the major second by a perfect octave), minor ninth (differing from the major ninth by a flat), diminished octave (differing from the perfect octave by a flat).

Addition Law for Note Intervals

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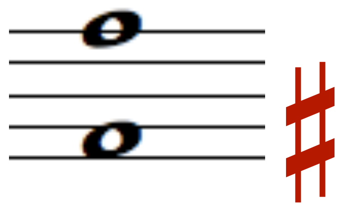
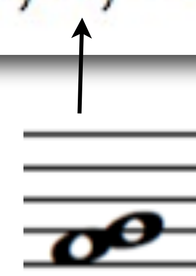


P8



M2

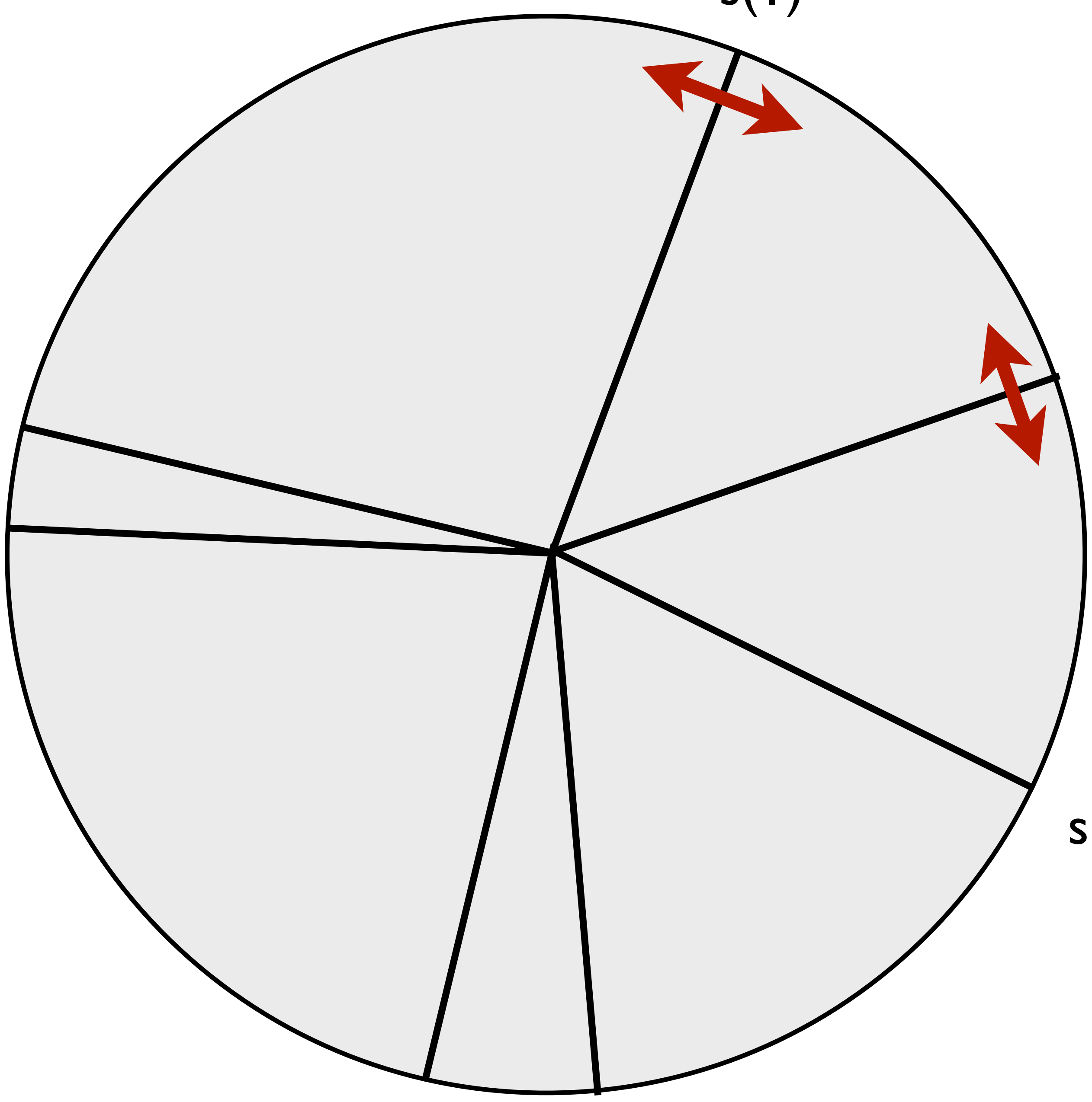
$$(o_1, s_1, d_1) +_{(1,2)} (o_2, s_2, d_2)$$

$$= \begin{cases} (o_1 + o_2, s_1 + s_2, d_1 + d_2) & \text{if } d_1 + d_2 < 7, \\ (o_1 + o_2 + 1, s_1 + s_2 + 2, d_1 + d_2 - 7) & \text{if } d_1 + d_2 \geq 7. \end{cases}$$

And here is the addition law in a kind of “Abacus”-formulation. It describes the notated intervals as a group extension of Z_7 by Z^2. The result is also isomorphic to Z^2. Whenever the sum of the step components is less then 7, the three coordinates are simply added separately. With every overflow one has to subtract 7 in the last coordinate, and in compensation one has to add 1 on the octave coordinate and 2 in the accentals. This is because seven Major seconds yield a doubly augmented octave. Our next goal is the study of a linear map in Z7. To that end we make a little detour into the investigation of musical scales.

An “Arbitrary” Scale

$$s : \mathbb{Z}_n \xrightarrow{\sim} S \subset \mathbb{R}/\mathbb{Z}$$



represented in ascending ordering with: $s(0) < s(1) < .... < s(n-1) < s(0) + 1$

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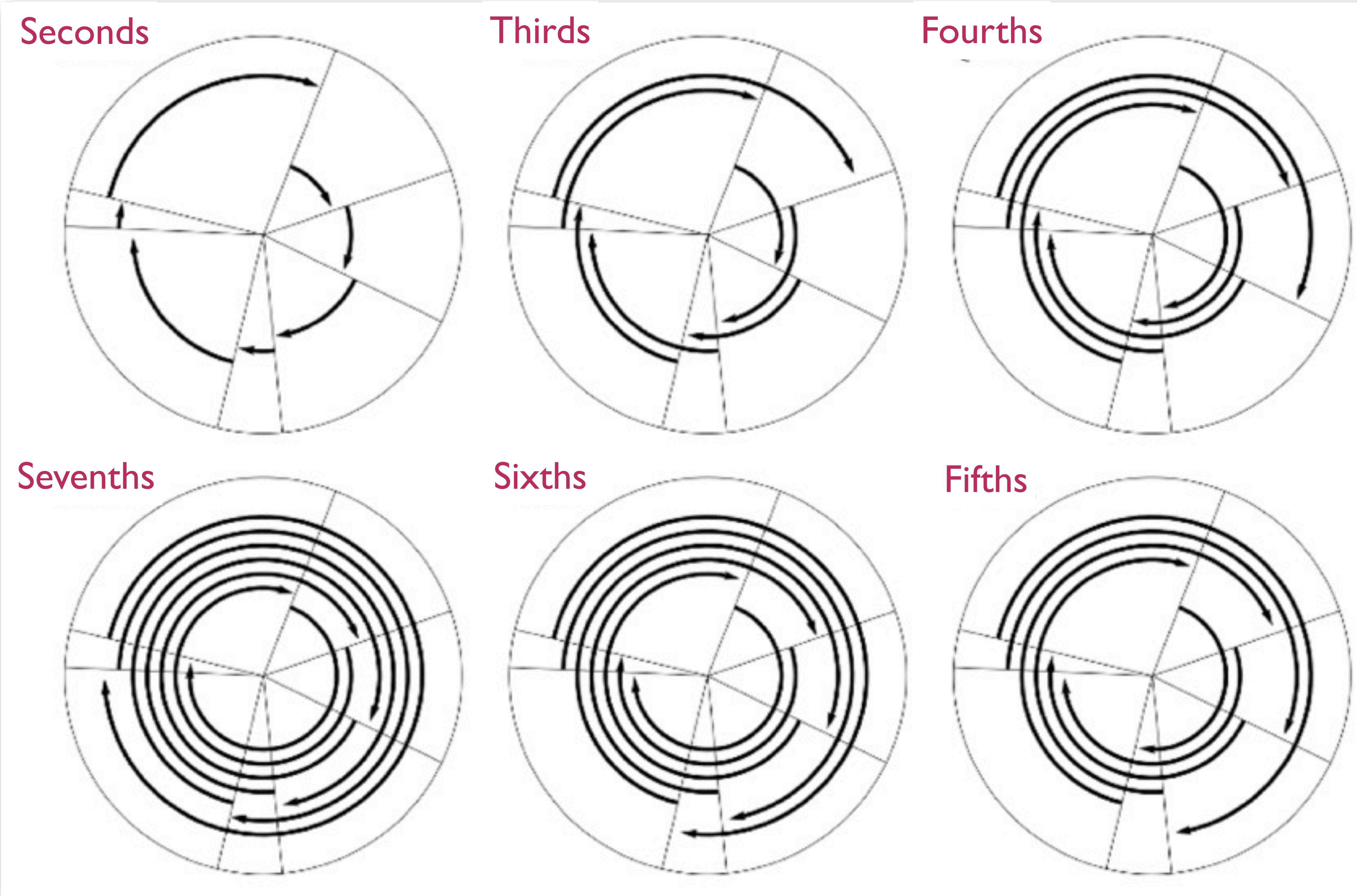
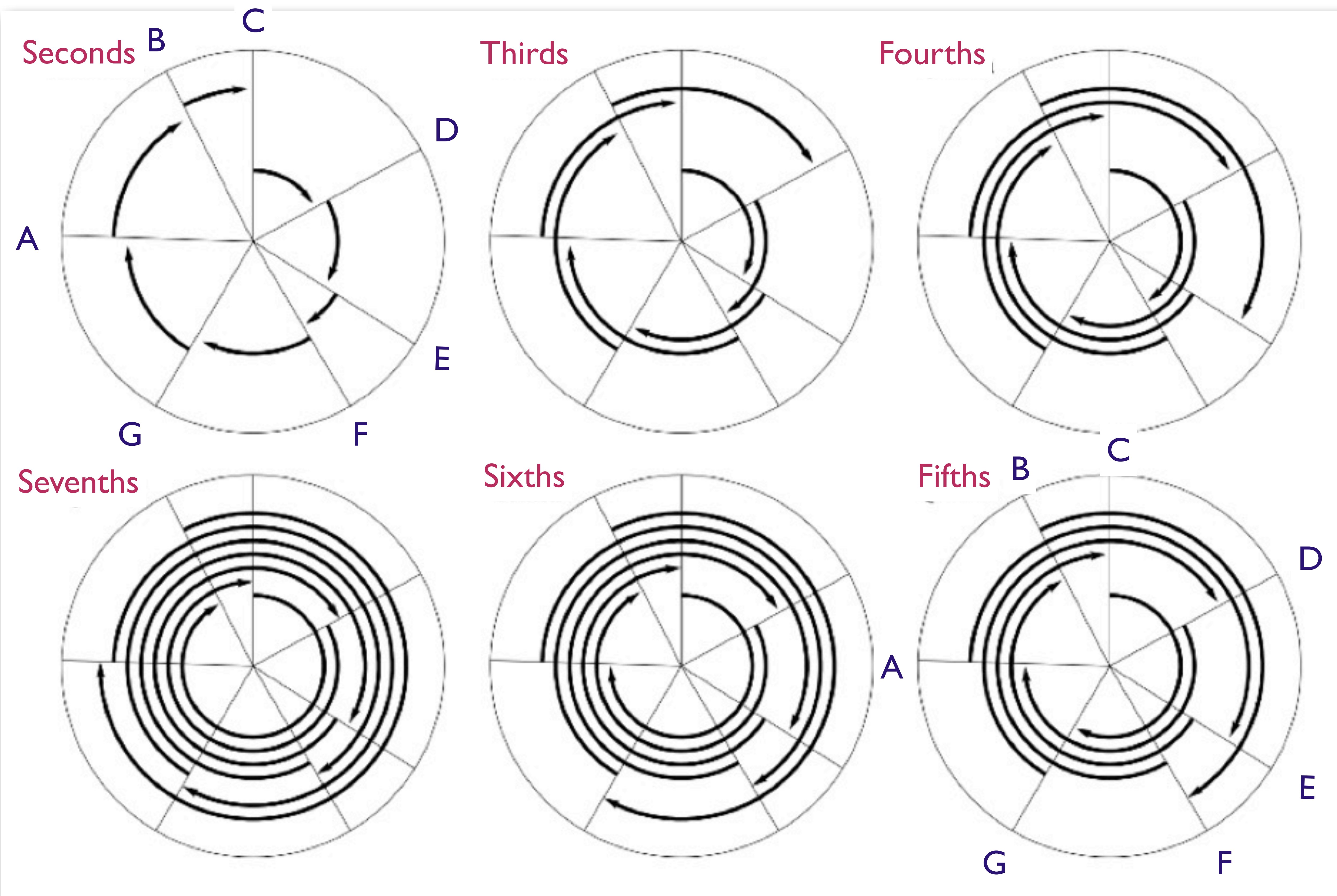
We leave - for a minute - the domain of musical notation and suppose a situation where a scale of 7 (or n) tones modulo octave is arbitrarily chosen. We may describe this as a set S of points on the circle which are indexed in ascending order by the residue classes 0 till n-1 mod n.
By measuring intervals along the indexes we have a generic level of description. This means, that the traditional interval names prime, second, third etc. can be applied here as well. But as we are on the circle, there is no distinction between the prime and the octave (for n=7). By measuring the specific intervals on R/Z we have a specific level of description. Scale theory investigates the interaction of these two levels. Among other things one may look for so called “contradictions”. This is where - for example - a specific third is smaller than a specific second. In our example this happens between the very large prime from s(0) to s(1) and the considerably smaller third from s(3) to s(5).

An "Arbitrary" Scale



In the case of an arbitrary scale it is quite unlikely to observe something like Myhill's property. Every generic interval can appear in as many species as there are tones in the scale. In my drawing you see that all the segments ("cake pieces") are of different size and that property also holds for groups of adjacent segments.

A Diatonic Scale: Myhill's Property



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Here we see again a diatonic scale consisting of five major seconds and two minor seconds. Allow me to draw your attention to the multiplicities of the other intervals: In the case of the thirds there are four instances of minor thirds and three instances of major thirds. In the case of the fourths we find six instances of the perfect fourth and just one instances of augmented fourth. The other intervals behave analogously to their octave complements. In particular we have six instances of perfect fifths and only one instance of a diminished fifth. The perfect fifth consists of three major seconds and one minor second, while the diminished fifth consists of two major seconds and two minor seconds.

The occurrence of multiplicity 6, which is one less than the number of tones, actually implies that the scale is generated by a single specific interval. Concatenating 6 perfect fifths starting from F (see lower right cake diagram) yields all 7 notes as an arithmentic sequence along the circle: F, C, G, D, A, E, B. Music theorists call this the (diatonic) circle of fifths.

To find an interval of multiplicity n-1 in a scale with Myhill's property is not accidental. Closer inspection shows that Myhill's property implies the existence of a generating interval. I shall make this more precise.

The Well-formedness Property

Scale:

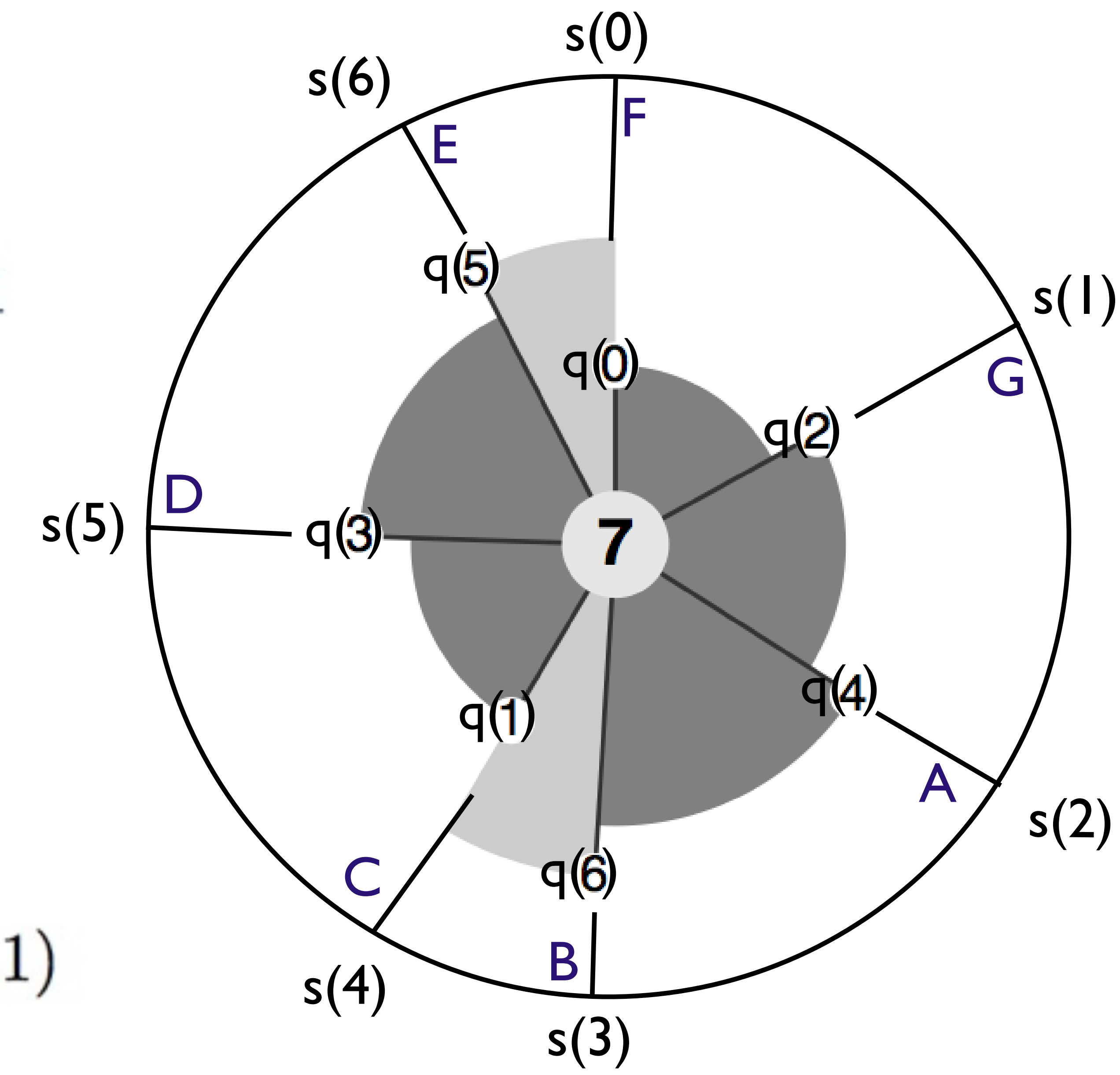
$s : \mathbb{Z}_n \xrightarrow{\sim} S \subset \mathbb{R}/\mathbb{Z}$

$s(0) < s(1) < .... < s(n-1) < s(0) + 1$

Generated Scale:

$$\begin{array}{ccc} \mathbb{Z}_n & & \\ p \downarrow \wr & \searrow^{q}_{\wr} & \\ \mathbb{Z}_n & \xrightarrow[s]{\sim} & S \subset \mathbb{R}/\mathbb{Z} \end{array}$$

$q(k) = g \cdot k \ mod \ 1$ for some $g \in (0, 1)$



Well-formed Scale:

The permutation $p = s^{-1}q : \mathbb{Z}_n \xrightarrow{\sim} \mathbb{Z}_n$ is a linear map.

It is known that for any finite arithmentic sequence on the circle of any length and any size of the generator one has at most three different distances between adjacent points. This finding is commonly referred to as the "Three Gap Theorem". The possibility to have three different "gap sizes" rather than just two, suggests that Myhill's property is still stronger than generatedness.

It is quite enlightening for music theorists to understand this stronger condition directly in terms of the generator.
For a generated scale we have reason to consider two bijective maps s and q from Z_n to the scale S. These bijections parametrize the scale S in scalar and generation order respectively. In the drawing the generation order is depicted inside of the cicle and the radii of the cake pieces exemplify this order: q(0), q(1), ..., q(6). The scale order is depicted outside of the cicle. By concatenating q with the inverse of s we obtain a permutation of Z_n.

If we look at this permutation in the diatonic case we see that it is actually a linear map: 0 goes to 0 (coordinates of the note F), 1 goes to 4 (see note C), 2 goes to 1 which is the same as 8 modulo 7 (at note G), 3 goes to 5 (at note D), etc. So this map is multiplication by 4 modulo 7.

This observation gives reason to the following definition: A generated scale is called well-formed if and only if the conversion from generation order to scalar order is a linear map.

For better musical understanding is useful to inspect a violation of the well-formedness property: Suppose we abandon the last generated tone of the diatonic scale: B. In the resulting generated six-tone scale C, D, E, F, G, A the perfect fifth from C to G is still divided into four scale steps: C to D, D to E, E to F and F to G. The fifth from G to D, however, contains only three steps: G to A, A to C and C to D. Literally speaking, this interval would not longer deserve the name "fifth". It is a "fourth" with a large step from A to C.
Returning after this detour to the complete diatonic scale (again with the tone B included) we may now appreciate the well-formedness property as a structural presupposition for the consistent usage of the term "fifth" by the musicians.

Theorem (Carey & Clampitt 1989, 1996):

Consider an n-tone scale $S = \{k \cdot g \bmod 1 \mid k = 0, ..., n-1\} \subset \mathbb{R}/\mathbb{Z}$, generated by $g \in (0,1)$. Let $q, s : \mathbb{Z}_n \to S$ denote the associated generation order and scalar order encodings of S, respectively. The following three properties are equivalent:

(i) S is non-degenerate well-formed, i.e. $s^{-1}(q(k)) = m \cdot k \bmod n$ for a suitable $m \in \mathbb{Z}_n$ and $S \neq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

(ii) (Myhill property) Each non-zero generic interval comes in precisely two specific sizes.

(iii) The ratio $\frac{m}{n}$ is a semiconvergent of the generator g with $\frac{m}{n} \neq g$.

Example g = $\mathrm{Log}_2(3/2)$:
with semiconvergents 1/2, 2/3, 3/5, 4/7, 7/12, ...

Here I wish to present you the underlying theorem which has been achieved by the two music theorists Norman Carey and David Clampitt. The concept of well-formedness includes the special case, where all n points from a regular n-gon. If one excludes this degenerate case in turns out that well-formed generatedness and Myhill property are equivalent. There is also an equivalent number-theoretic condition: The ratio m/n (with m being the number of steps in the generator) and n the number of the notes in the scale needs to be a semi-convergent of the generator g. It is precisely the degenerate case, when this ratio coincides with the specific size of the generator.

This third property also has two music-theoretically important consequences. Below the theorem I list the semiconvergents of the number Log(3/2) with respect to the basis 2, which may refer for the "just tuned fifth" with frequency ratio 3/2. The first consequence is that the theorem puts a hierarchy of musically prominent scales (division of the octave, tetractys, pentatonic, diatonic, chromatic) in direct connection with the well-formedness property. The second consequence is a certain robustness or stability of the result. A small perturbation of the "just fifth" in the role of the generatpor has no impact in the structural properties of the theorem.

[NB: I should emphasize that the present approach is not intended to be interpreted in psycho-accoustic terms. Nonetheless it is epistemologically interesting here to remind you about a classical achievement of Hermann von Helmholtz in his psycho-accoustic approach to the study of musical intervals. On the one hand he confirmed the Phytagorean emphasis of small integer ratios for the consonance of musical intervals. He found them as local minima of a smooth dissonance curve. But thereby - in contrast to Euler's approach (based on the prime decompositions of rational numbers) - the consonances turn out to be stable in a neigborhood of these minima. Carey and Clampitt's theorem similarly confirms the prominence of the interval of the fifth, but at the same time it puts the precise value Log(2, 3/2) into perspective.]

Is the automorphism musically relevant?

$$\cdot m : \mathbb{Z}_n \to \mathbb{Z}_n$$

$$\begin{array}{ccc} \mathbb{Z}_n & & \\ \cdot m \downarrow \wr & \searrow & \\ \mathbb{Z}_n & \xrightarrow{\sim} & S \subset \mathbb{R}/\mathbb{Z} \end{array}$$

The diatonic case:

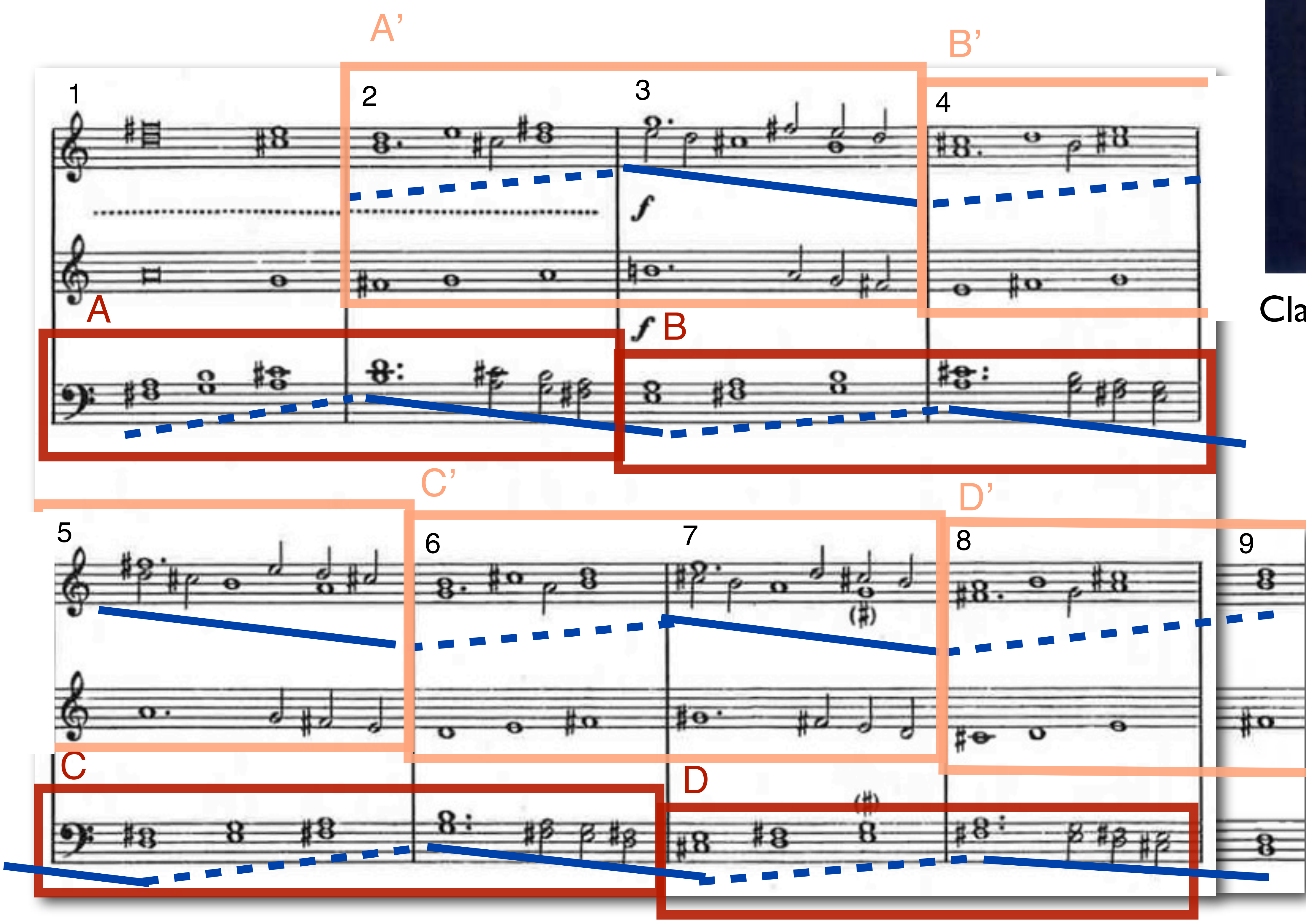
n = 7
m = 4

Every step can be divided into two fifths.
Every fifth can be divided into four steps.

4 . 2 = 1 mod 7

Now, as psycho-acoustics is not our concern, I shall now sketch some musical and music-theoretical interpretations, which illustrate the relevance of this theorem, and in particular of the linear map in Z_7: Multiplication by 4 and its inverse mod 7, namely by 2.
To begin with, you may interpret these two maps in terms of a circular definition of the diatonic scale:
What is a fifth? A fifth consists of four steps. But what is a step? A step consists of two fifths.
A solution to this “equation system” can be found in Z7, where ninths (double fifths) are identified with seconds.
In the next slides we look at exemplifications of this equation system in actual music from different periods.

Excerpt from the madrigal “Questi vaghi”



Claudio Monteverdi

The passage is from the beginning of the 17th. century. It is an excerpt from the opening “sinfonia prima” of Monteverdi’s madrigal “Questi vaghi” from his 5th book of madrigals (published in 1605).

Our passage begins at measure 12 of the piece, but for simplicity we begin counting from 1.
Let us begin our little analysis by identifying the interval of a descending fifth between the first and the last note in the lowest voice (namely between the initial note f# in measure 1 and the note b at the downbeat of measure 9). At this level of analysis the interval is just an “empty” descending fifth.

Then let us have a look at the segmentation of the lower system of the passage into two-measure-segments as indicated by the red boxes, labeled A, B, C, D. The segments are shifted downwards each time one step. Without yet analysing the content of these segments we just choose the first bass note in each segment as an anchor: f# - e - d - c# - (b). These anchors as well as their associated segments exemplify the filling of the initially “empty” fifth with four steps.

In our next level of the analysis we look at the first notes (of the bass melody) in each measure. From measure 1 to measure 2 the bass ascends a fourth (from f# to b) and from measure 2 to measure 3 it descends a fifth (from b to e). In Z_7 an ascending fourth is abstractly identified with a descending fifth (remember that the octave is identified with the prime). In other words: every descending step on the level of 2-measure-segments is decomposed into two descending fifths (up to octave identification).

At this point it is useful to observe that the upper three voices imitate the lower two voices with an offset of one measure. This is indicated by the orange boxes, labeled A’, B’, C‘ and D’. The third voice imitates the bass one octave higher and the upper two voices imitate the lower two voices an octave and a fourth higher. (To make this claim, we disregard the temporal displacements in the second voice.) A consequence of this configuration is, that the upper two voices always start their imitation from the same two notes which the lower two voices have just reached (an octave lower or two). The descending fifth from b to e between the first notes in measures 2 and 3 (lower system) is followed by a descending fifth from e to a between the first notes in measures 3 and 4 (upper system), This is followed by a descending fifth from a to d between the first notes in measures 4 and 5 (lower system), etc. In other words: Instead of counting the ascending fourths in the lower system as abstract substitutions of descending fifths we may instead alternate between the upper and lower systems and find effectively a sequence of descencing fifths. The jumps between the systems are octaves or double-octaves.

This last observation becomes even more relevant, if we look into the melodic content of each single voice in each single measure of the segments. Each melody either ascends a fourth stepwise (dotted blue lines) or it descends a fifth stepwise (continuous blue lines). Looking at the descending fifths, the balance of our analysis reads as follows: The descending fifth consists of four steps, each of which (except in measure 1) consist of two descending fifths, each of which consists of four descending steps. An analogous stament can be made about ascending fourths and ascending steps.

Beginning of the “Passacaille”
(Suite for Harpsichord in G-Minor)

Georg Friedrich Händel

This harpsichord piece from Georg Friedrich Händel dates from 1720. The passacaglia is a series of variations of the same underlying harmonic pattern. The bass notes at the downbeat of each measure descend by steps: g – f – e-flat – d. The corresponding notes in the highest voice descend analogously, but one fifth higher: d – c – b-flat – a. One has to take into consideration, though, that the D-major triad on the downbeat of measure 4 is not the final harmony of this progression (as it actually would be the case in the so-called andalusian cadence). On the the last beat of measure 4 there is – in the in the upper voice – a note g. This g completes the stepwise descend of a fifth: d – c – b-flat – a – g. So the upper voice exemplifies: A fifth consists of four steps. The bass also concludes with a note g (on the third beat of measure 4). The balance of the bass line is a descending octave, which is divided in to a fourth from g down to d (at the downbeat of measure 4) and a fifth from d down to g. That upper fourth is filled with steps while the remaining fifth is empty.

If you look at the chords in the left hand, you see that the harmonic progression is articulated at a level of half notes (= half measures). I should mention that music theorists distinguish between the real bass (the lowest voice) und the fundamental bass (a virtual voice, implied by the harmonic progression). The roots (or fundaments) of these chords G – C – F – B-flat– E-flat – A – D – G almost coincide with the real bass notes. Exceptions are the chords on C and A which appear in first inversion. The property that every step is composed by two fifths is here exemplified by the fundamental bass (rather than the real bass).

In case you are a little irritated by the concept of the fundamental bass, I propose to look into the first variation (measures 5 – 8), where the real bass literally exemplifies our linear automorphism. The red boxes identify 1-measure-segments which descend by 1 step each time. The harmonies are the same as before and now the real bass notes c – f – b-flat – e-flat – a – d –g form a zigzag of descending fifths and ascending fourths. Between the end of measure 2 and the beginning of measure 3 the bass is actually raised an extra octave and so it ends on the same register as it started.
At the level of eighth-notes we observe that every descending fifth is filled by 4 descending steps (and every ascending fourth by three steps). In the case of the ascending fourths Händel uses the vacant fourth metric position in each quadrupel of eight notes for a brief reminder of the preceding fundament. (c – d – e-flat – inserted c – f).

1st Theme from a Piano Sonata

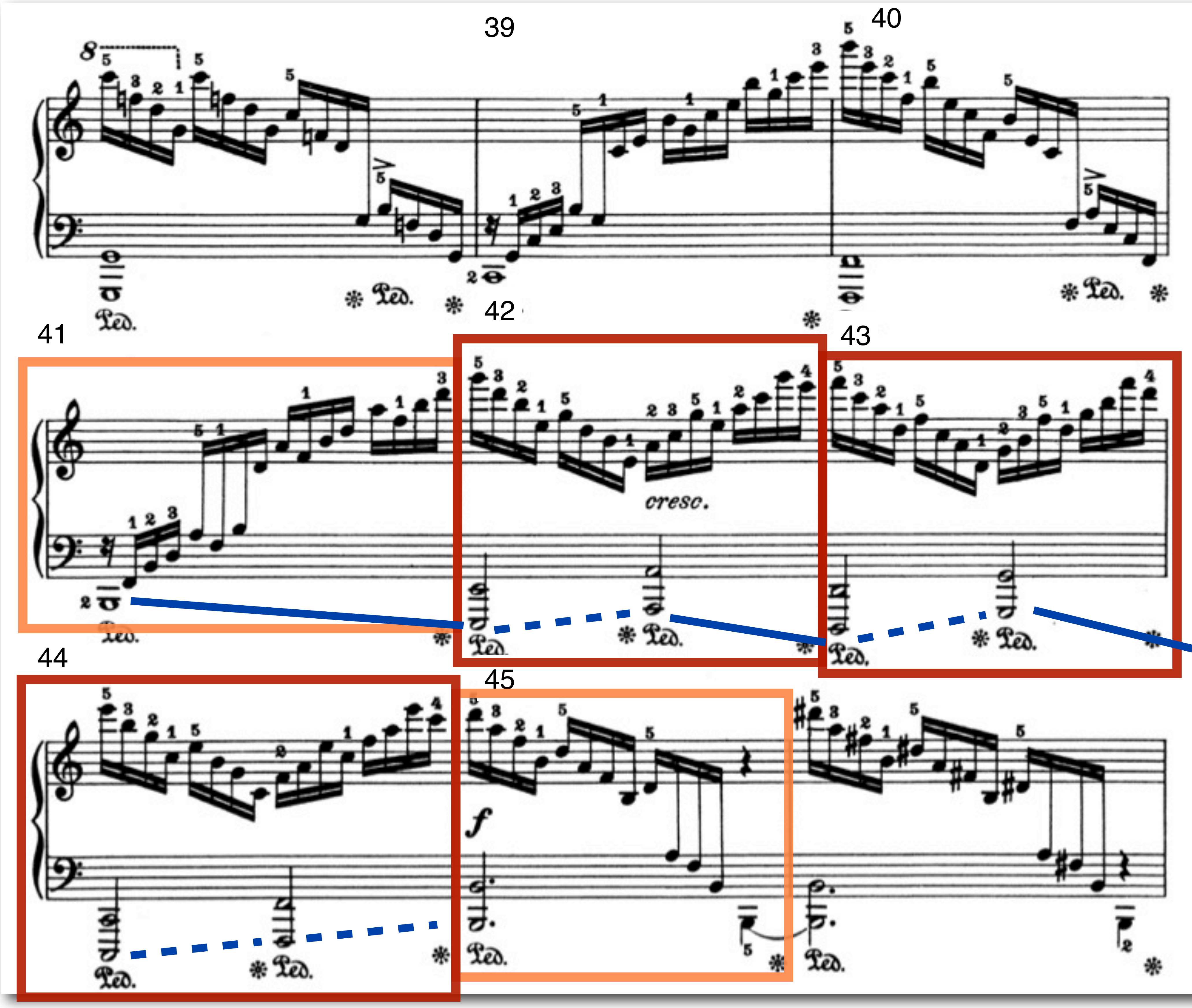
W.A. Mozart

A D G C F B E A

(Fundamental Bass)

Here you see the first theme of the 1st movement from Mozart's A-minor Sonata (K310, 1778). It is of a so called sentence-form and consists of two 4-measure phrases. The first phrase presents a 2-measure idea, which is immediately repeated. The second phrase fullfilles a continuation function and consists of four fragments of approximately 1-measure length. A stepwise descend is manifest in the A-minor-, G-major-, F-major- and E-major triads of the left hand. The right hand does instead a wave-like movement around the third c-e. The fundamental bass divides each of the three descending steps a-g, g-f, f- e into two fifths/fourths a-d-g, g-c-f, f-b-e. The cycle of fifths a - d - g - c - f - b - e - a is completed at the downbeat of bar 9.

Excerpt from a Study for Piano

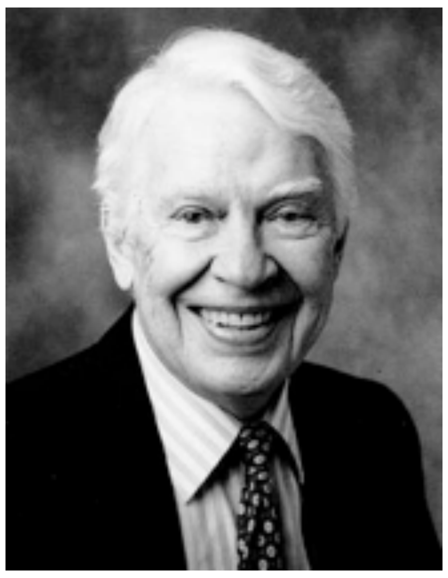


F. Chopin

This is a passage from (the end of) the middle part of the C-major study (Op. 10 No. 1) by Chopin (published in 1833). It is heading towards a fulminant arrival on the dominant E of A-minor. [NB: Chopin thereby misguides the listener, as the formally “proper” goal would be the dominant G of C-major.]

We will start our little analysis with measure 41 (framed with an orange box). All the preceding 40 measures form units of two measures, where the right hand always ascends for the span of one measure and symmetrically descends in the following one. In the excerpt you may see this in measures 39/40. After measure 41, however, there is no such symmetric descend. A corresponding descending pattern with the same bass and the same harmony can later be found in measure 45. In the three measures (42, 43, 44) marked with red boxes we see instead right-hand-patterns within a smaller height ambit descending and ascending within one measure each. Measures 43 and 44 are literal copies of measure 42 which are shifted downwards one step each time. The first half of measure 45 still complies to this scheme. So we depart from the bass b in measure 41 into the stepwise descending bassline e-d-c-b, which returns to b. This is then subdivided into a folding of ascending fourths and descending fifths, which also supports the harmony. Real bass and fundamental bass coincide. Starting from b in measure 41 this yields a complete diatonic cycle of fifths: b - e - a -d - g -c - f - b.

[NB for musicians: This example consists purely of white notes until the halfdiminished chord $B^{ø}$ (measure 45) is turned into a secondary dominant B^7 (dominant of the dominant) of A-minor (measure 46). But in the music before our passage we find a complete cycle of fiths which is continuously modulating flatwards. In measure 22 there is also a similar secondary dominant B7b5 (with d-sharp) with real bass f and fundamental bass b. And starting from there the harmony falls in fifths and modulates at the same time flatwards over seven fifths till it reaches the same generic scale degree, but this time the form of b-flat. To be more precise: It is the real bass of the last chord which reaches b-flat. In Chopins respelling the Bb7 chord turns out to be an augmented sixth chord with real bass b-flat, but fundament e. These things can hardly be catched by a generic diatonic perspective. Yet it is insightful to include it as a significant piece of puzzle into a more sophisticated approach.]



Bart Howard

“Fly me to the moon” is a popular song by Bart Howard which was first published under the name “In other words” in 1954.
The harmony forms a complete cycle of fifths. The note-names in the chord symbols denote the fundamental bass progression: a – d – g – c – f – b – e – a. The melody forms a Zigzag which goes in parallel thirds (or tenths = thirds + octaves) with the fundamental bass: c – f – b – e – a – d – g-sharp – c. The zigzag groups these 8 notes into 4 pairs c – f, b – e, a – d, and g-sharp – c. The pair are stepwise descending. The two notes forming a pair are always connected through four descending steps (as indicated by the red boxes).

NB: The Zigzag of the melody is quite similar to the Bassline in the 1st variation of Haendel’s Passacaglia. But while in the song the melody goes in parallel tenths with the bass, Haendels upper melody goes in countermotion to the bass. The two voices alternate between fifths (+ octaves) and tenths (thirds + octaves).

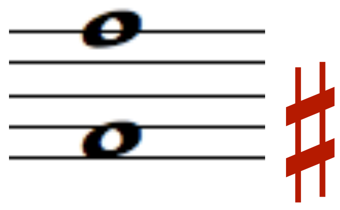
$$A5 = M2 + M2 + m2 + A2$$

The consideration of the linear maps in Z_7 is quite robust. But – at first sight – it seems to imply a complete ignorance of octaves and accidentals. This requirement has caused skepticism among working music theorists and has therefore diminished their enthusiuasm for the well-formedness property in particular with respect to the analysis of actual music.
In “Fly me to the moon”, for example one of the fifths is an augmented fifth between c and g#. From below it is stepwize filled by two major seconds (c-d and d-e), a minor second (e-f) and an augmented second (f-g#).

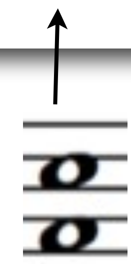
[NB: One may tend to regard the – so called – “harmonic minor scale” a-b-c-d-e-f-g#-a as an instance of scale which is not well-formed (and not even generated). Therefore some theorists might hesitate to interpret the stepwise filling of the augmented fifth as an instance of our automorphism mod 7. But as I will show now there is nevertheless a strong argument in favor of such an interpretation.]

P5-based Addition Law for Note Intervals

$$E_{(4,1)}: \ 0 \ \to \ \mathbb{Z}\times\mathbb{Z} \ \to \ \mathbb{Z}\times\mathbb{Z}\times\{0,1,\ldots,6\} \ \to \ \mathbb{Z}_7 \ \to \ 0$$



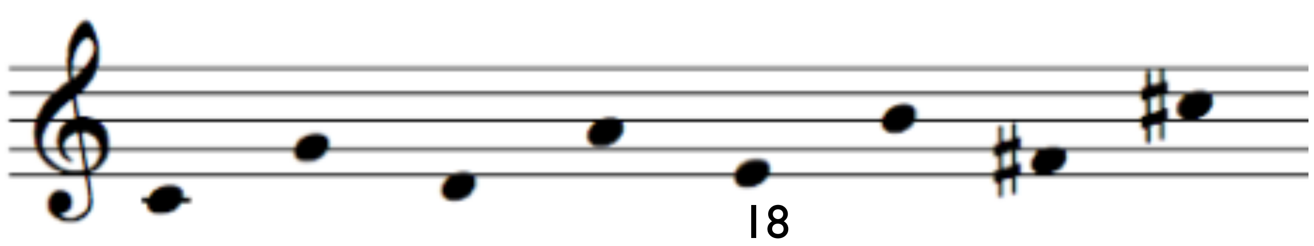
P8



P5

$$(o_1, s_1, q_1) +_{(4,1)} (o_2, s_2, q_2)$$

$$= \begin{cases} (o_1 + o_2, s_1 + s_2, q_1 + q_2) & \text{if } q_1 + q_2 < 7, \\ (o_1 + o_2 + 4, s_1 + s_2 + 1, q_1 + q_2 - 7) & \text{if } q_1 + q_2 \geq 7. \end{cases}$$



At this point the mathematician has to interveen, as he/she may find out whether it is really unavoidable to ignore the octaves and accidentals. On slide 5 we introduced a group extension of Z_7 by Z^2 (the latter being generated by the octave P8 and the augmented prime A1). That addition law on the extension was based on the major second M2 as a generator, i.e. the numbers 0, 1, ..., 6 represented multiples of the major second M2. Now on this slide we see an analogous “addition law” for the case, where the generator is the perfect fifth P5. The “abacus” law is now indexed by the pair (4, 1) as the octave winding number of seven fifths is 4 and the sharp winding number is 1.

Extending the Linear Automorphism

$$\begin{array}{cccccccccc} E_{(4,1)}: & 0 & \rightarrow & \mathbb{Z}\times\mathbb{Z} & \rightarrow & \mathbb{Z}\times\mathbb{Z}\times\{0,1,\ldots,6\} & \rightarrow & \mathbb{Z}_7 & \rightarrow & 0 \\ & & & \downarrow id & & \downarrow \tilde{\alpha} & & \downarrow \alpha & & \\ E_{(1,2)}: & 0 & \rightarrow & \mathbb{Z}\times\mathbb{Z} & \rightarrow & \mathbb{Z}\times\mathbb{Z}\times\{0,1,\ldots,6\} & \rightarrow & \mathbb{Z}_7 & \rightarrow & 0 \end{array}$$

Proposition 2.1. *The map* $\tilde{\alpha} : \mathbb{Z}\times\mathbb{Z}\times\{0,\ldots,6\} \rightarrow \mathbb{Z}\times\mathbb{Z}\times\{0,\ldots,6\}$ *which is given by the formula*

$$\tilde{\alpha}(o,s,q) = (o + \lfloor\frac{4q}{7}\rfloor, s - q + 2\lfloor\frac{4q}{7}\rfloor, 4q \; mod \; 7)$$

defines a group isomorphism with respect to the addition laws $+_{(4,1)}$ *and* $+_{(1,2)}$*, respectively.*

So the question naturally arises wether there is a unique and music-theoretical meaningful lifting of the linear automorphism on generic intervals to the full group of note intervals.

The affirmative answer is: There is a unique group isomorphism between the two group extensions E_(4,1) and E_(1,2), extending the linear multiplication map on Z7 and yielding the identity on octaves and sharps.

[NB: This is the special case of a general fact about the lifitng of linear automorphism of Z_n to such cyclic extensions by Z^2.]

Note Intervals with Basis {M2, m2}

$$E_{(1,2)}: \; 0 \rightarrow \mathbb{Z}\times\mathbb{Z} \rightarrow \mathbb{Z}\times\mathbb{Z}\times\{0,1,\ldots,6\} \rightarrow \mathbb{Z}_7 \rightarrow 0$$

$$\downarrow \phi$$

$$0 \rightarrow \mathbb{Z}\left[\begin{pmatrix}5\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right] \hookrightarrow \mathbb{Z}[M2, m2] \xrightarrow{+} \mathbb{Z}_7 \rightarrow 0$$

$$\begin{aligned}\phi(o,s,d) &:= o\cdot(5,2)+s\cdot(1,-1)+d\cdot(1,0),\\ \phi^{-1}(k,l) &:= (\lfloor\frac{k+l}{7}\rfloor, -l+2\lfloor\frac{k+l}{7}\rfloor, k+l \bmod 7).\end{aligned}$$

As I already mentioned, the group of note intervals is a free commutative group of rank 2. For several purposes it is therefore convenient to represent it with respect to corresponding bases.
On the one hand we choose the basis {M2, m2} (major second and minor second). The projection map to Z_7 is then given by the sum of the major- and minor second coordinates.

M2 = (0,0,1)
M9 = (1,0,1)
m9 = (1, -1, 1)
d8 = (1, -1, 0)

↓ ϕ

M2 = (1,0)
M9 = (6,2)
m9 = (5, 3)
d8 = (4,3)

$$\begin{aligned} \phi(o,s,d) &:= o\cdot(5,2)+s\cdot(1,-1)+d\cdot(1,0), \\ \phi^{-1}(k,l) &:= (\lfloor\frac{k+l}{7}\rfloor, -l+2\lfloor\frac{k+l}{7}\rfloor, k+l \bmod 7). \end{aligned}$$

Here we see how the tiples – considered earlier (slide 4) – are turned into {M2, m2} coordinates.

Note Intervals with Basis {P5, P4}

$$E_{(4,1)}: \; 0 \to \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \times \{0, 1, \ldots, 6\} \to \mathbb{Z}_7 \to 0$$

$$\downarrow \psi$$

$$0 \to \mathbb{Z}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \end{pmatrix}\right] \hookrightarrow \mathbb{Z}[P5, P4] \xrightarrow{-} \mathbb{Z}_7 \to 0$$

$$\begin{aligned} \psi(o, s, q) &:= o \cdot (1,1) + s \cdot (3, -4) + q \cdot (1, 0), \\ \psi^{-1}(u, v) &:= (v + 4\lfloor \frac{u-v}{7} \rfloor, \lfloor \frac{u-v}{7} \rfloor, (u - v) \bmod 7). \end{aligned}$$

On the other hand we choose the basis {P5, P4} (ascending perfect fifth and anscending perfect fourth). Here the projection to Z_7 is given by the difference of the two coordinates. This is because descending fourths are equivalent to ascending fifths under octave identification.

Extending the Linear Automorphism

$$\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{Z}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \end{pmatrix}\right] & \hookrightarrow & \mathbb{Z}\,[P5, P4] & \stackrel{-}{\longrightarrow} & \mathbb{Z}_7 & \rightarrow & 0 \\
 & & \downarrow \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} & & \downarrow \cdot 4 & & \\
0 & \rightarrow & \mathbb{Z}\left[\begin{pmatrix} 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] & \hookrightarrow & \mathbb{Z}\,[M2, m2] & \stackrel{+}{\longrightarrow} & \mathbb{Z}_7 & \rightarrow & 0
\end{array}$$

This commutative diagram shows the extended linear automorphism with respect to these new bases. An analogous diagram can be drawn for any matrix ((a b) (c d)) in SL(2, N). The order of the cyclic group Z_n is then given by the sum of its entries n = a + b + c + d. The multiplication factor m mod n is the sum a + c of its left column. Likewize we may start from the automormism .m mod n to construct the entire diagram.

$$
\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}
\qquad
\begin{array}{ccccccccc}
A5 & = & M2 & + & M2 & + & m2 & + & A2 \\
\begin{pmatrix} 4 \\ -4 \end{pmatrix} & = & \begin{pmatrix} 1 \\ -1 \end{pmatrix} & + & \begin{pmatrix} 1 \\ -1 \end{pmatrix} & + & \begin{pmatrix} -2 \\ 3 \end{pmatrix} & + & \begin{pmatrix} 4 \\ -5 \end{pmatrix} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\begin{pmatrix} 4 \\ 0 \end{pmatrix} & = & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & + & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & + & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & + & \begin{pmatrix} 2 \\ -1 \end{pmatrix}
\end{array}
$$

$$
\begin{array}{cccccccccc}
1 & = & 2 & + & 2 & + & 2 & + & 2 & \mod 7 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
4 & = & 1 & + & 1 & + & 1 & + & 1 & \mod 7
\end{array}
$$

Here we revisit the situation with the augmented fifth in “Fly me to the moon”. The lifted automorphism commutes with the “robust” multiplication by 4 mod 7. Subtracting the fifth/fourth-coordinates 4 and -4 of the augmented fifth, we otain 4 + 4 = 8 = 1 mod 7. Adding the “step-coordinates” 4 and 0 of its image under the linear mapon Z^2 we obtain 4 + 0 = 4 mod 7. And 4 = 4 times 1. The same kind of commutativity argument holds for all the summands M2, M2, m2, and A2.

Two ramifications of the investigation:

(1) Refinement to algebraic combinatorics on words
(2) “Little Devil”: a prediction of the theory.

The main question of this talk remains, whether the linear automorphism on Z7 or its refinement on Z^2 is music-theoretically relevant. I hope that the musical examples were helpful to illustrate and further motivate this question. While in the early Monteverdi example the bass-melody still behaved analogously to the others, we observe in the later compositions chains of falling fifths predominantly in the bass. Often they are not further connected through steps at a finer level.

While fifth progressions in the real bass and/or fundamental bass are indeed typical for the entire - so called - common practice period (baroque, classic, romantic music), it is not the case that we always find six such progressions in a row, completing an entire diatonic cycle - as in our examples. In the last part of my talk I will therefore inspect a scale with just three notes within the same mathematical context of fifth-generated well-formed scales. In particular I will musically interpret a certain interval, which - according to the theory - should play a prominent role. My collegue Karst de Jong and I called it “little devil”.

Another approach to this question allows for the fifth/fourth-coordinates to be hidden behind the musical surface. A resolute advocat of such an approach was the musicologist Jacques Handschin. I wish to mention the work of this extraordinary scholar as a valuable source, where an activity of mathematization may help to resurrect a deadlocked and meanwhile abandoned discussion.

Jacques Handschin
(1886 - 1955)

Der Toncharakter

eine Einführung in die Tonpsychologie

Zürich, 1948

The character of a tone within the "Tone Society" is determined by its position on the line of fifths.

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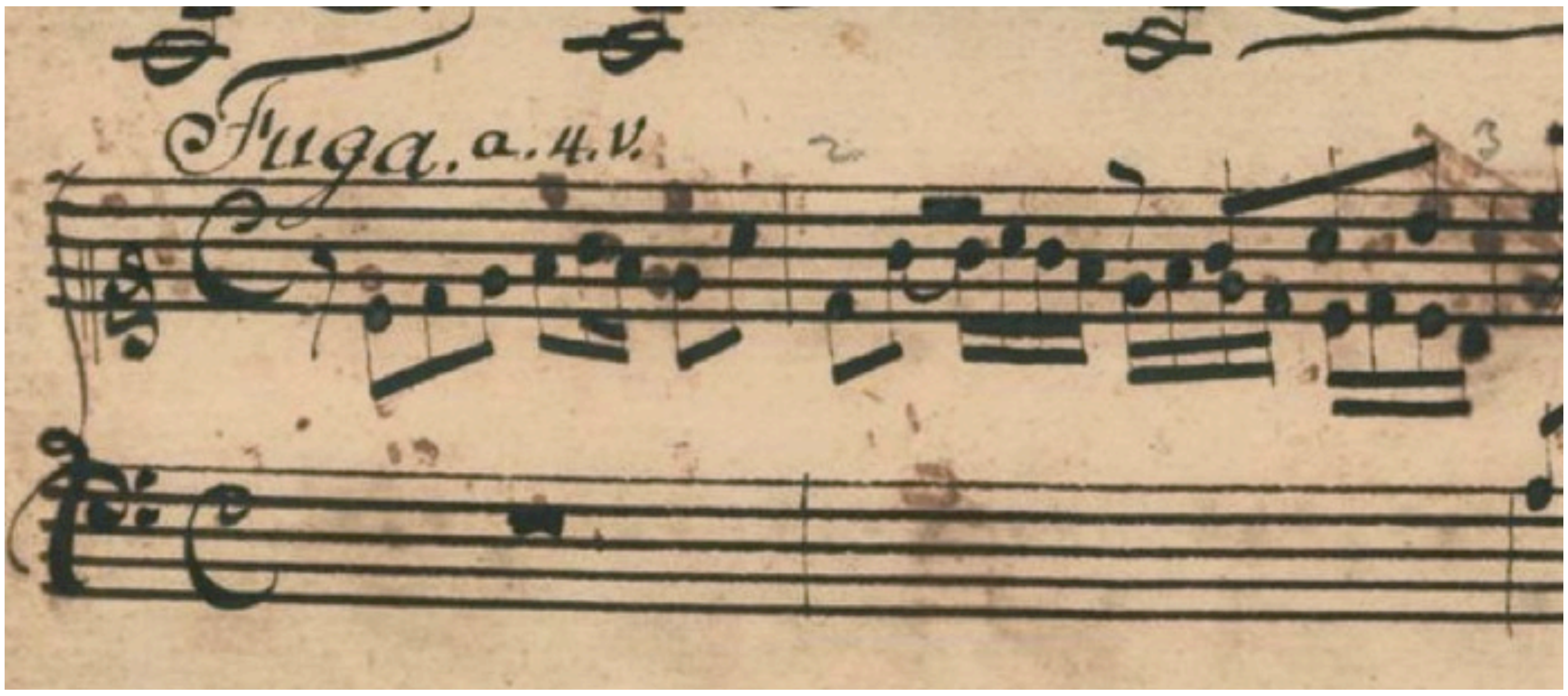
Modes, the Height-Width Duality, and Handschin's Tone Character (1)

David Clampitt and Thomas Noll

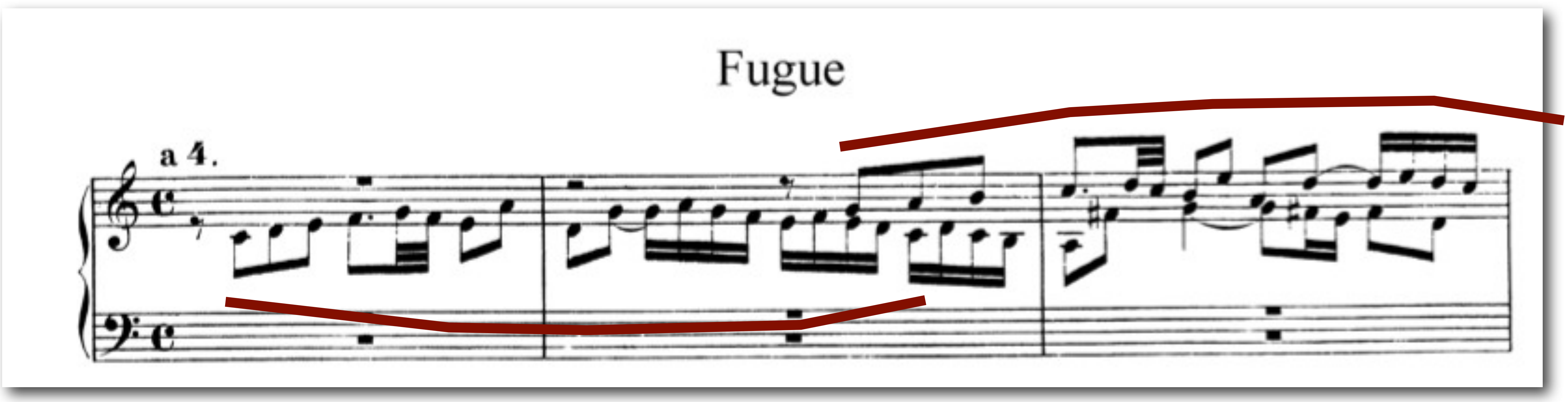
As a trained organist the young Handschin was familiar with the common practice literature. Political turbulences during the Russian revolution hindered him to realise his original goal, namely to advance organ music in St. Petersburg. He turned into musicology with particular focus on 13th century polyphony. So it happend that he percieved the 19th and early 20th century approaches to tone psychology from the viewpoint of an experienced musician on the one hand and as an expert of late medieval music and music theory on the other.

This made him quite critical about the predominant role of the pitch height parameter in the psychological debates. In addition to this external psycho-accoustically motivated parameter of pitch height he assumed the existence of a second internal tone parameter. Handschin argued that the generation order of the diatonic scale along the line (or cycle) of fifths reflects a musically relevant property of any tone, challed the tone's character.

One is tempted to speculate, that in such an audacious psychological hypothsis the linear automorphism should play a fundamental role. In a joint paper David Clampitt and I therefore aimed at reconnecting Handschin's ideas with mathematical refinements of the underlying theoretical concepts. This leads into a non-commutative variant of the note interval group and associated automorphisms.



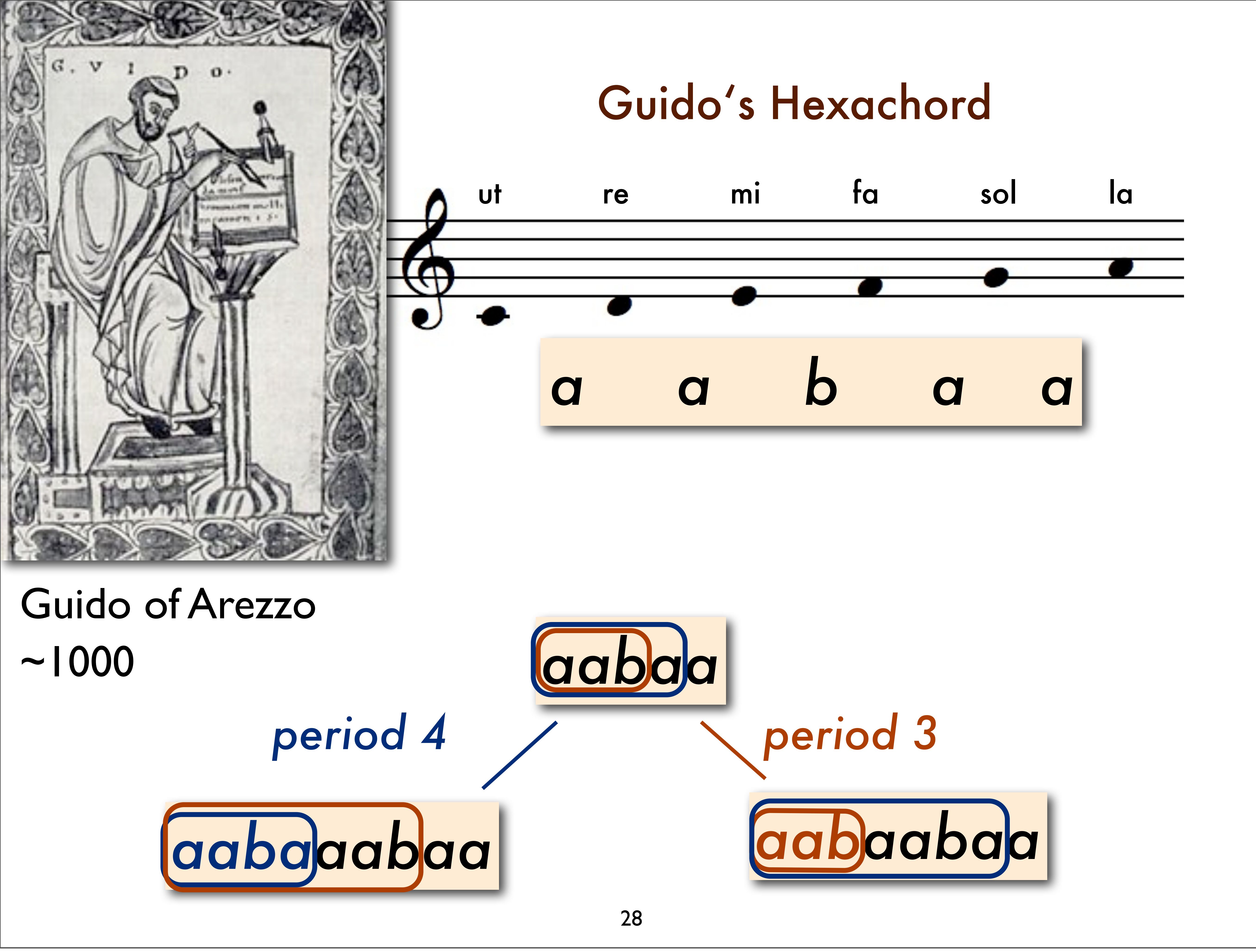
ut re mi fa sol la:



So we jump now into a non-commutative theory of musical intervals and tone relations and start with a prominent Region of tones, which is called the Guidonian hexachord. This involves a terminological switch from notes to tones. By tone I mean an enriched concept of note, which includes a position in a step interval pattern as well as a scale degree.

The first fugue of Bach’s programmatic collection of “Das Wohltemperierte Klavier” has a theme, which can be labeled with the six syllables of “Guido’s Hexachord”: ut – re – mi – fa – sol – la. This is a step interval pattern of the form M2 – M2 – m2 – M2 – M2 with a minor second between the tones with the syllables mi and fa. The first entry of this theme starts and ends on C (see the lower red arc) and the second entry starts and ends on G (higher red arc). In fugue theory one would call this a “real answer” as every note of the theme is transposed up a perfect fifth. Note, however, that – at the same time – the transposed theme fully remains in the original tonality of C-major. The note f# occurs only in the accompanying counter-subject, not in the transposed fugue theme.

Let us isolate the relevant property behind this finding:
The 7-note diatonic scale contains a 6-note subset c – d – e – f – g – a together with its transposition up a perfect fifth g – a – b – c – d – e.



This remarkable property has already been observed by Guido of Arezzo about 1 millenium ago. He emphasized that all the four notes d, e, f, and g have neighborhoods (e.g. at least one note below and one above) which can be found again one fifth higher in each case: around a, b, c' and d' (or likewise one fourth below). He called this property affinity.

The step pattern of Guido's hexachord can be modeled by a word in the two letters a and b (standing for the intervals M2 and m2). The word aabaa is an element of the free monoid, generated by a and b. Guido's affinity property at the fifth above can be then described as a certain property of double-periodicity of the word aabaaabaa. This is the step interval pattern formed by the tones c – d – e – f – g – a – b – c' – d' – e'. It includes the step-neighborhoods from d up to g and from a up to d'. This word aabaaabaa has the two periods 4 and 7. The perodicity 4 stands for the affinity as described by Guido and the periodicity 7 which stands for the implicitness that all this takes place within the octave-periodic diatonic scale.

Before we explore more music-theoretical details about these and other two-letter words, it is useful to put these words aabaa and aabaaabaa into a mathematical context.

Central Words

Limiting Cases for the Theorem by Fine and Wilf

Example: bonbon or bonnbonn or bonnbon

Fine and Wilf's theorem on words from PlanetMath.org (Theorem)

Let w be a word on an alphabet A and let $|w|$ be its length. A period of w is a value $p > 0$ such that $w_{i+p} = w_i$ for every $i \in \{1, 2, \ldots, |w| - p\}$. This is the same as saying that $w = u^k r$ for some $u, r \in A^*$ and $k \in \mathbb{N}$ such that $|u| = p$ and r is a prefix of u.

Fine and Wilf's theorem gives a condition on the length of the periods a word can have.

Theorem 1 *Let w be a word on an alphabet A having periods p and q. If $|w| \geq p + q - \gcd(p, q)$, then w has period $\gcd(p, q)$. The value $p + q - \gcd(p, q)$ is the smallest one that makes the theorem true.*

Example for a maximal doubly-periodic word

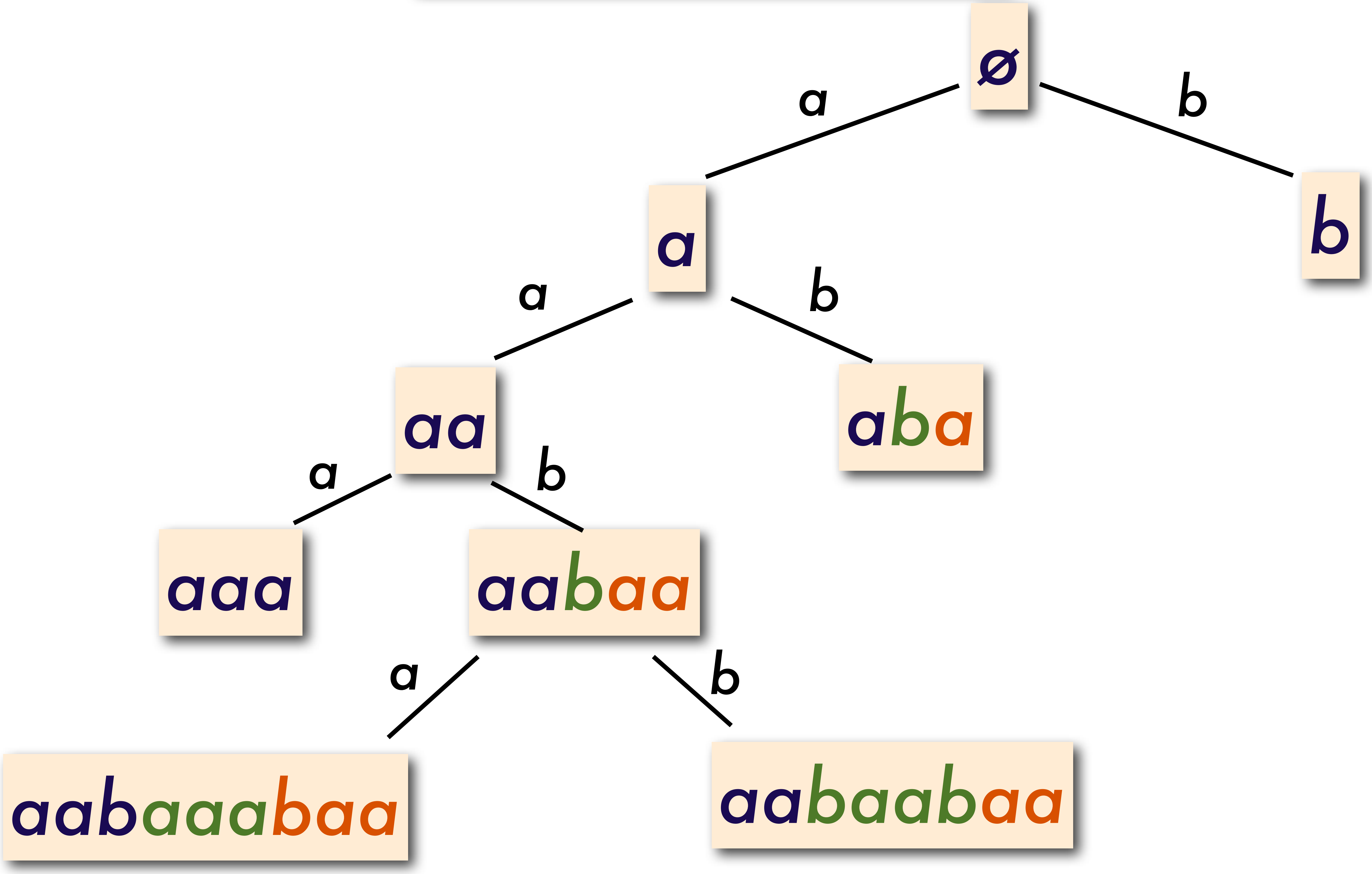
aabaa

There is a particular family of two-letter words which are known under the term “central words”. There are also called central palindromes, as they are palindromes of a certain kind. Let us first look at the rather exotic property of double-periodicity.
Within word theory (algebraic combinatorics on words) there is a known result - called “Fine and Wilf’s Theorem”. Before stating this theorem let me make sure that the concept of a periodic word is clear. Abvioulsly “bonbon” is periodic with period 3 and “bonnbonn” is periodic with period 4. These words are powers of a certain factor, whoose length is the period. A more liberal concept of periodicity allows that the last instance of this factor is incomplete. In “bonnbon” the suffix “bon” is only a prefix of the factor “bonn”. The missing letter “n” is not regarded as an obstacle to speak of a periodic word.

Fine and Wilf’s theorem says: If a word (with any number of different letters) has two periods p and q, then it either has also gcd(p, q) as a period or it is very short: It’s length must be less than p + q - gcd(p, q) in order to allow two “independent” periods. Central words represent the limiting case for this theorem: The maximal truly doubly-periodic words. They are two-letter words of length p+q-2, while p and q are co-prime.

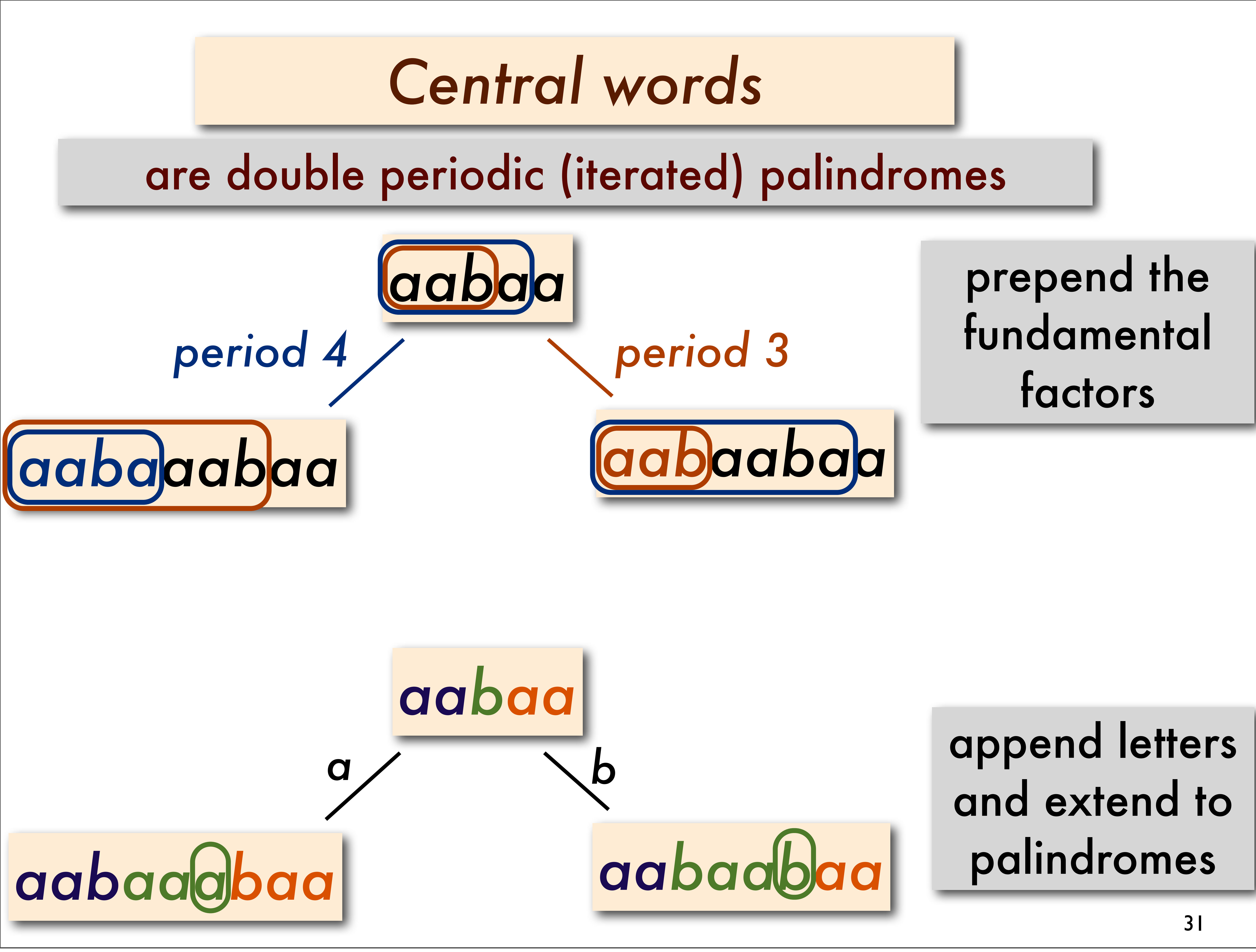
Central words

form a binary tree

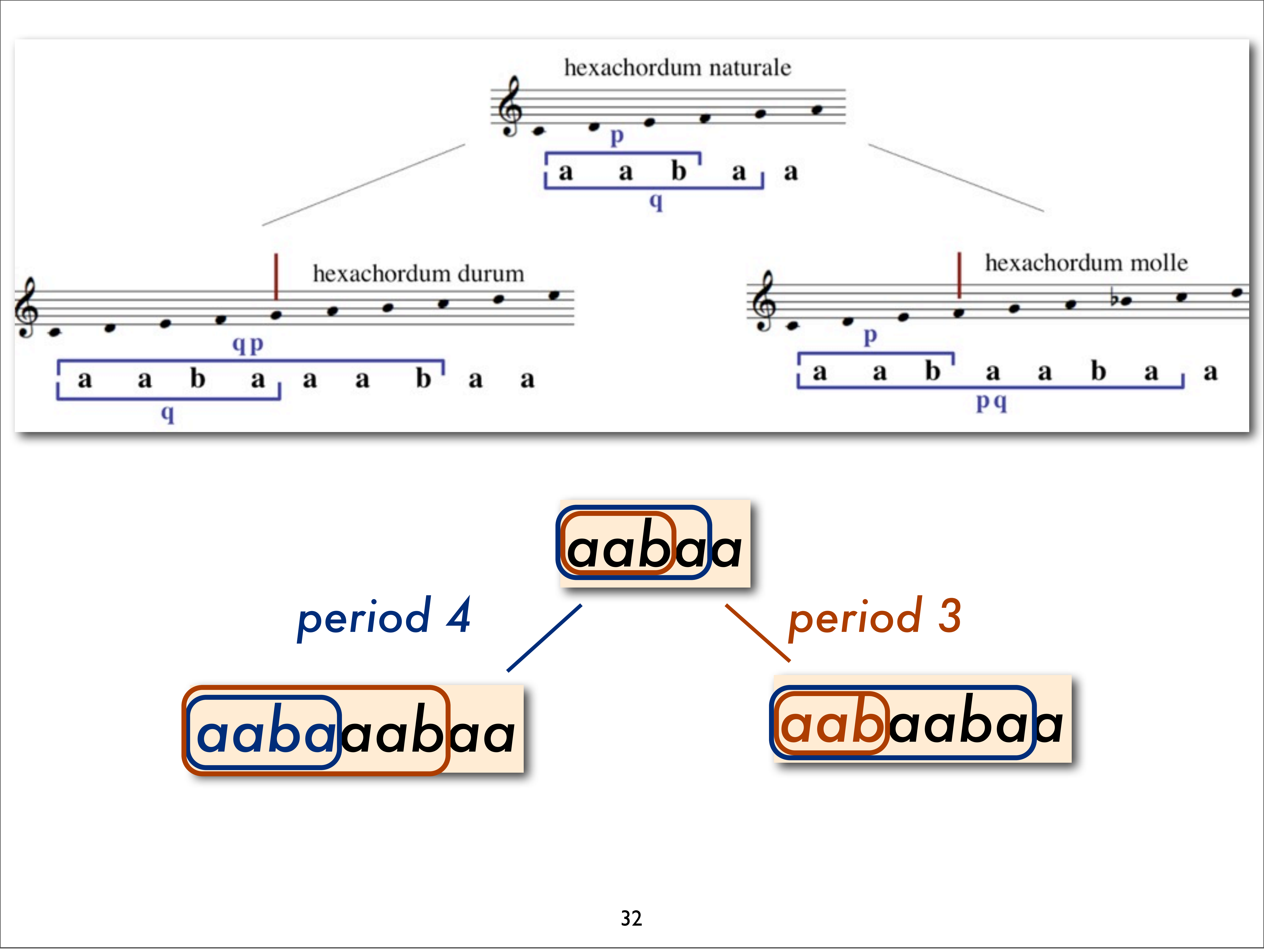


The central words form a binary tree and can be constructed iteratively in the following way:
On top we start with the empty word w = ø. Suppose now, we have already constructed some node w of the tree.
In order to construct the left successor, we append the letter a to the word w and decompose the word wa = uv into the maximal palindromic suffix v (green color) and the remaining prefix u. Then we append the retrograde u~ (red color) of u (reading u backwards) and obtain wau~. In order to construct the right successor, we append the letter b and follow the same proceedure.

The two periods can be calculated from the letter frequencies. For example the periods 4 and 7 in our example aabaaabaa (bottom left) are the multiplicative inverses of 3 and 8 modulo 11, where 3 equals 1 plus the multiplicity of letter b and 8 equals 1 plus the multiplicity of letter a; and where 11 = 8 + 3 which equals 2 + the length of the word.

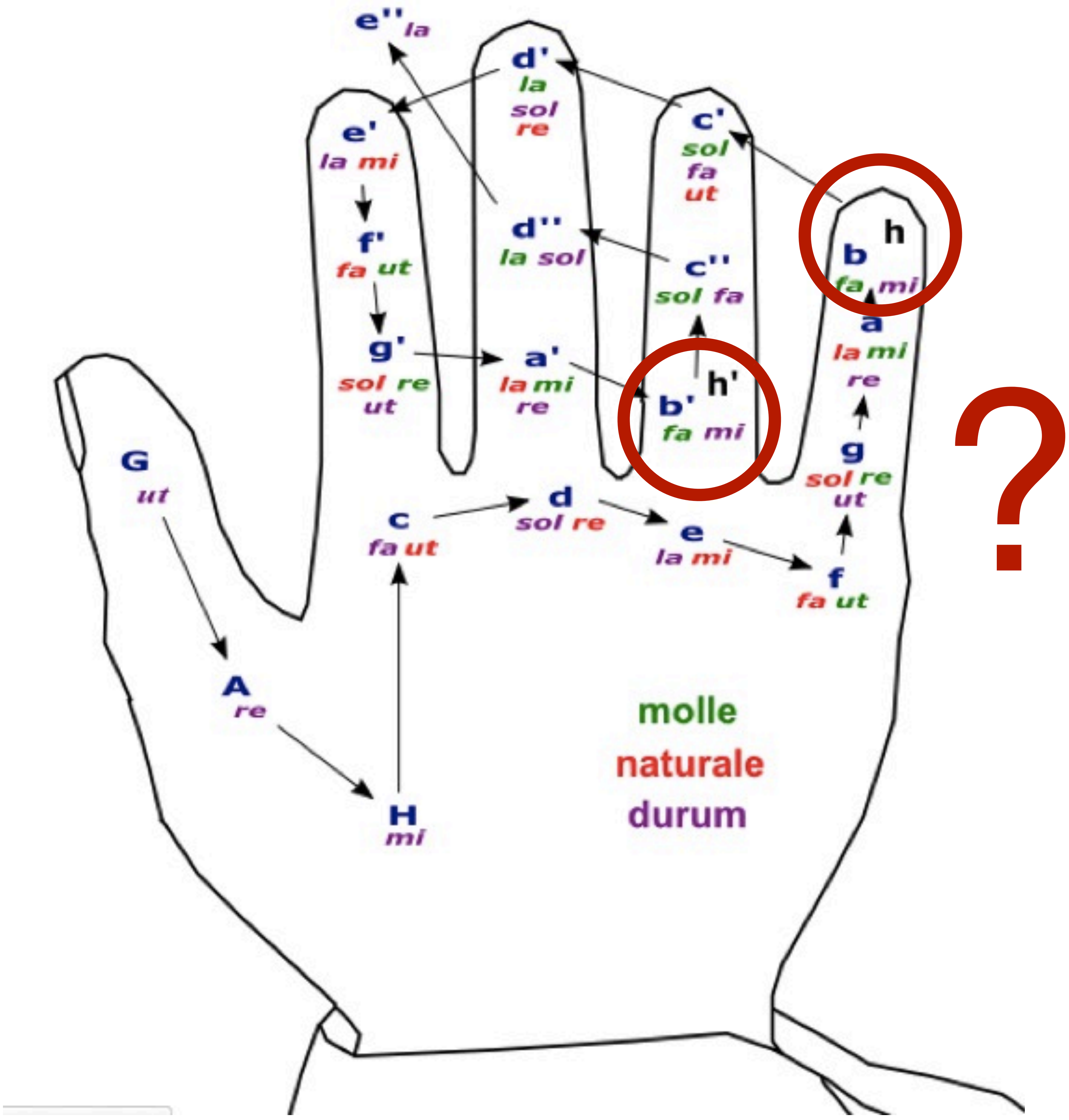


Once the periods of a central word are known one may alternatively construct its successors on the binary tree by prepending the two "fundamental" factors which represent the periods. The two periods of (the step intervall pattern of) Guido's hexachord aabaa are 4 and 3 and they correpsond to its prefixes aaba and aab. Prepending these two words to aabaa we obtain the left successor (aaba)aabaa and the right successor (aab)aabaa.



This slide shows a musical interpretation of the hexachordal pattern aabaa and its two successors on the tree of central words.
If all three patterns start from the same note c, we obtain the so called natural hexachord as well as two tone regions which are covered by the natural hexachord and one other instance of the hexachord:
g – a – b – c – d – e is called the hard hexachord and f – g – a – b-flat – c – d is called the soft hexachord.

12. century



cited from Wikipedia

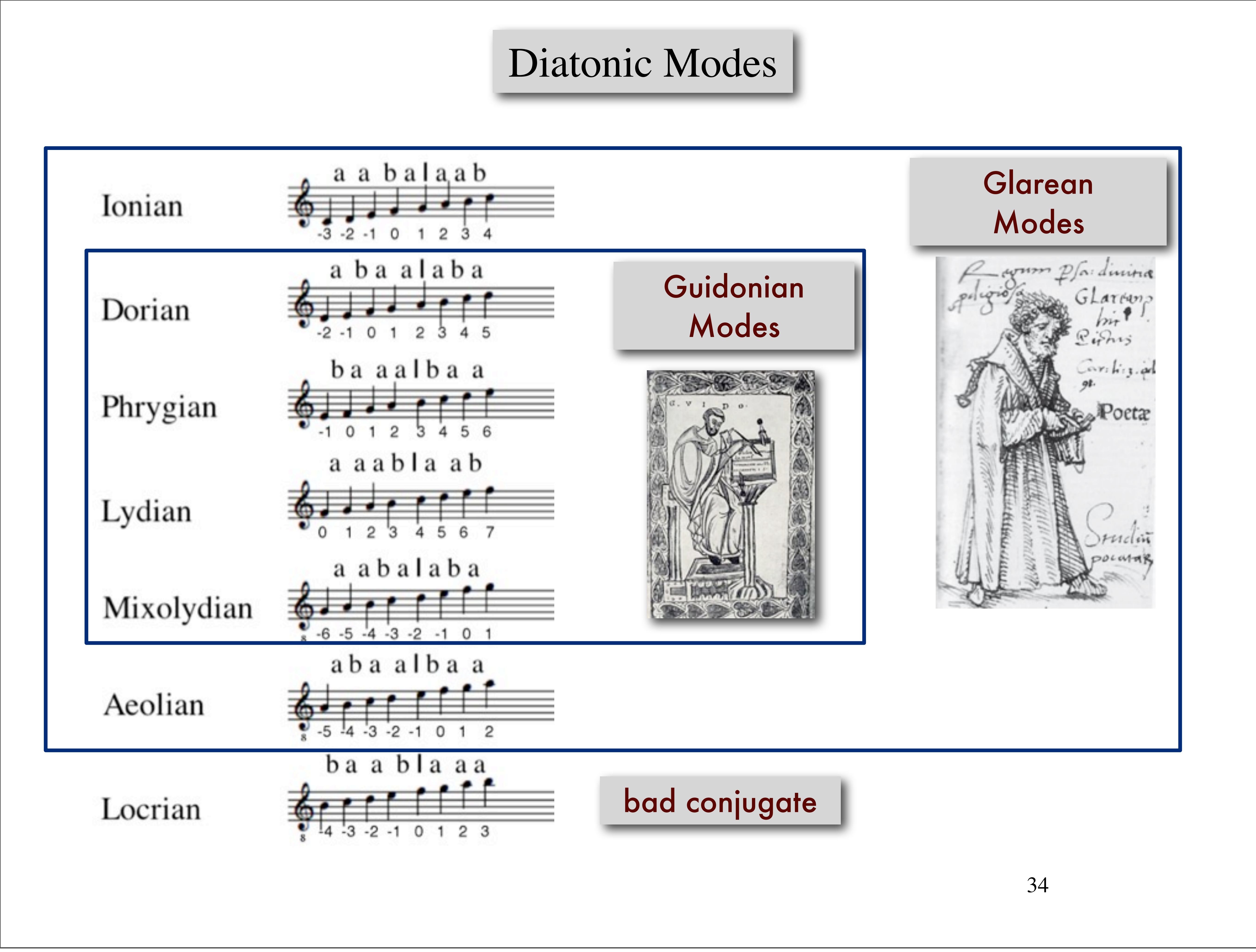
In the 12 century a mnemonic technique was developed, namely to navigate through the tone repertoire (the Gamut) with the help of covering hexachords. The so called “Guidonian Hand” became a gestural manifestation of this mnemotechnique. The three instances and partially overlapping instances of the hexachord are colored red, green and pink.
It is indeed remarkable that the hexachord served in this navigation function although(!!) the basic constitution of the musical tone relations was essentially diatonic. The next slides are dedicated to the so called “pseudo-classical” diatonic modes, which are theoretical counterparts to the traditional church tones (and psalm tones).

[NB: The term “pseudo-classical” has been choosen in order to acknowledge the intention of the medieval and Renaissance theorists to anchor their theoretical approaches in elements of anchient greek music theory. The negative connotation of the attribute “pseudo” (= not properly greek modes under the strict eye of modern philology) is compensated by a positive connotation namely that these theorists have been productive and innovative pioneers in music theory.]

On this slide I would like to direct your attention to an interesting circumstance in the musical Gamut, (i.e. in the basic medieval note repertoire) and its covering by the hexachords: The generic diatonic scale degree, which corresponds to the note b, occurs in two forms: as b and as b-flat. Actually one would not need all three hexachords (natural, hard and soft) in order to cover the diatonic scale. Why did the mediaval musicians and theorists nvertheless add b-flat and only b-flat to the basic note repertoire?

Allow me to just mention two points without further explaining them: One reason has to do with the combinatorial limitation of the mediaval classification of modes. The “missing” modes, which had been officially acknowledged by Glarean in the 16th century did practically already exist in transposed form the medieval system thanks to the availability of the b-flat. Somewhat related, but not “equivalent” to this limitation was the desire to avoid diminished fifths as succesive and simultaneous intervals.

In the sequel I would like to give a surprising mathematical argument in favor of b-flat as a more natural choice for a single added note to the seven “white notes” c, d, e, f, g, a, b over the alternative f#. We shall look into this in detail.



We will start with a quite simple combinatorial definition of the diatonic modes. Later we make them more sophisticated.
The simple definition describes a mode as a species of the octave, i.e. as a cyclic permutation of the word aabaaab.
i.e. as a word in the free monoid of words over the alphabet {a, b}, which is conjugate to aabaaab.
Music-theoretically these words are interpreted as step-interval patterns. The letter "a" designates the major step and "b" the minor step (as in hexchord a view slides ago).
In order to obtain a sequence of notes from a step interval pattern we need to specify an anchor note: a tonic, say.
The seven conjugates of aabaaab are shown here as a familiy of white-note modes. There is no sharp and no flat.

The frames and the images of Guido of Arezzo and of Heinrich Glarean indicate that these modes have been thematized and acknowledged in different historical periods. There is a group of four modes on d, e, f, and g which form the four authentic modes of the so called octenary system. To each one there is a plagal counterpart. The modes on c and a have been acknowledged as modes in their own right by Heinrich Glarean in 1547. Together with the previosly mentioned four modes they form the six authentic modes within the so called Dodecachordon, i.e. within a family of 12 modes.

To understand these distinctions mathematically we need to refine the definition of mode. An essential element in the pseudo-classical concept of mode is a division of the perfect octave into a perfect fifth and a perfect fourth. In the authentic modes the fifth occupies the lower range and the fourth the higher one. In the plagal modes it is the other way around. Today I restrict myself to the authentic modes.

Above the little scores you see inbetween the notes the letters a or b for the associated step intervals. Above the fifth note in each mode you see a vertical line which marks the divider. In addition the the species of the octave (a word with seven letters) we have now also a species of the fifth (the divider prefix) and a species of the fourth (the divider suffix). For example in the Ionian mode aabaaab the word is divided as aaba | aab. The Ionian species of the fifth is aaba and the Ionian Species of the fourth is aab.

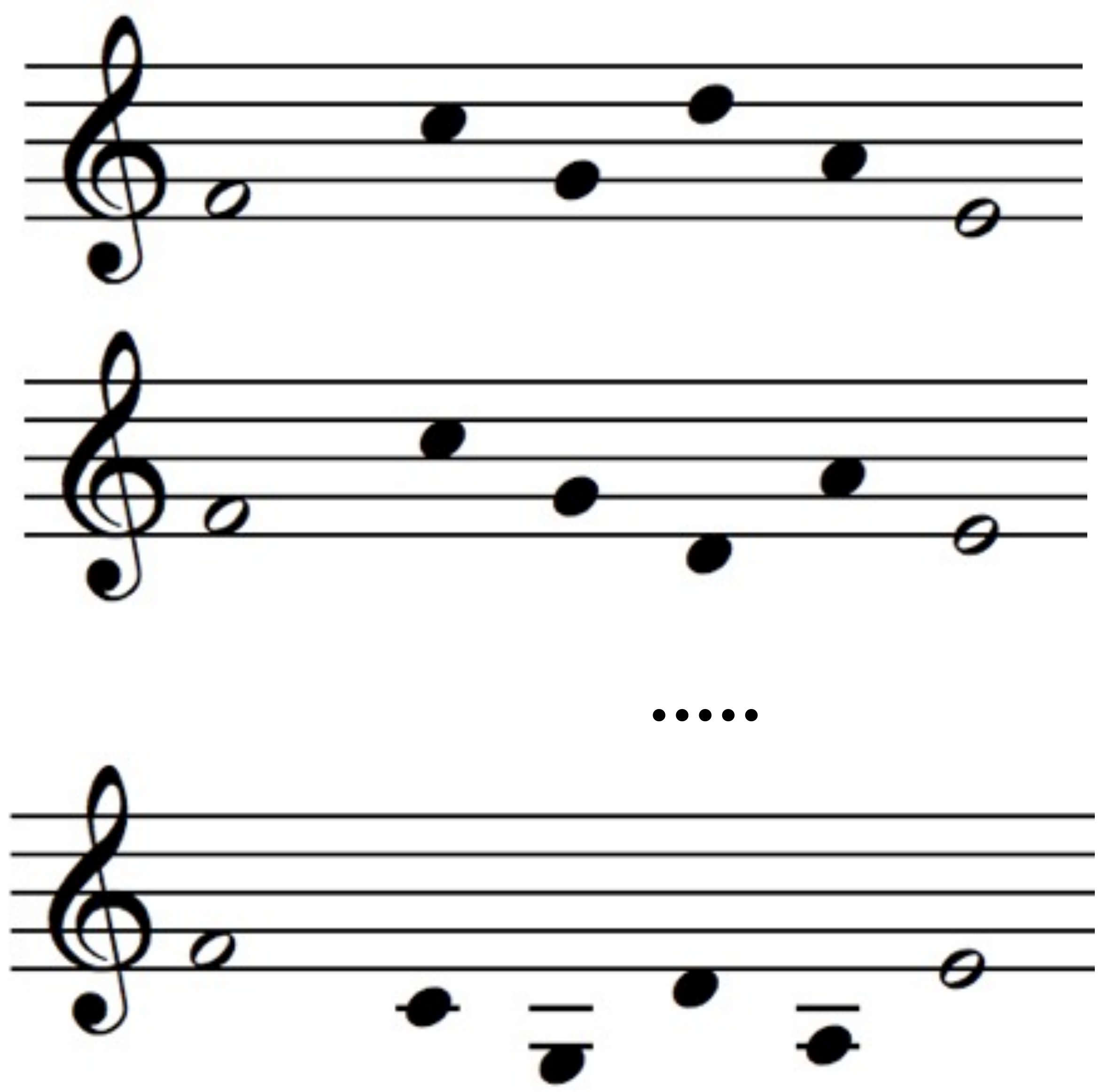
The locrian mode is labeled as the "bad conjugate". This term has been introduced by the word theorists. But also the music theorists avoided the Locrian mode. Observe that for all six Glarean modes the species of the fifth are conjugate to each other and also the species of the fourth are conjugate to each other. What distinguishes the Guidonian modes among the Glarean modes is the affinity property mentioned earlier. [NB: the "new" Glarean modes have a property which David Clampitt coined double-neighbor polarity: the step neighborhoods of the tonic and the divider differ from one another.]

Our goal is now to dualize these step interval patterns together with the subdivisions. Let me therefore draw a connection to the first part pf my talk. The perfect fifth and fourth served as elements for a basis of the note interval space. If we express them in step coordinates through the linear automorphism ((3 2) (1 1)) we obtain the "vectors" (3 1) and (2 1). The four species of the fifth aaba, abaa, baaa and aaab and the three species of the fourth aab, aba, baa are non-commutative refinements of these two vectors.

$$M2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad -m2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Species of the Major Second

Species of the Minor Second



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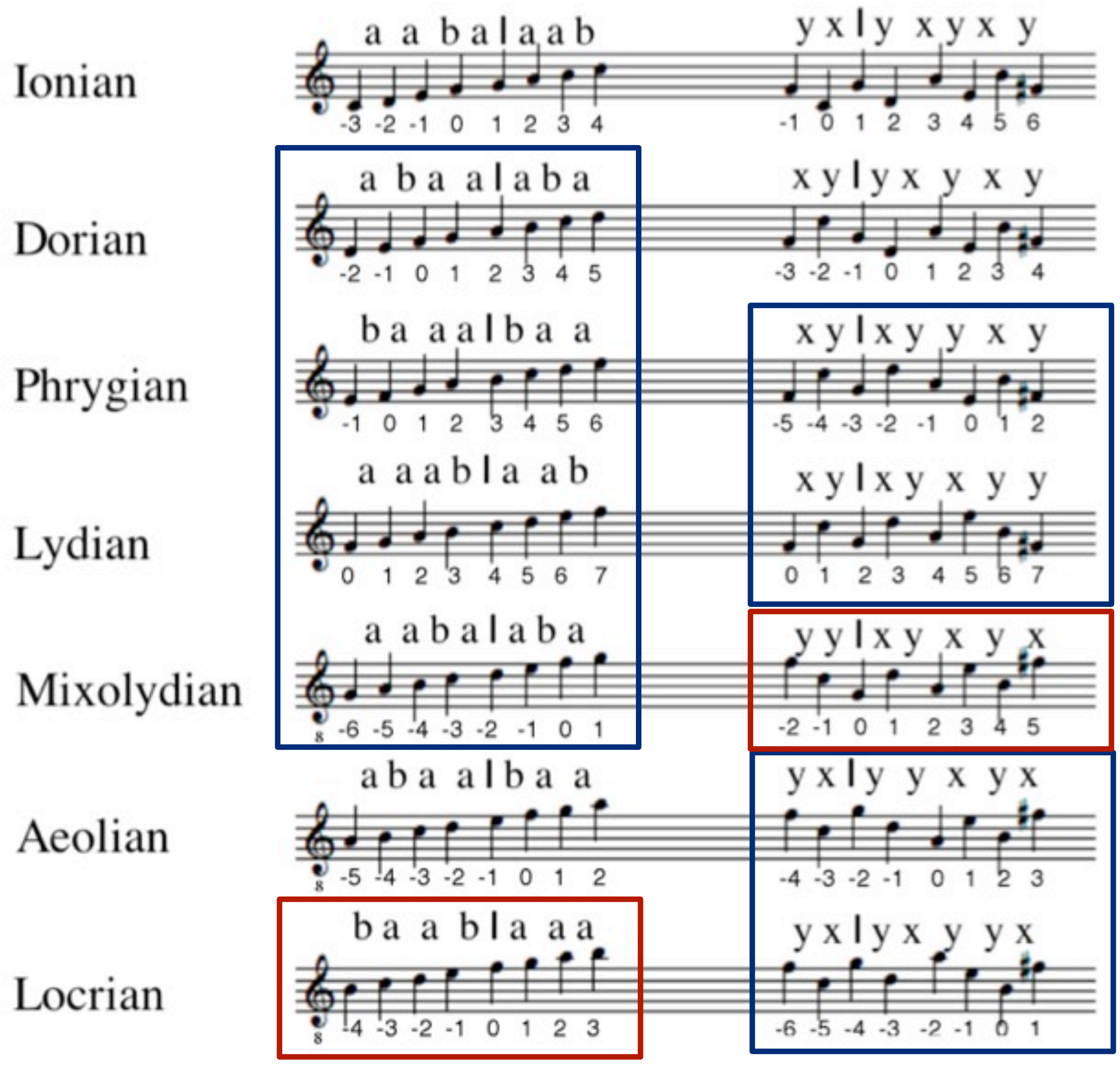
Dually, we expressed the ascending major and minor seconds in fifth/fourth coordinates (see slide 24): M2 = (1, -1) and m2 = (-2, 3).
On this slide here you see associated non-commutative refinements, which I term “species of the major second” and “species of the minor second”. The minor sencond goes downward here, though. We obtain theses species by dualizing the traditional concepts. For each species we require that the height ambit remains inside the range of an octave. On the next slide you see, why...

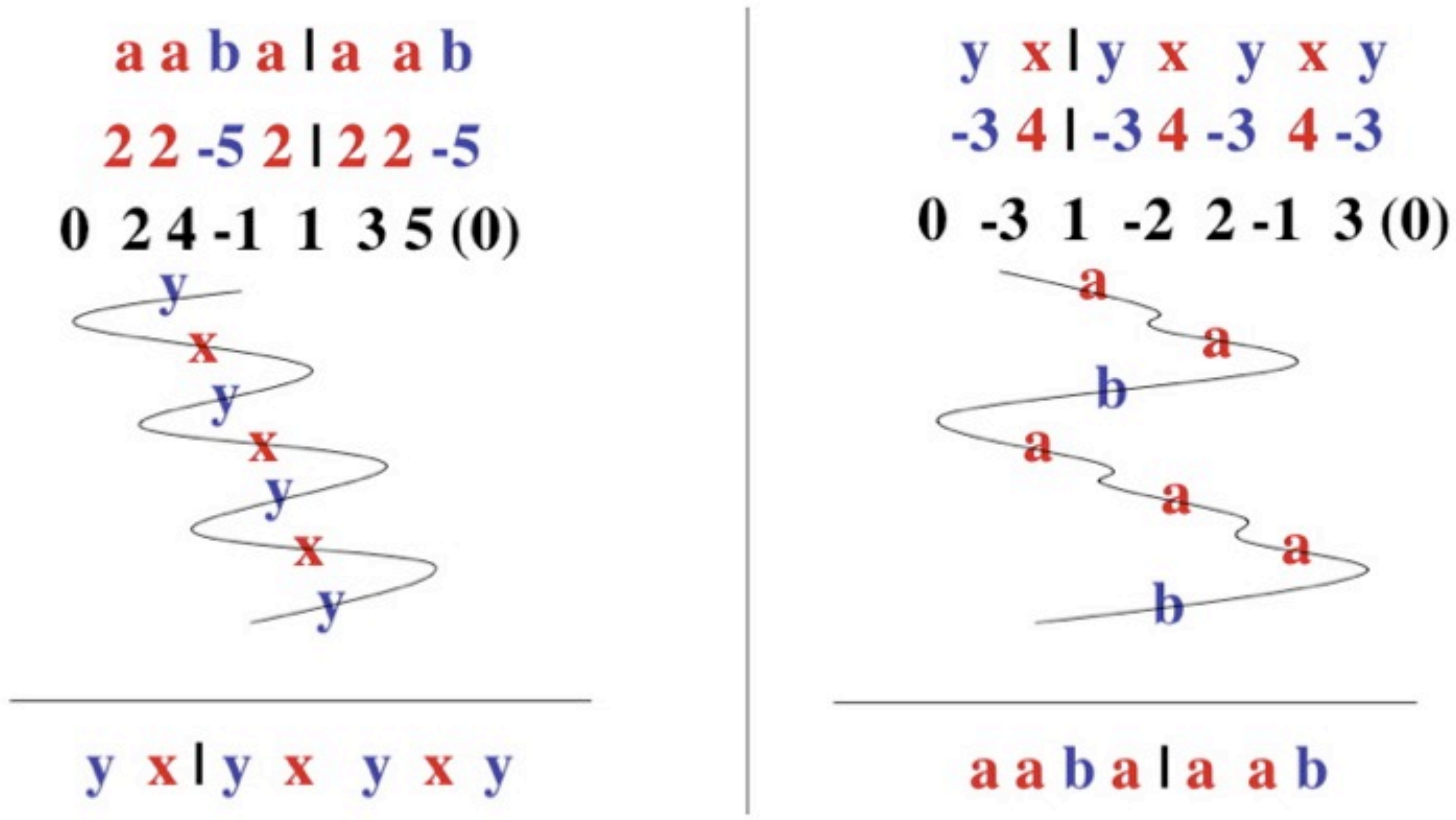
Modes and their Plain Adjoints



On the left side we see the step patterns of the diatonic modes as before. On the right side we see associated folding patterns with upward fifths and downward fourths. The dual concept to the octave (as a framing interval) is now the augmented prime (a sharp). Let us first give a music-theoretical definition of the folding patterns on the right side. They are uniquely determined by the condition that all the notes (except the last and highest one, which repeats the first note one octave higher) in the stepwise ascending modes on the left side are reached precisely at the same height on the staff, but in fifth/fourth ordering.

All these folding patterns start on f and end on f#. They all constist of three ascending fifths (letter x) and four descending fourths (letter y). It turns out that all the seven patterns form conjugates of the ionian folding yxyxyxyxy. With the exception of the mixolydian folding yyxyxyx they are all devided into a conjugate of yx and a conjugate of yxyxy. We obtain a dual classification of the seven foldings into (1) four ones with Guido's affinity property, two analogues to Glareans extensions and one analogue to the bad conjugate: the mixolydian folding.

Calculation of the plain adjoint (by example)



similarly: twisted adjoint

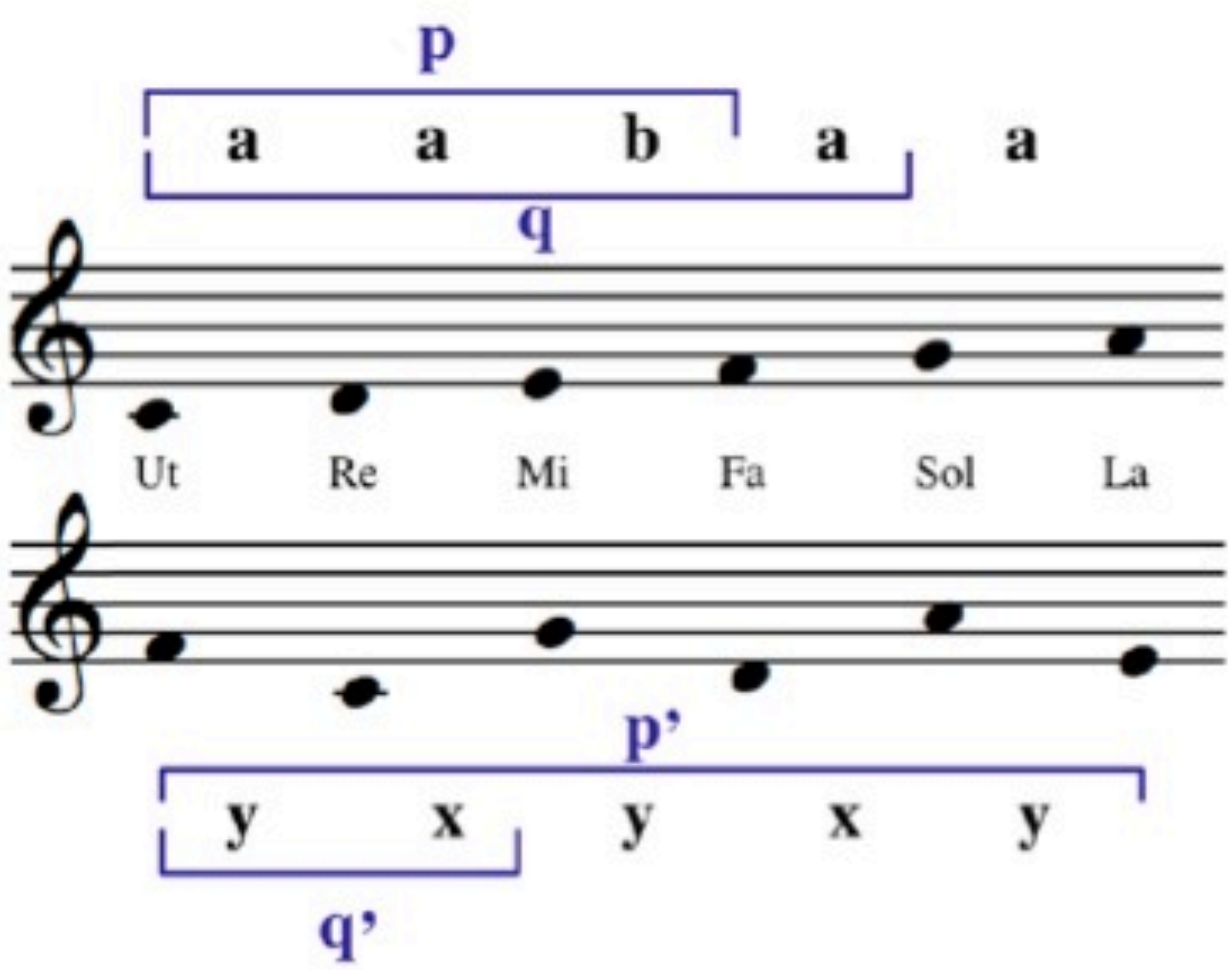
Each folding pattern is a two-letter-word and therefore it is desirable to calculate it directly from the associated step interval pattern.
To do that in appropriate generality we have to know the word-theoretical scope for this kind of “dual” (we say plain adjoint).
In extrapolation of the term “well-formed scale”, we say that a well-formed word is any two-letter word which is conjugate to the step interval pattern of any non-degenerate well-formed scale. Mathematically this is equivalent to say that it is conjugate to any fundamental period (c.f. slide 31) of a central word. [In the literature the fundamental periods of central words are called standard words.]
Here is a simple algorithm to calculate the plain adjoint:
(1) substitute every occurrence of the letter “a” by the multiplicity of the the letter “b”: here 2.
(2) substitute every occurrence of the letter “b” by minus the multiplicity of the the letter “a”: here -5.
(3) calculate partial sums starting from 0.
(4) parse these numbers in ascending order starting from the minimum: here -1
(5) from the maximum return to the minimum
(6) interpret the resulting zigzag path as a two-letter-word: x = move to the right, y = move to the left.

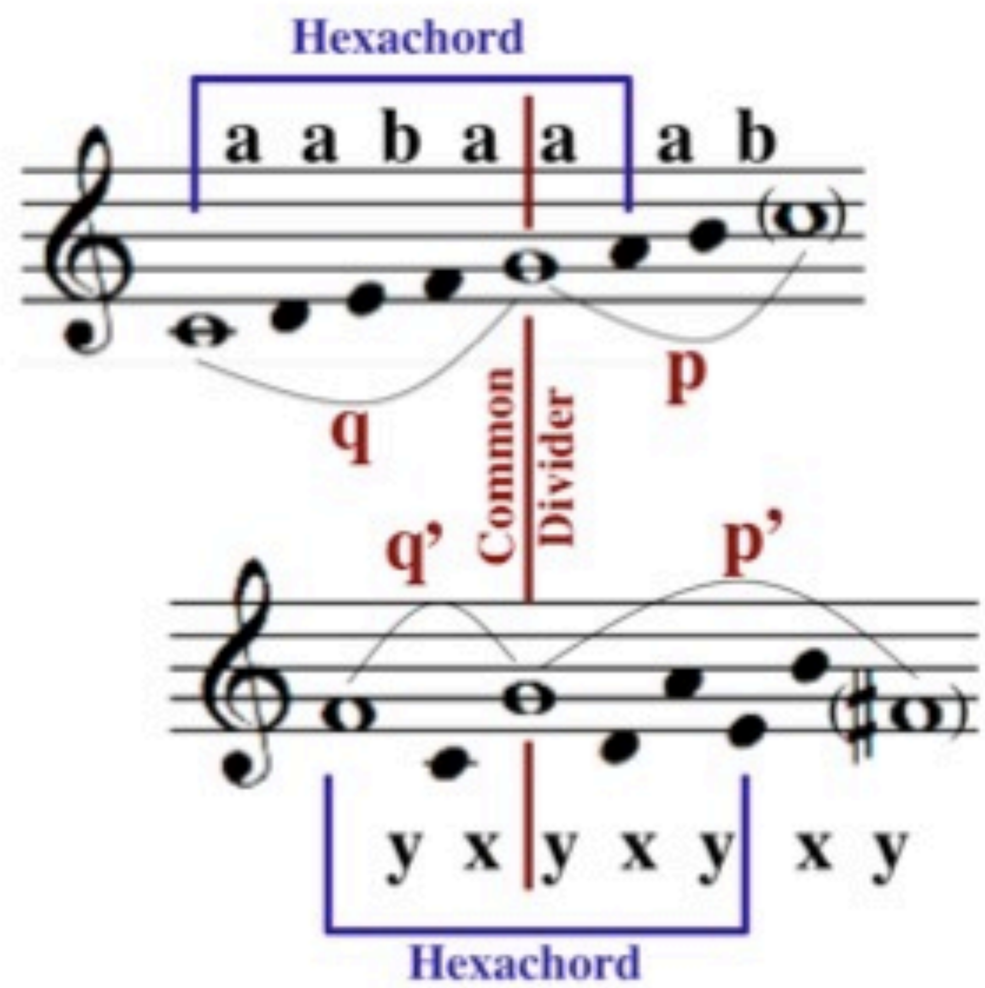
There is an analogous proceedure to obtain a folding pattern consisting of decending fifths and ascending fourths. We call it the twisted adjoint.
(1’) here we substitute every occurrence of the letter “a” by minus the multiplicity of the the letter “b”: here -2.
(2’) and accordingly we substitute every occurrence of the letter “b” by the multiplicity of the the letter “a”: here 5.
(3 - 5) are the same as in the plain case
(6’) in the zigzag path now x = move to the left, y = move to the right.

I shall show you this twisted adjoint a few slides later. The reason why I anticipate it here is the following: From the viewpoint of these two algorithmic definitions there is no evident reason, why one of the two types of adjoint (plain or twisted) should be preferred. They look quite symmetric to each other.
David Clampitt and I - when we started our word-theoretic investigations onto modes in 2007, were tempted to favor the plain adjoint in connection with one particular mode: The Ionian mode. The temptation was not unbiased: The Ionian mode corresponds to the “modern” major mode.

Hexachord and Ionian Mode:

periods 4 and 3
frequencies 2 und 5





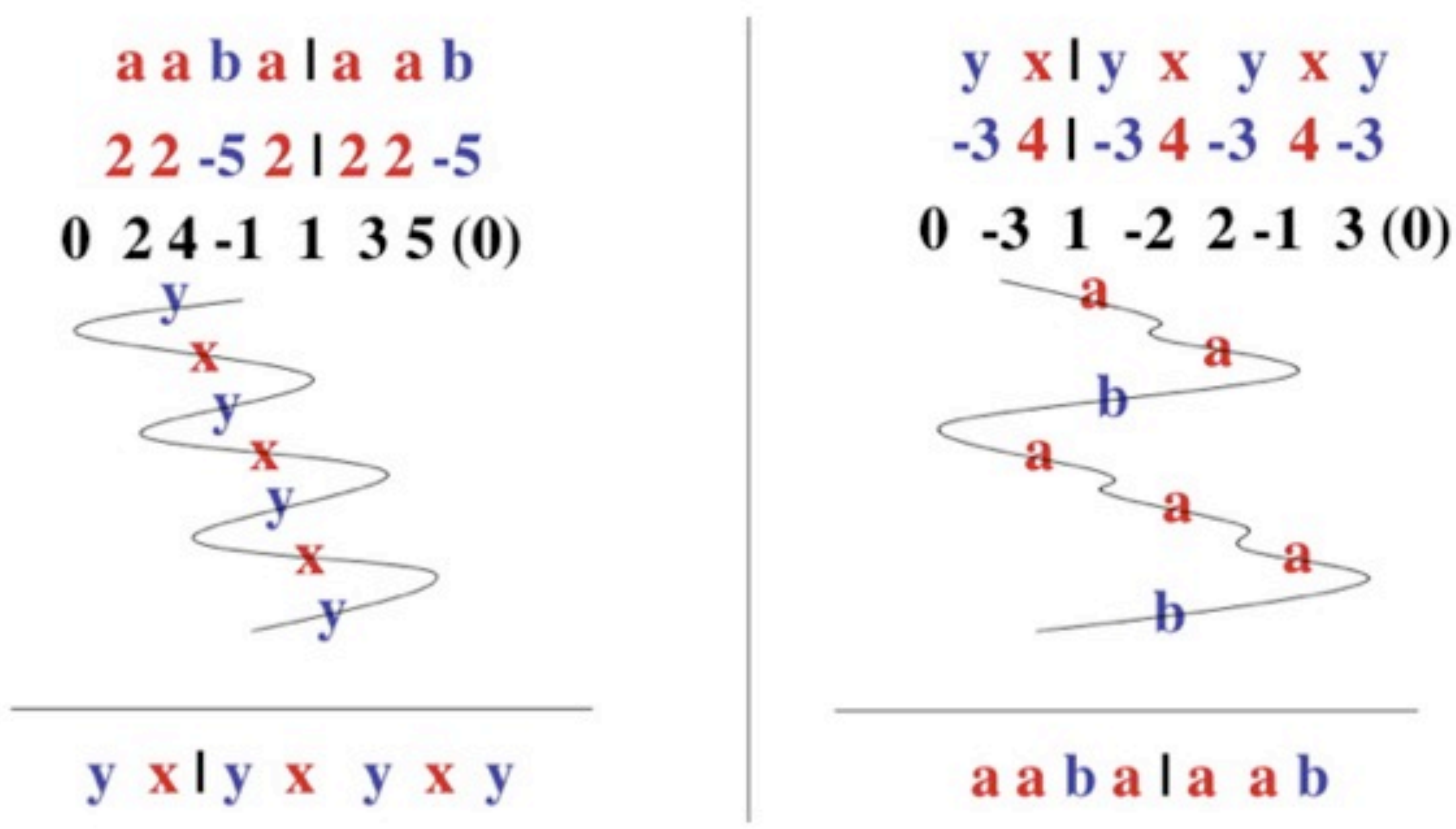
frequencies 4 and 3
periods 2 und 5

One has a duality for central words, which is a refinement of the duality between the Stern-Brocot-Tree and the Raney-Tree for the positive rationals. The duality is given by the reversal of pathways down the tree. We reached the node aabaa on the central tree along the path left-left-right (this corresponds to the so-called directive word aab). The dual of the step pattern aabaa of the Guidonian hexachord is yxyxy which is the fifth/fourth folding pattern of that hexahord. It is reached on the cenral tree along the path right - left - left (directive word yxx). In accordance with the palindromic structure of both words aabaa and yxyxy there is just this one kind of “adjointness”, namely duality of central words.
What attracted our music-theoretical attention was the observation that the Ionian species of the fifth aaba and of the fourth aab coincide with the fundamental periods of the Guidonian hexachord aabaa. The same is true for the major second species yx and the minor second species yxyxy in the division of the plain fifth/fourth folding of the Ionian mode yx | yxyxy. These are the fundamental periods of the folding of the hexachord yxyxy.

Ionian Theorem:

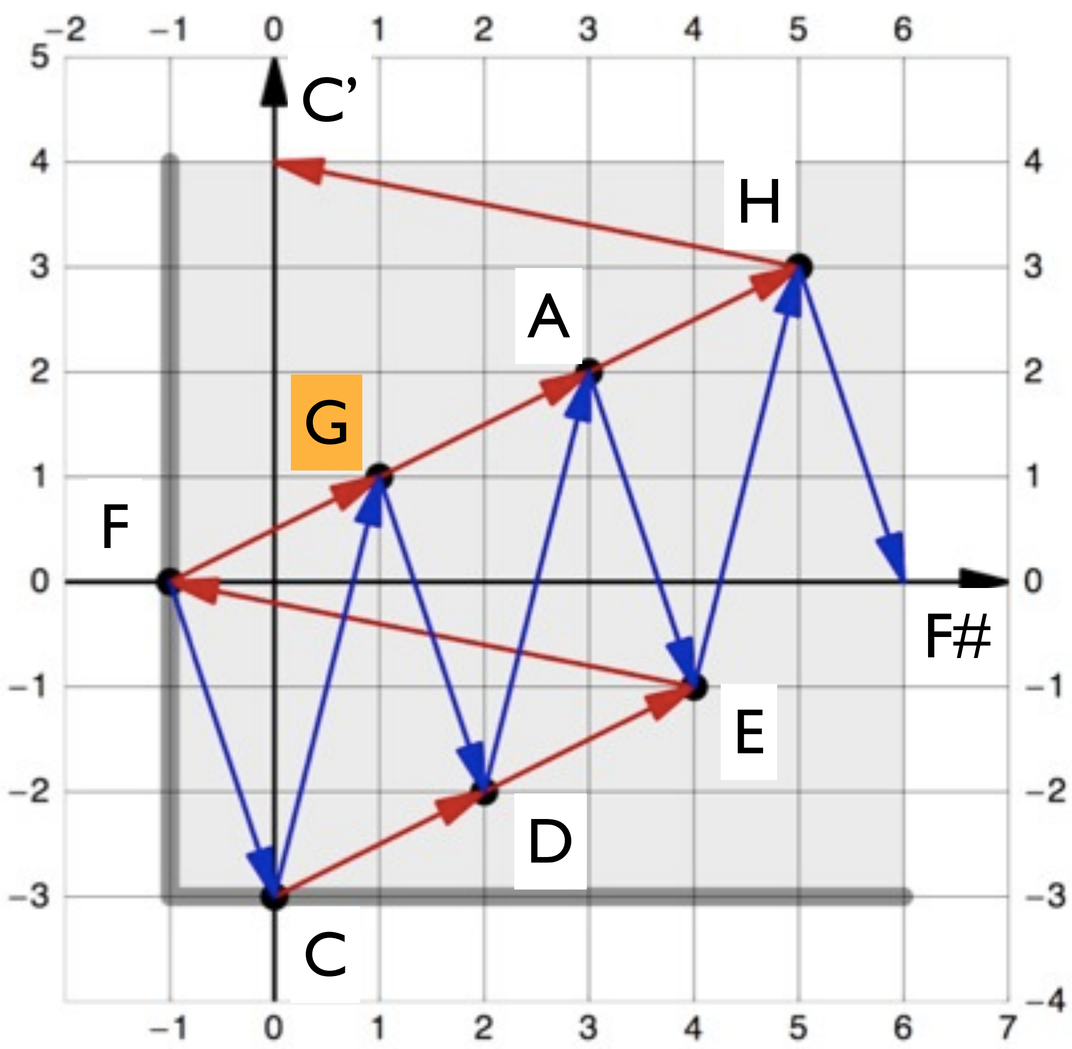
COROLLARY 6.2 *(Ionian Theorem) Let $w = f(ab) \in \{a < b\}^*$ denote a positive standard word with $f \in \langle G, D \rangle$. Along with w we consider its plain adjoint $u = w^{\square} = f^{\boxslash}(xy) \in \{x < y\}^*$. The authentic divider of w on the height trajectory Φ_w and the authentic divider of u on the width trajectory Ψ_u coincide:*

$$AuthDiv(w) = \Phi_w(|u|_y) = (1,1) = \Psi_u(|w|_b) = AuthDiv(u).$$



My interest to understand the musical prominence of the Ionian mode among its conjugates eventually led to a theorem about standard words (= fundamental periods of central words). To be more precise. It is a theorem about the trajectories in Z^2 which we may assoociate with these words. This theorem states that the points which represent the dividers in a standard word and its plain adjoint (which is also standard) coincide. I called this property “divider incidence”.

aabalaab yxlyxyxy



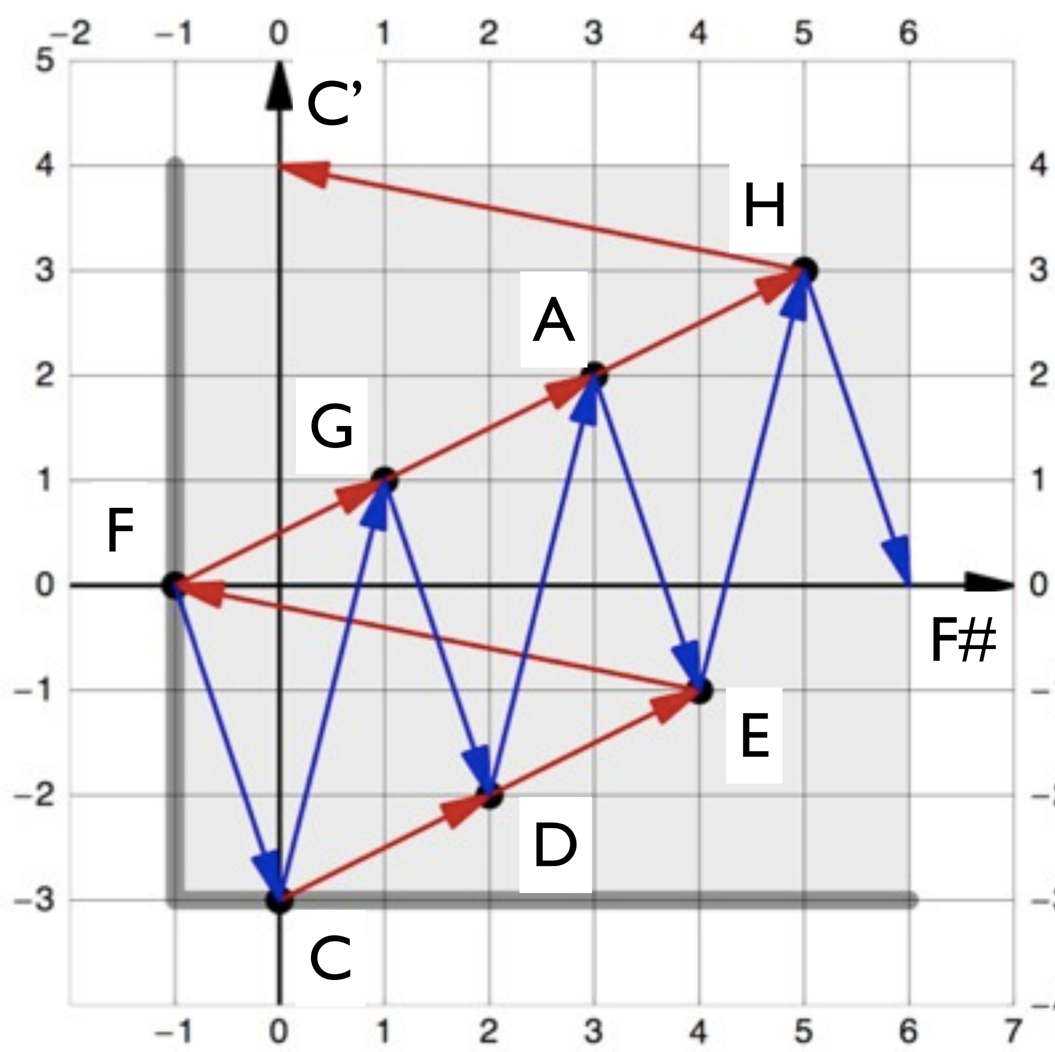
plain adjoint

Let me illustrate this:
The coordinates of the points (notes) are obtained from the partial sums in the calculation of the adjoints.
In the C-Ionian mode it is the note G = (1,1) which is – on the one hand – the authentic divider of the octave c – c’ into perfect fifth and perfect fourth, and which – on the other hand – is the divider of the ascending augmented prime f–f# into and ascending major second and a descending minor second.

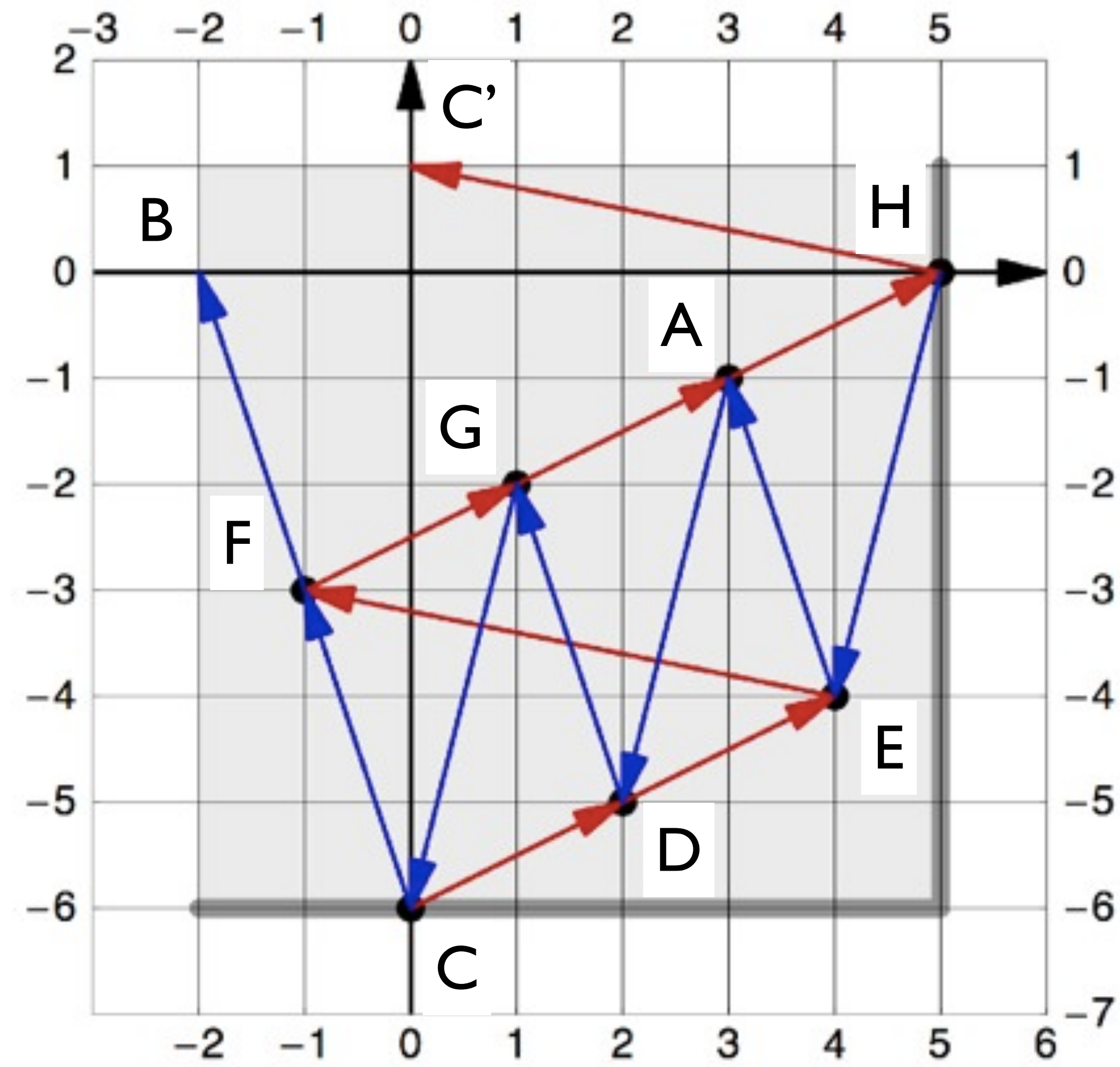
Comparing the two Adjoints in F_2

Automorphism f of F_2: x = f(a) = aaba y = f(b) = aab

aabalaab $y^{-1}xly^{-1}xy^{-1}xy^{-1}$ aabalaab $x^{-1}ylx^{-1}yx^{-1}yy$

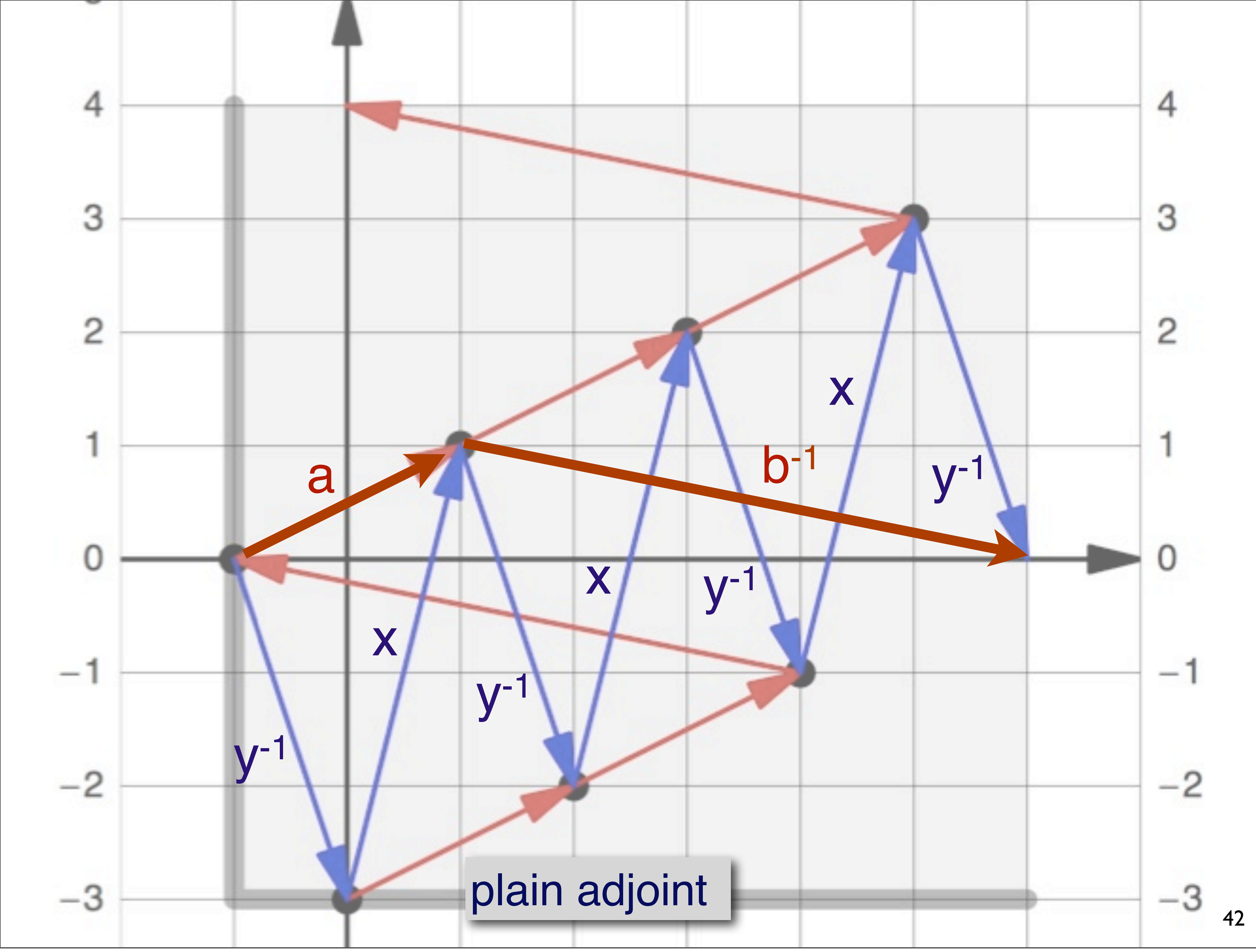


plain adjoint



twisted adjoint

So far we used letters as generators of a free monoid. But musical intervals can be ascending or descending, they may be directed sharpward or flatward. Therefore it is promising to study their concatenation within the free group F_2. Here you see the asending C-Ionian mode (red trajectories) together with their plain and twisted adjoints (blue trajectories).
The fact that the species x = aaba and y = aab are the fundamental periods of a central word implies that the map f(a) = aaba and f(b) = aab defines an automorphism of the free group. Furthermore, we may investigate the plain and twisted adjoints in connection with this automorphism as follows:



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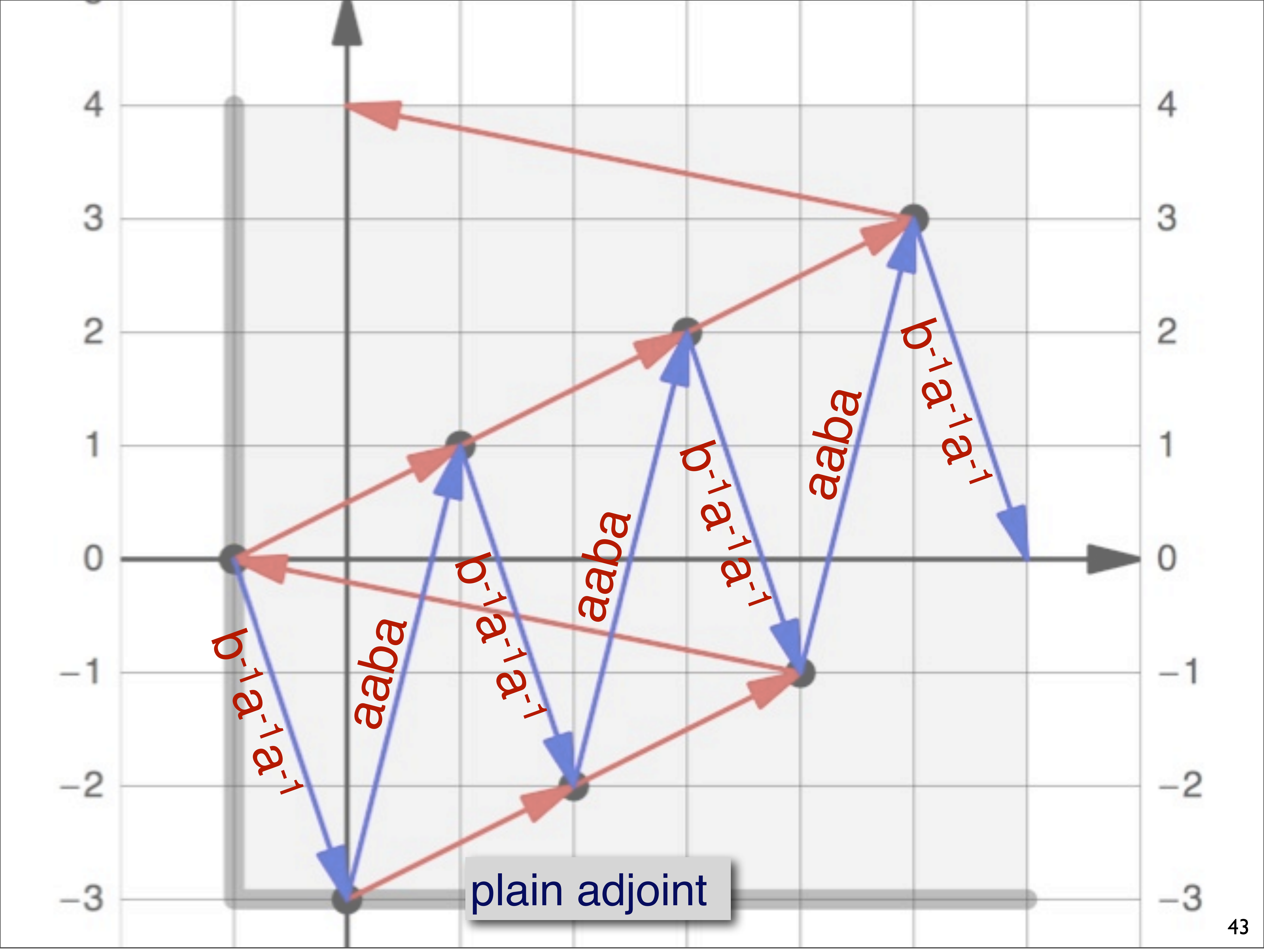
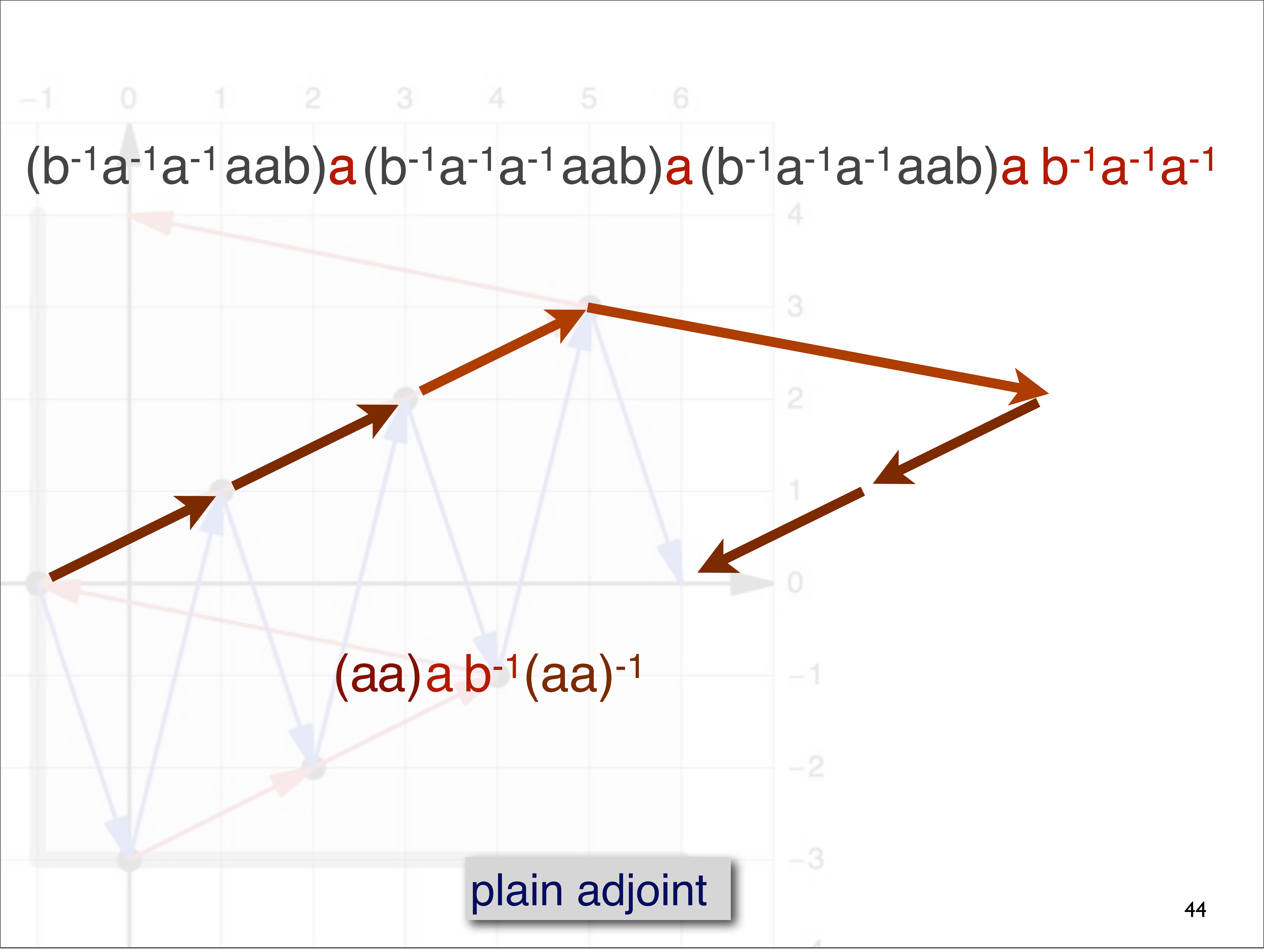
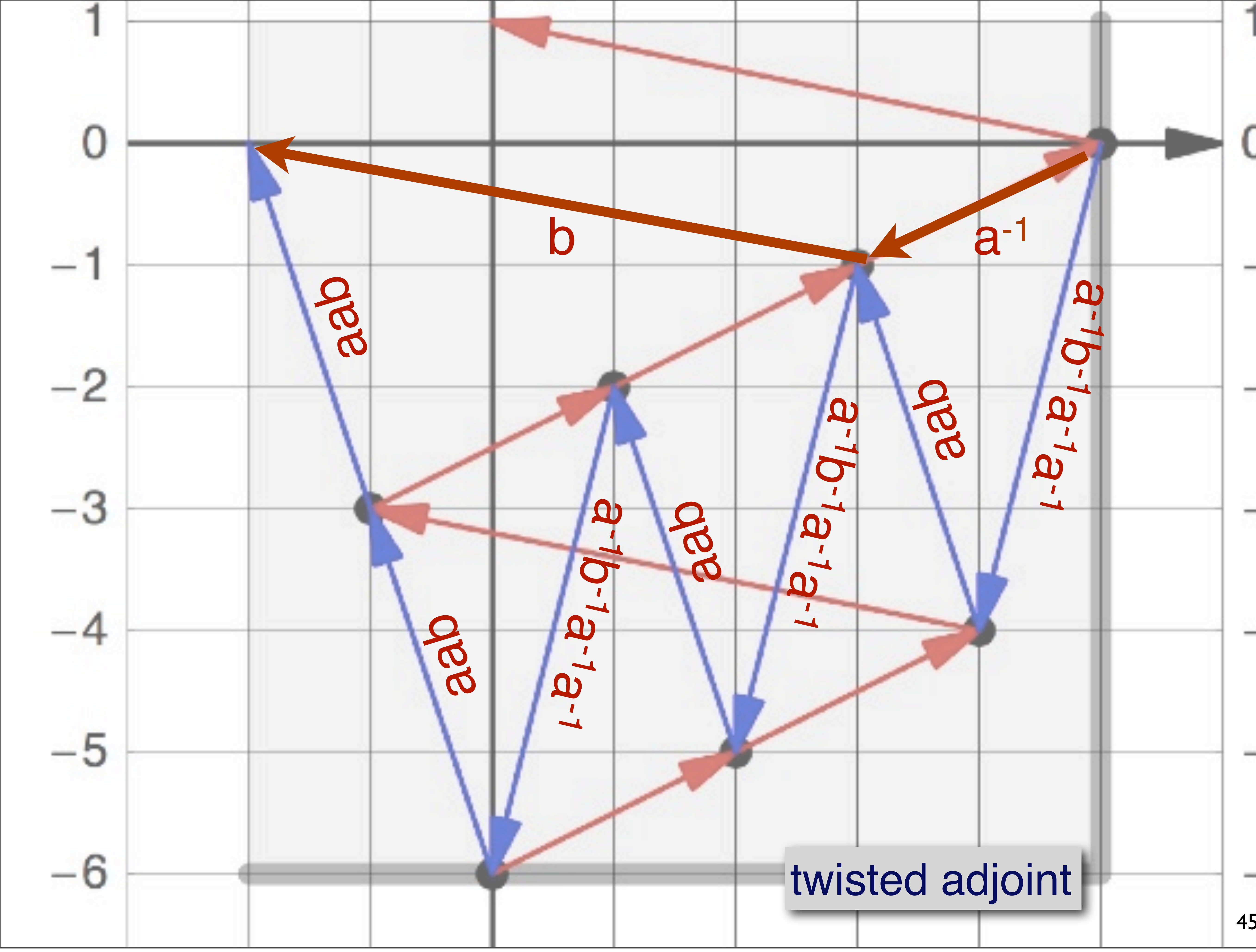
We start with the word ab^(-1) which describes the division of the (sharp and ascending) augmented prime into an ascending major second and a descending minor second.
Next we substitute a and b^(-1) for the associated folding patterns y^(-1)x and y^(-1)xy^(-1)xy^(-1) respectively.



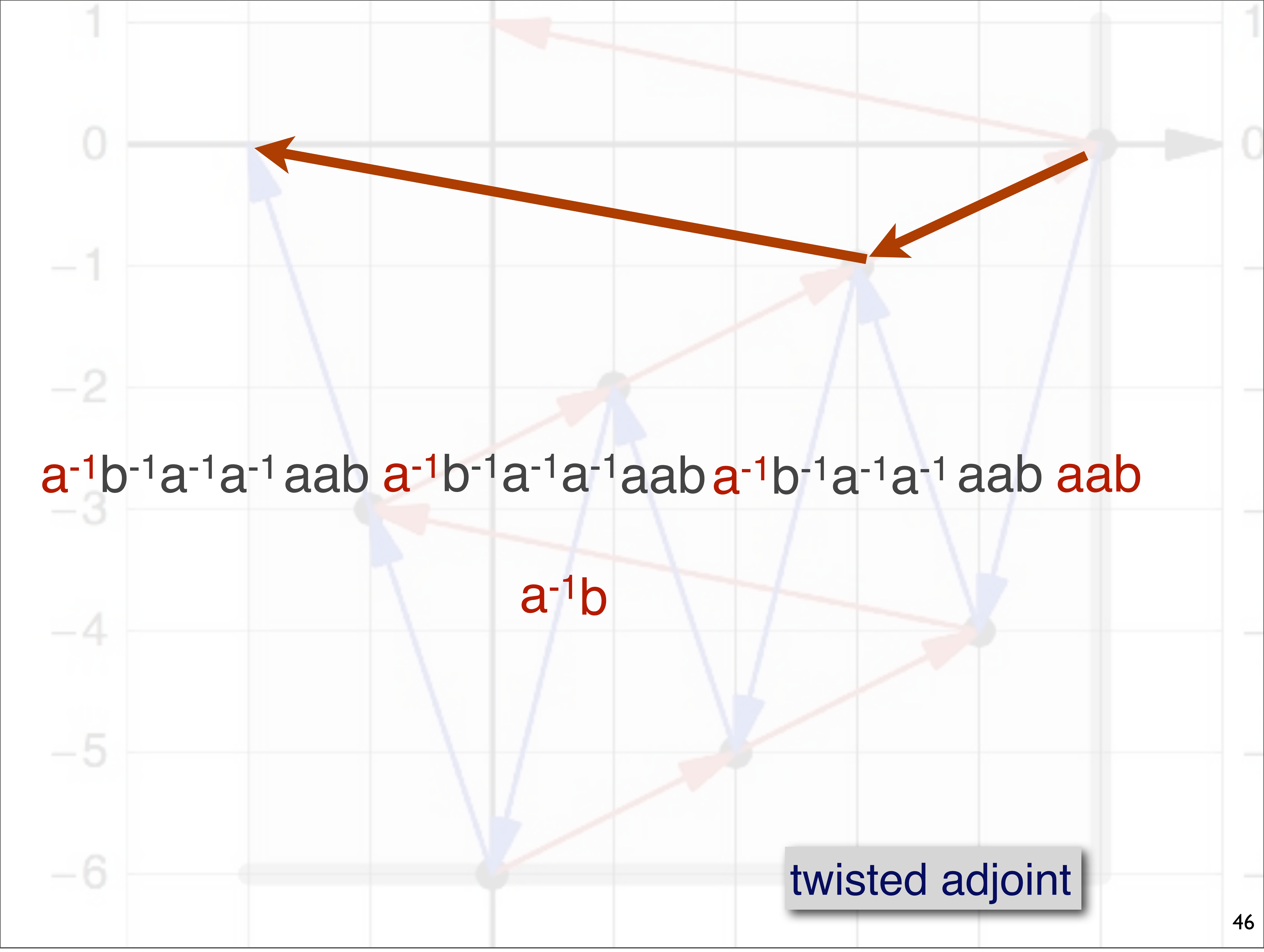
Next we substitute every instance of x for the Ionian species of the fifth aaba and every instance of y^(−1) for the inverse (aab)^(−1) = b^(−1)b^(−1) a of the Ionian species of the fourth aab.



After reduction we obtain the word aa ab^(−1) (aa)^(−1), which is conjugate to the word ab^(−1) where we started.



Now we apply the analogous substitutions to the division a^(−1)b of the (flat and descending) augmented prime into a descending major second and an ascending minor second. Substitue a^(−1) for x^(−1)y and b for x^(−1)yx^(−1)yy and obtain thereby the twisted adjoint x^(−1)yx^(−1)yx^(−1)yy. Every x^(−1) is then substituted for the inverse of the Ionian species of the fifth (aaba)^(−1) = a^(−1)b^(−1)a^(−1)a^(−1) and every y for the Ionian species of the fourth aab.



After reduction we obtain the same word a^(-1)b, from which we departed.
The Ionian mode serves here only as an example. The observation illustrates a general result about the twisted adjoint.

The group $Aut(F_2)$ is (redundantly) generated by the following automorphisms: $G, \tilde{G}, D, \tilde{D}, E : F_2 \to F_2$ with

$$G(a) = a, \quad G(b) = ab, \quad \tilde{G}(a) = \widetilde{G(a)} = a, \quad \tilde{G}(b) = \widetilde{G(b)} = ba,$$
$$D(a) = ba \quad D(b) = b, \quad \tilde{D}(a) = \widetilde{D(a)} = ab, \quad \tilde{D}(b) = \widetilde{D(b)} = b,$$
$$E(a) = b, \quad E(b) = a$$

Proposition 4.1 *The mapping $f \mapsto f^* = (x^{-1}, y) f^{-1} (x^{-1}, y)$ is an involutive anti-automorphism of the special Sturmian monoid, that exchanges D and $\tilde{D}$ and fixes G and $\tilde{G}$. It sends conjugacy classes of morphisms onto conjugacy classes. The involution on Christoffel words that it induces is the same as the one of Section 2.*

Berthé, V., de Luca, A., Reutenauer, C.: On an involution of Christoffel words and Sturmian morphisms. European Journal of Combinatorics 29(2), 535–553 (2008)

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This is a theorem from a paper written by three leading authors in the discipline of algebraic combinatorics on words: Valerie Berthe, Aldo de Luca and Christophe Reutenauer. It characterizes an involutive anti-automorphism of the special Sturmian monoid. This is the submonoid of the special automorphism group, generated by the positive special automorphisms (sending a and b to positive words).

Above the theorem you see a set of five generators of the automorphism group of the free group. The four automorphisms G, G~, D and D~ (without the exchange automorphism E) generate a subgroup of index 2, namely the special automorphisms. If we look only at the generated submonoid (rather than the group) generated by these four we obtain the special Sturmian monoid. Sturmian involution leaves G and G~ fixed and exchanges D and D~. It is an anti-automorphism, i.e. in concatenations one has the revert the order of the generators.

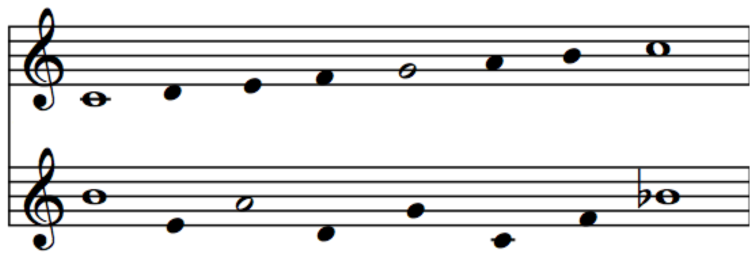
The theorem (apologies: it uses letters x and y rather than a and b) states that Sturmian involution coincides with the antiautomorphism of inversion (in Aut(F_2)) up to conjugation with the inversion of the first letter x. It further states something about Christoffel words which can be refined to the following statement about twisted adjoints: For every special Sturmian morphism f the twisted adjoint of the word f(xy) is the word f*(xy).

$f_4 = GG\tilde{D} = (xxxy, xxy)$

C-Lydian

$f_4^* = DGG = (yx, yxyxy)$

$f_1 = GGD = (xxyx, xxy)$



C-Ionian

$f_1^* = \tilde{D}GG = (xy, xyxyy)$

$f_5 = G\tilde{G}\tilde{D} = (xxyx, xyx)$

C-Mixolydian

$f_5^* = DG\tilde{G} = (yx, yxyyx)$

$f_2 = \tilde{G}GD = (xyxx, xyx)$

C-Dorian

$f_2^* = \tilde{D}G\tilde{G} = (xy, xyyxy)$

$f_6 = \tilde{G}\tilde{G}\tilde{D} = (xyxx, yxx)$

C-Aeolian

$f_6^* = D\tilde{G}\tilde{G} = (yx, yyxyx)$

$f_3 = \tilde{G}\tilde{G}D = (yxxx, yxx)$

C-Phrygian

$f_3^* = \tilde{D}\tilde{G}\tilde{G} = (xy, yxyxy)$

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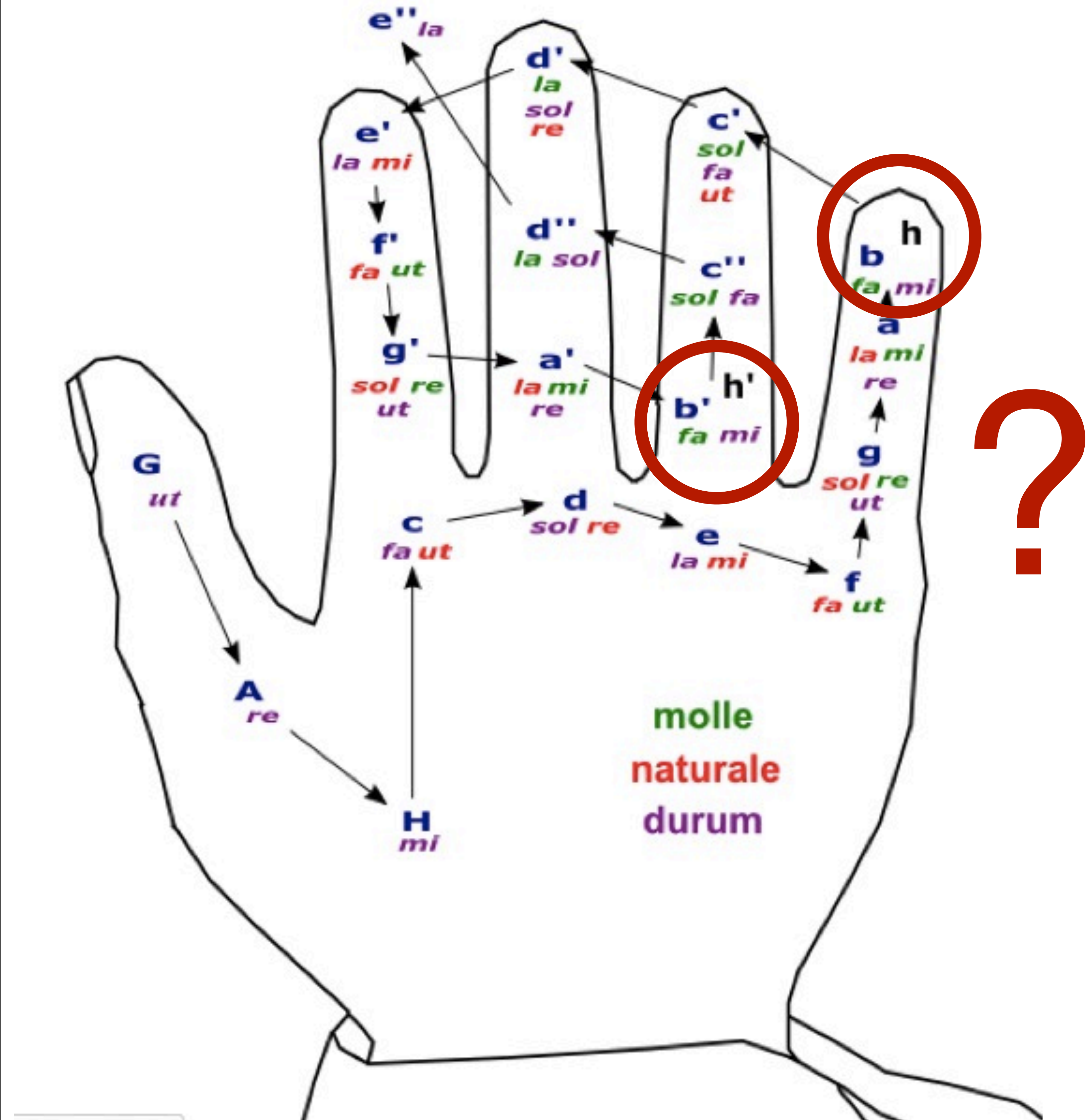
For the diatonic modes we have six conjugate Sturmian morphisms f_i, whose images f_i(x) of x and f_i(y) of y yield the species of the fifth and of the fourth, respectively. The Sturmian involutions f_i* of these morphisms yield the associated species of the descending major seconds and ascending minor seconds, respectively.

Note, that on this slide I show the 6 authentic modes all starting from the same fixed tonic c. Therefore you see a sharp in the Lydian case (top) and four flats in the phrygian case (bottom). This collection of modes illustrates that the word-theoretic approach alone does not yet give access to the notes. Before discussing this problem I wish to music-theoretically interpret the finding about the twisted adjoints.
To begin with, the bad conjugates (yxxy, xxx) and (yy, xyxyx) - which are both(!) mssing on the slide - are related to each other under twisted adjointness. All the "good" ones correspond to conjugate Sturmian morphisms. We will see later how this supports to study the authentic modes as one family of modes.



The twisted adjoint is the canonical choice. For ascending modes this distinguishes the flatward direction.

The "canonical" border interval is B-Bb and not F-F#.

Another interpretation concerns the role of the automorphisms. I mentioned Jacques Handschin as a proponent of the idea that the line of fifths provides a fundamental dimension for the constitution of musical tone relations. One complaisant criticsm with a dialectical attitude suggests to view the two tone-orderings, i.e. with resepct to the chain of fifths on the one hand and with respect to pitch height on the other as two sides of the same coin. None of them should be regarded as foundational and the other as derived.

To my mind it is the concept of an autmorphism (on Z_7, Z^2 and now F_2) which may productively replace the vague image of a coin with two sides. More concretely, we just learned that in the non-commutative case the twisted adjoint (corresponding to Sturmian involution) makes the passage from one side to the other traversable in both directions without defect. The plain adjoints allow this passage only up to conjugation.
In addition there is a transformational formulation which elegantly matches the passage from one side to the other in terms of Sturmian involution.

The flatward augmented prime from b to b-flat is therefore the canonical candidate for a "border interval", dual to the ascending octave in the step interval patterns.

A Speculation:

musica ficta =
plain adjoint

musica recta =
twisted adjoint

From a perspective of structural semiology one would characterise orientation upward+flatward as the unmarked constellation of directions. The other possibility, namely upward+sharpward, should then be the marked (i.e. exceptional) one. This reminds of a dichotomy in the late medieval conceptualisation of the tone repertoire. The unmarked side is occupied by the Gamut, as we just saw it on the Guidonian hand. It is also called "music recta". The marked side is called "musica ficta" and containes predominantly sharpward altered notes.
The main motivation for the extension of the Gamut has to be seen in counterpoint, in particular in cadences, as you see it in this standard configuration between the tenor and soprano. The favored note to be altered is the so called "subsemitonium modi" (the semitone below the tonic) and occasionally the semitone below the divider. The plain adjoint of ascending Ionian mode contains these notes.
An integration of modal theory with issues of counterpoint is an actual challenge for future work in mathematical music theory.
The possible role of the upward+flatward versus upward+sharpward orientations in the cadence is thereby a driving motivation for such a study. ("Thrust reversal hypothsis").

The full conjugacy class (“Half of the Dodecachordon”)

Question:
The morphisms nicely generate the interval patterns.
But we don’t have access to the notes yet.
Is there a transformational approach?

Answer is here:

Arnoux, P., Shunji, I.: Pisot substitutions and Rauzy fractals. Bulletin of the Belgian Mathematical Society Simon Stevin 8, 181–207 (2001)

Berthé, V., de Luca, A., Reutenauer, C.: On an involution of Christoffel words and Sturmian morphisms. European Journal of Combinatorics 29(2), 535–553 (2008)

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Let me conclude this part of my talk with a remark on the transformation of notes. The Dodecachordon of Glarean from 1547 is a family of “white note” modes. Among the authentic modes the height ambit raises from mode to mode D-Dorian, E-Phrygian, F-Lydian, G-Mixolydian, A-Aeolian and C-Ionian.

Glarean appends A-Aeolian and C-Ionian to the already established list and so he therefore obtains a gap of a minor third between the tonic A and tonic C of the Aeolian and Ionian modes respectively. Zarlino argues in 1571 that Ionian should better be prepended to the Dorian mode. The tonics notes of the authentic modes then form a Guidonian hexachord, he argues. Let me share with you a transformational argument, which supports Zarlino’s point.

The underlying mathematics is due to Pierre Arnoux and Ito Shunji as well as to the three authors mentioned earlier: Berthe, de Luca and Reutenauer.

The music-theoretical interpretation of this work is joint work with Mariana Montiel and has recently presented at our bi-annual conference MCM2013 (“Mathematics and Computation in Music”).

Lattice Path Transformations and their Linear Adjoints

Let $e_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = V(x)$ and $e_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = V(y)$ denote the base vectors of $\mathbb{Z}^2$. Consider the set $\mathcal{B} = \{(W, e_x) \mid W \in \mathbb{Z}^2\} \sqcup \{(W, e_y) \mid W \in \mathbb{Z}^2\}$ and consider the linear space

$$\mathcal{F} = \{v : \mathcal{B} \to \mathbb{R} \mid v(W, e_z) = 0, \text{ for all but finitely many } (W, e_z) \in \mathcal{B}\} .$$

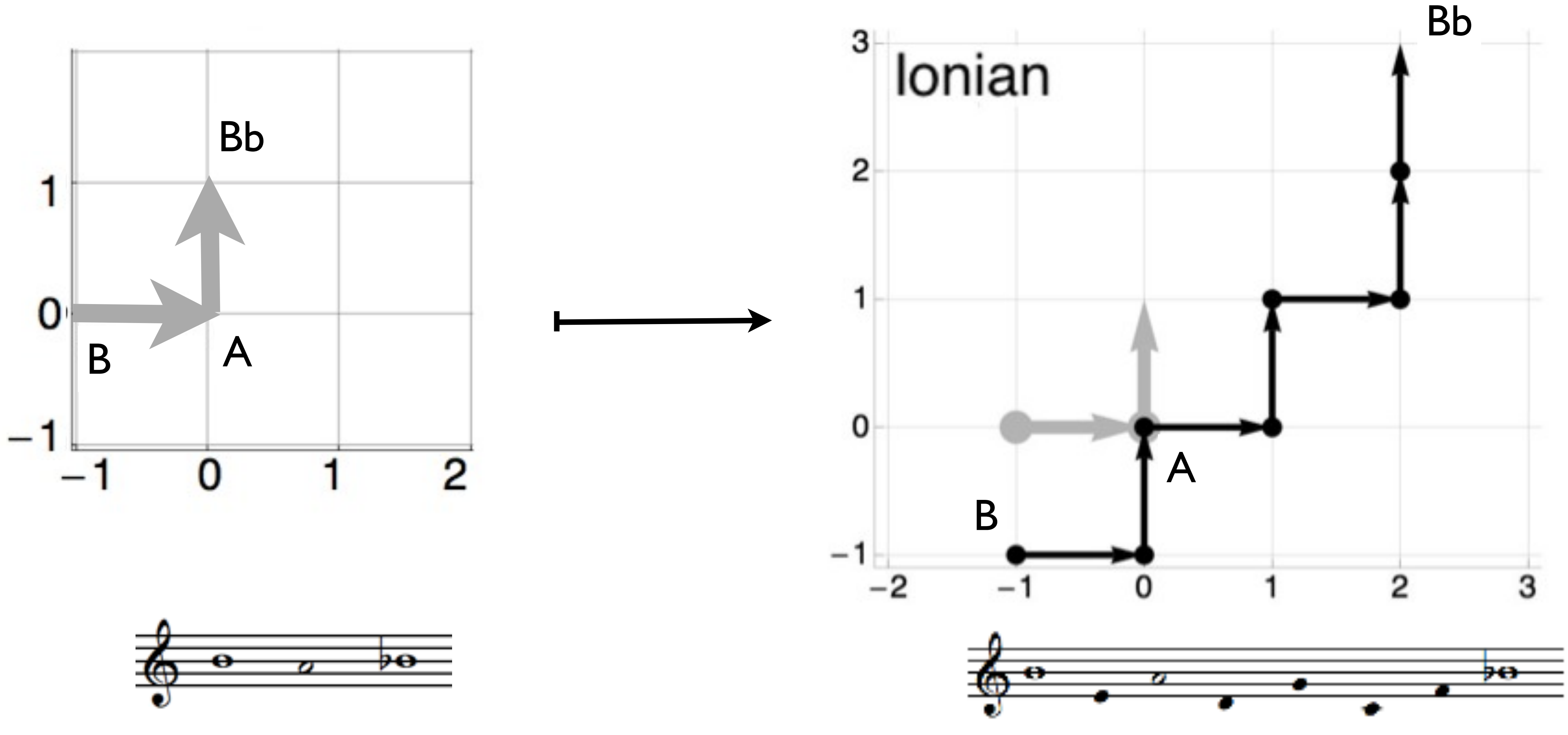
$$E(f)(W, e_z) := \sum_{k=1}^{|f(z)|} (M_f \cdot W + V(P_k(f(z))), e_{L_k(f(z))})$$

$$E(f)^*(W, e_z)^* = \sum_{L_j(f(x))=z} (M_f^{-1}(W - V(P_j(f(x)))), e_x)^* + \sum_{L_j(f(y))=z} (M_f^{-1}(W - V(P_j(f(y)))), e_y)^*$$

We consider an infinite-dimensional real vector space “fancy F” whoose basis is formed by all horizontal and vertical lattice segments in Z^2. Each of these basis elements is an ordered pair (W, e_z) consisting of a point W in Z^2 and one of the two unit vectors e_x or e_y. The Sturmian monoid acts on this vector space by virtue of linear transformations. They are called lattice path transformations. This name is motivated by the fact, that these transformations map lattice paths (connected Zigzag lines) into lattice paths. Every Sturmian morphism f defines a lattice patch transformation E(f).

[The formula should be read es follows: Under E(f) the lattice segment (W, e_z) (z being either x or y) is sent to a formal sum of lattice segments. The anchor point for each summand is a sum M_f.W + V(P_k(f(z))). Here M_f.W is the image of the point W under the linear transformation in Z^2 associated with f. It is given by the so-called incidence matrix of f, an element of SL(2,N). For every k from 1 to the length of the word f(z) the expression V(P_k(f(z))) denotes the commutative image in Z^2 of the prefix P_k(f(z)) of length k-1 of the word f(z). The direction of the associated unit vector is given by the k-th letter L_k(f(z)) of the word f(z).]

The formala for the dual map is also given. In accordance with Arnoux and Shunji we interpret the covectors (W, e_z)* geometrically as hyperfaces, which in dimension 2 are “also” lattice segments.



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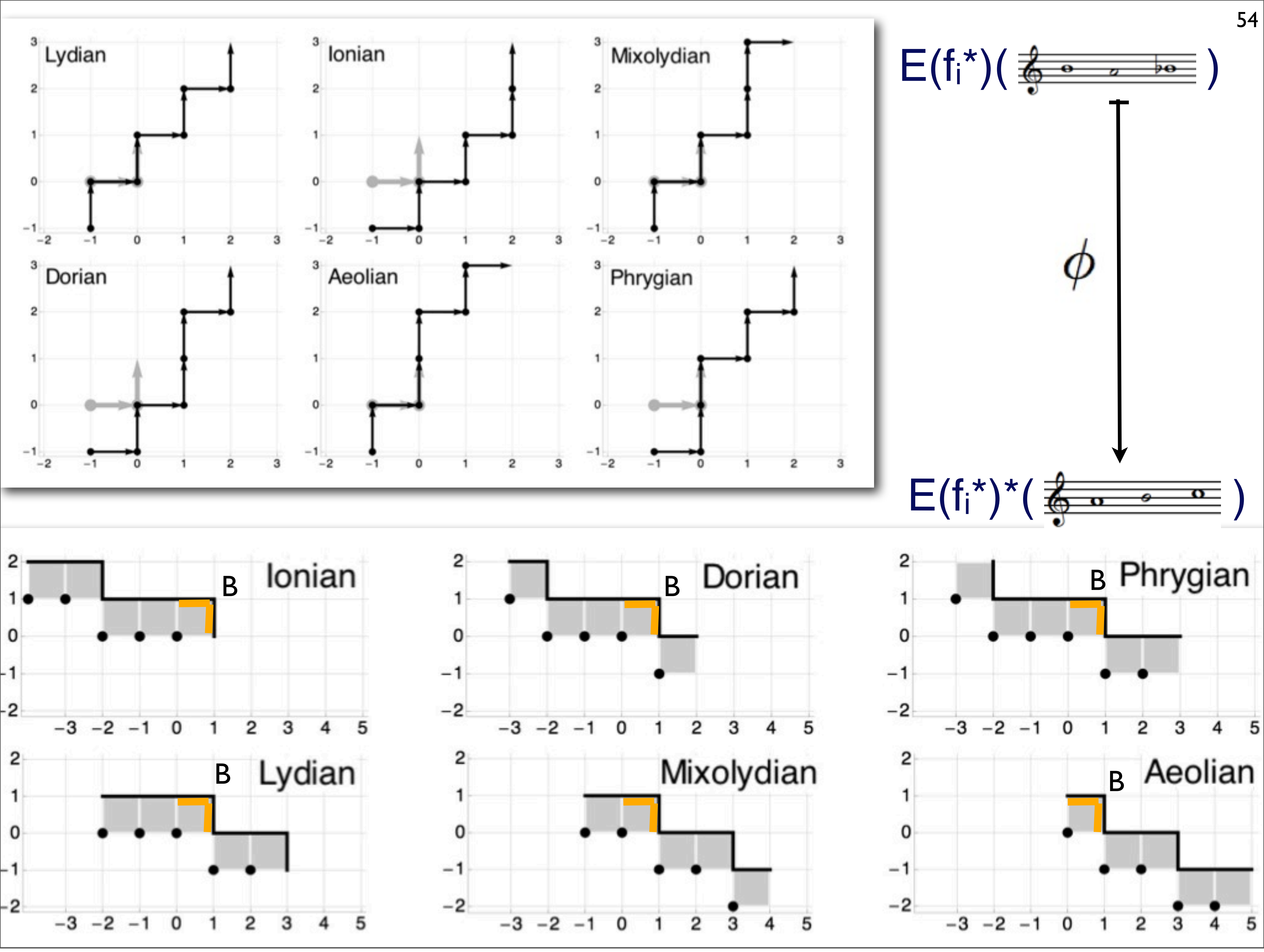
Here the folding of the white-note Ionian mode from B to B-flat is described as a lattice path (right side). It can be obtained as the image of the short lattice path b - a - b-flat (left side) under the application of the linear map E(f_1*) which is associated with the Sturmian involution f_1* = (xy, xyxyy) of the Sturmian morphism f_1 = (xxyx, xxy).



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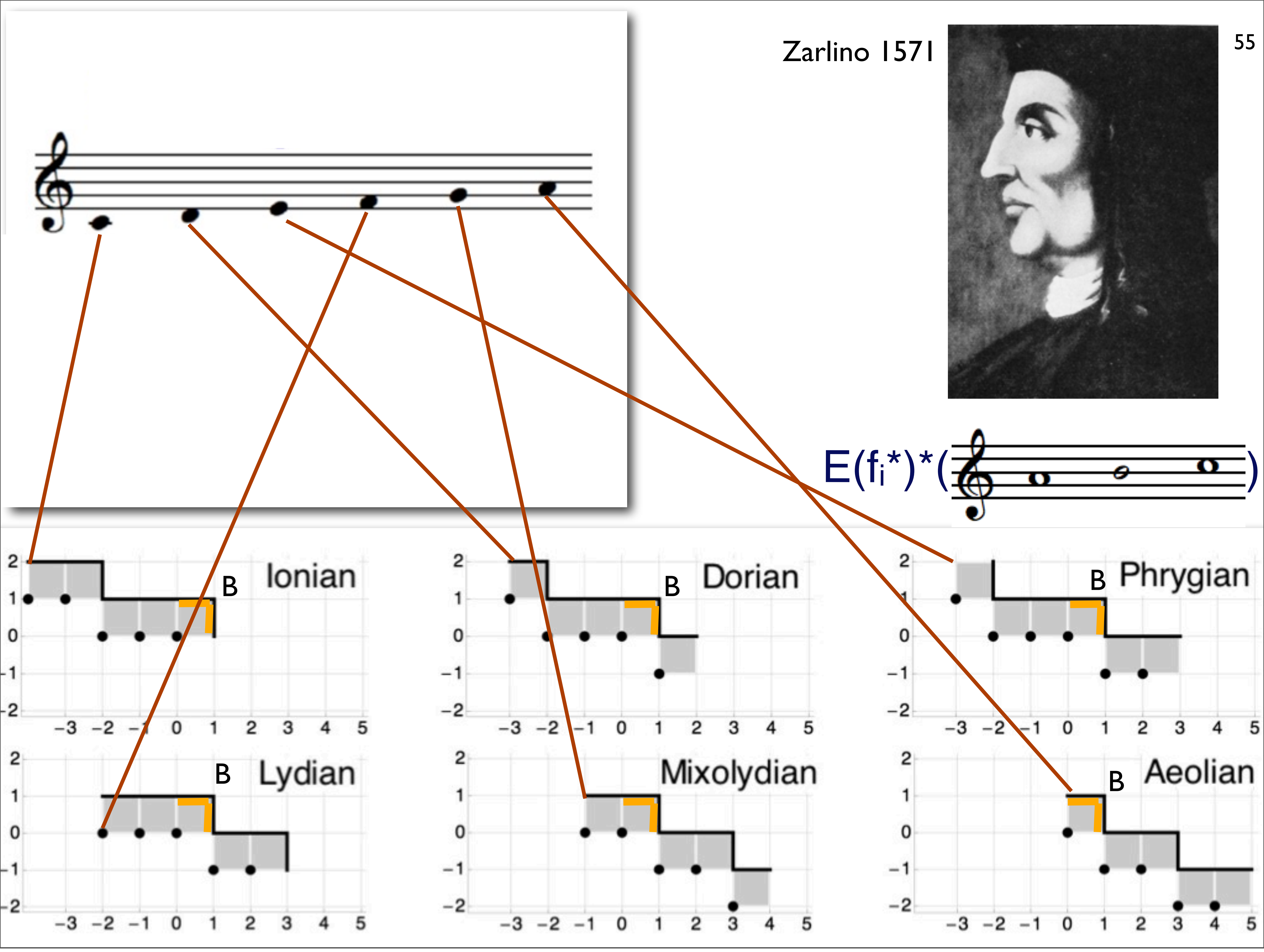
By applying the six conjugates f_1*, ..., f_6* to the same initial lattice path, we obtain the complete family of the foldings of the authentic white-note modes as lattice paths (see the 6 images on the top left).

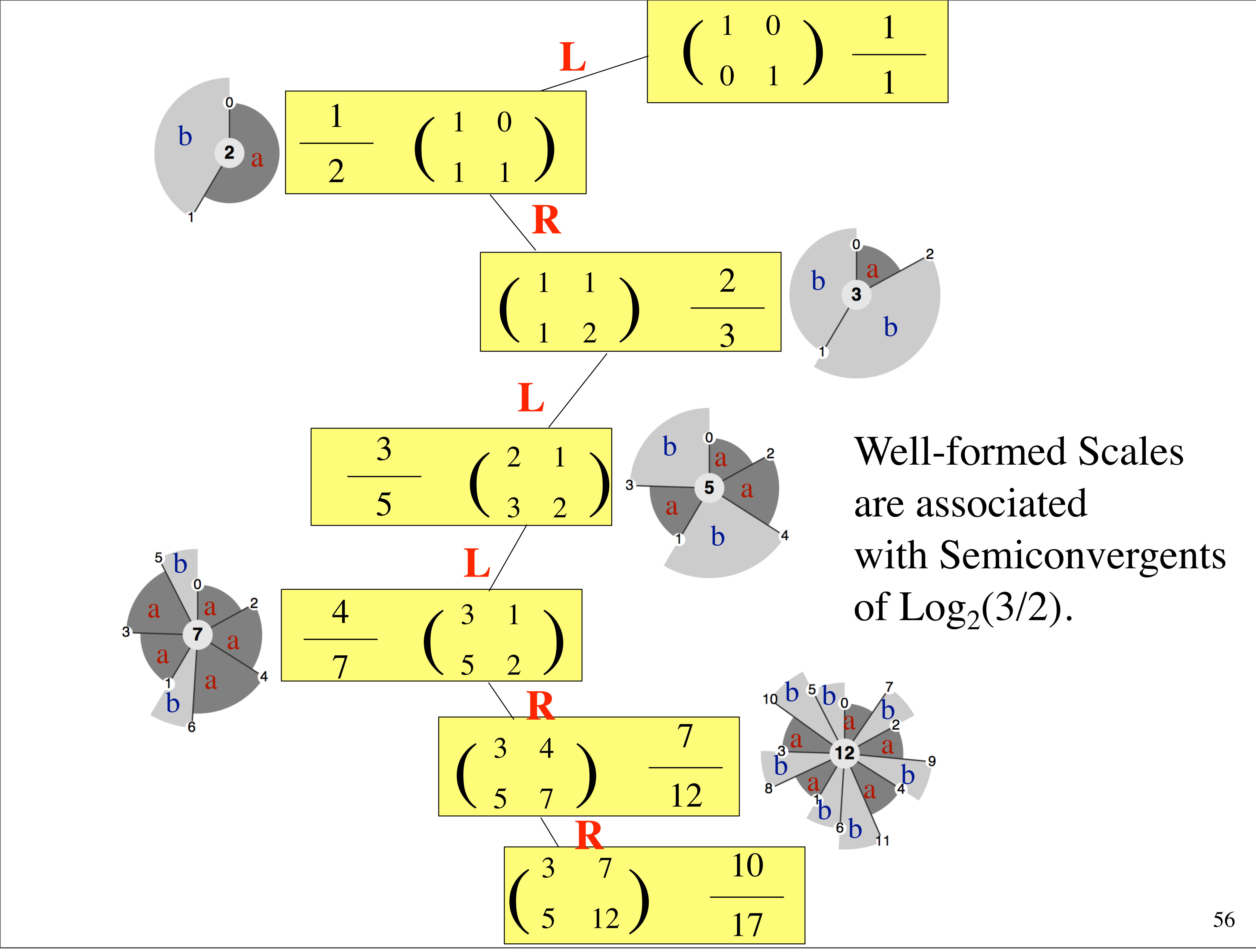
The next ingredient is a map phi from the big vector space “fancy F” to its dual. It maps the step-folding b - a - b-flat of the augmented prime to the step pattern a - b - c.

The linear duals E(f_i*)* are systematically related to the linear maps E(f_i), but they are not the same! It turns out that the application of the six dual maps E(f_i*)* to the step pattern a - b - c yields the step patterns of the six authentic white note modes as dual lattice paths.



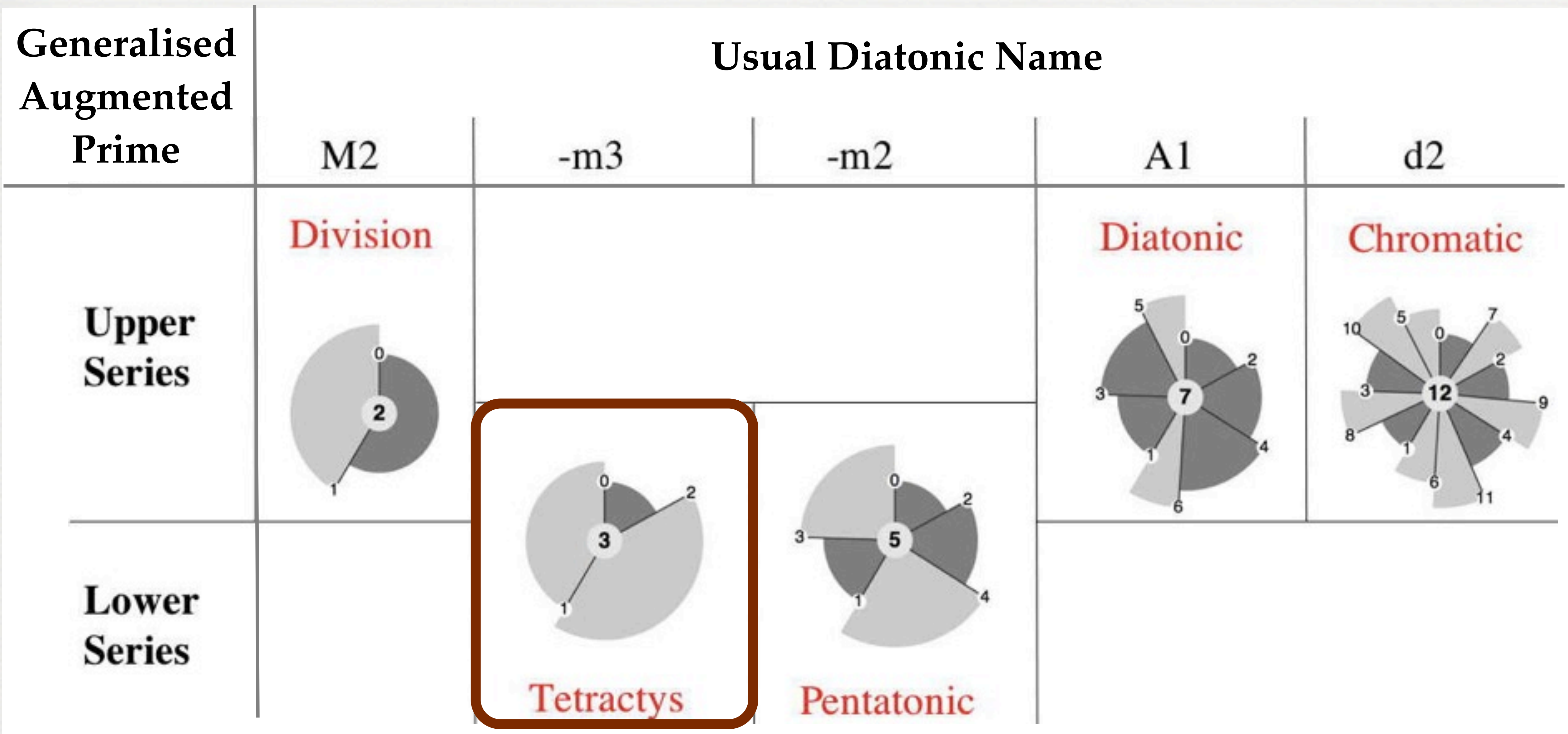
They line up to a cover of a longer lattice path. And their beginnings form the step pattern of a Guidonian hexachord. This nicely supports Zarlino's argument.

This completes the main part of my talk. As a coda allow me to append a short musical portrait of the “little devil”.



Let us review the semiconvergents of Log(2, 3/2). All the content of this talk so far emerged from the ratio 4/7. But in fact, we may do completely analogous music-theoretical investigations at any node of this path. Our attention is now dedicated to the ratio 2/3, describing a fifth-generated scale with three notes. Each instance of the generator is divided into two steps.

Hierarchy of the fifth-generated Well-formed Scales



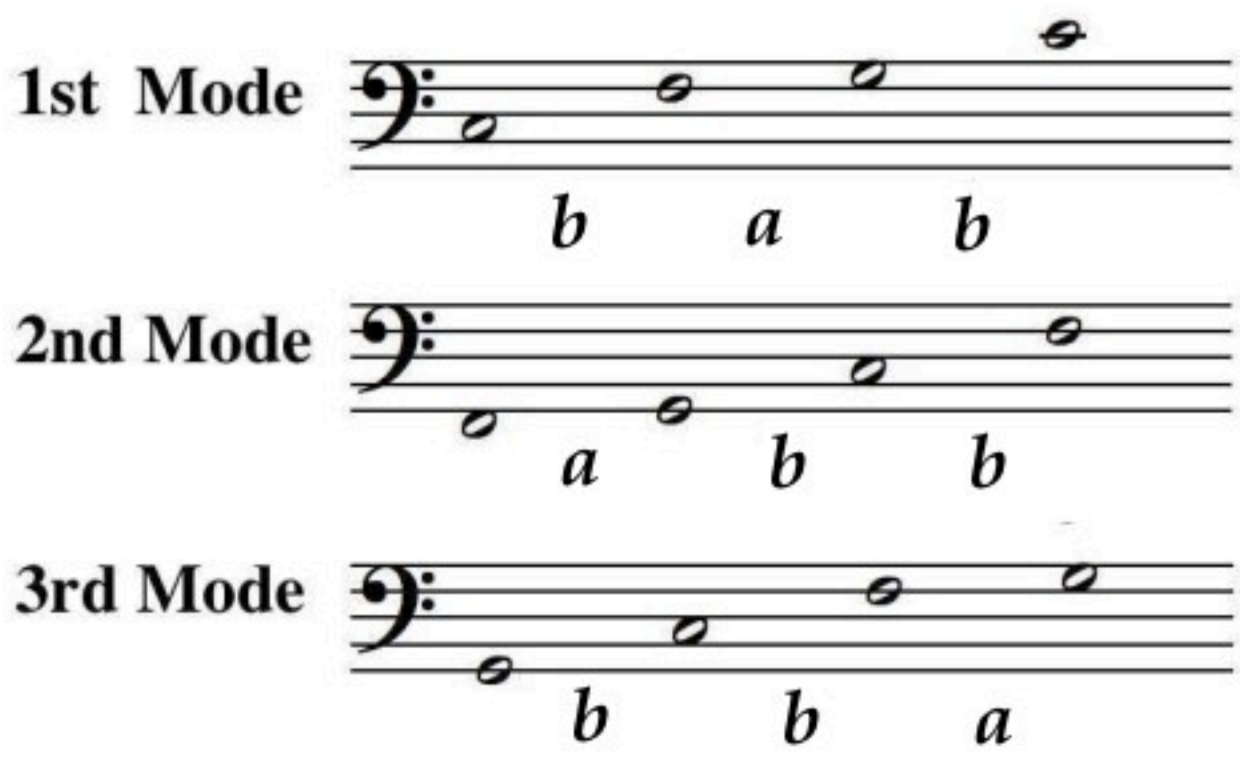
Position of the tetractys scale (also: structural scale) within this hierarchy Carey and Clampitt (1989).

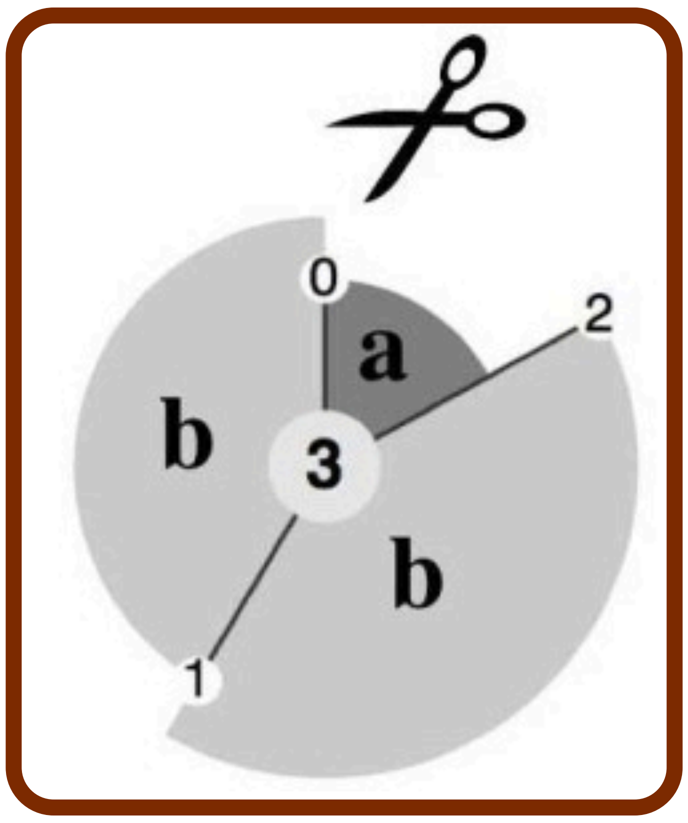
These last slides briefly introduce joint work with Karst de Jong. Karst teaches classical improvisation and composition techniques in Barcelona and in den Haag.

Our theoretical contribution centers around the interpretation of certain three-note modes in the analysis of fundamental bass progressions. On the slide the three note scale is marked with the red frame and is named “Tetractys”. This name was used in antiquity not only for the “sacred” triangle (with the numbers 1 to 10) but also for the “musical tetractys” 6 : 8 : 9 : 12, a sequence of string length ratios. In ascending order we may designate it with the note names c – f – g – c’.

In the first row of this table you see a series of intervals which are said to be “generalized augmented primes”. Please remember from the beginning of my talk, that in accordance with Myhills property the two species of the same interval genus differ by an augmented prime. In each of the scales along this hierachy a different interval occupies this role. In the case of the tetractys this is what we usually call the minor third. I will explain this on the next slide.

Modal Varieties of the Tetractys

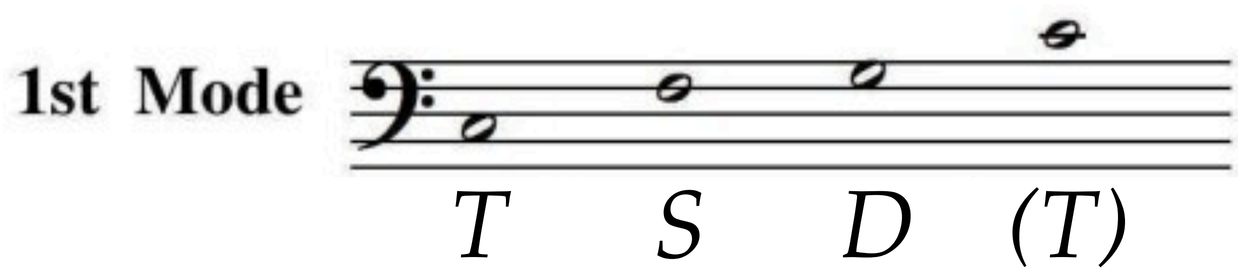


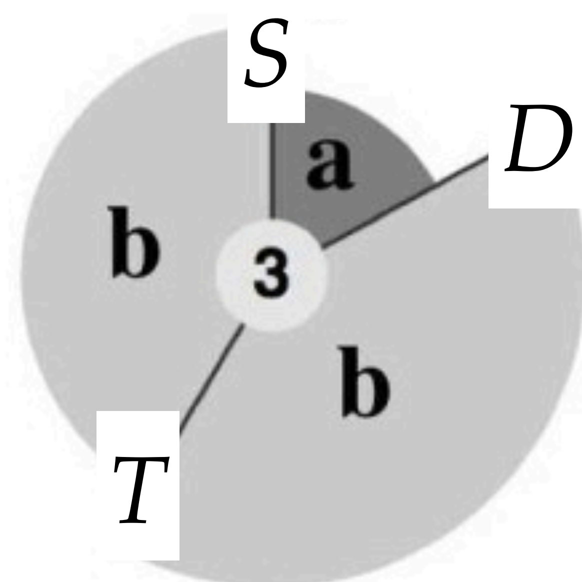


When speaking about this three-note scale we still use the common diatonic names for the intervals. There are two perfect fourths (designated with the letter b) and one major second (designated with the letter a). With respect to the structure of the three note scale these are large and small steps, of course. The difference between Perfect fourth P4 and Major second M2 is the minor third m3. Karst de Jong and I call it the “little devil” [NB: it is half of a diminished fifth - which has been called “diabolus in musica”].

Before searching for the little devil let me briefly portrait the “innocent” situation:
There are three species of the octave. We simply call them 1st mode, 2nd mode, 3rd mode.
With three musical examples I would like to familiarize you with the musical interpretation of these modes in the fundamental bass.

Tetractys Modes as Functional Modes: Examples



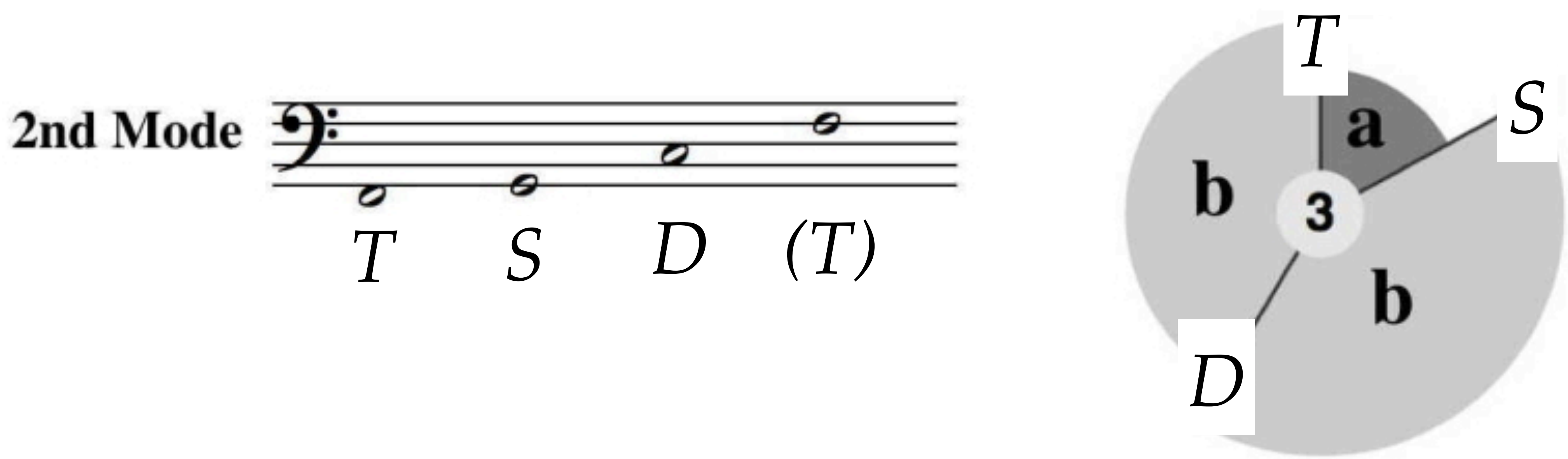


Haydn, Sonata Hob. XVI No. 33

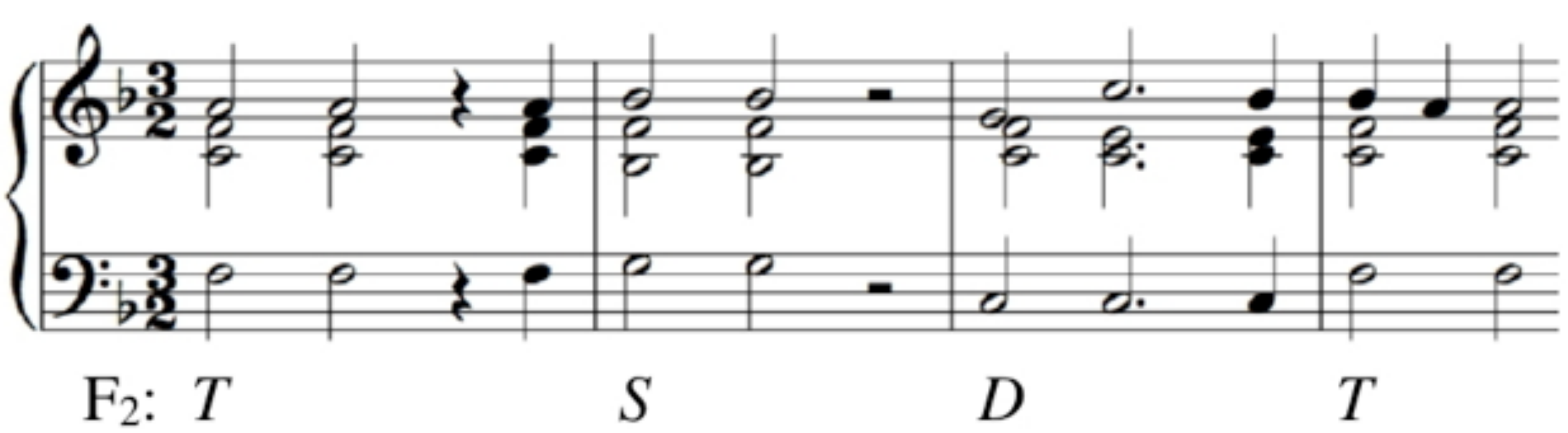
In the tradition of Functional Harmony one speaks of a trias of harmonic functions, namely tonic, subdominant and dominant.
In our approach we interpret the fundamental bass in terms of these modes and associate the three scale degrees 1, 2, 3 of any of the three modes (of the previous slide) with the three tonal functions in this order 1 = T (tonic), 2 = S (subdominant), 3 = D (dominant). In this interpretation we call this a functional mode and we interpret the notes of the fundamental bass as “melodies” in this mode.

Here we have a 1st mode example. It is the beginning of a Haydn Piano Sonata and the symbols below the musical excerpt refer to a 1st mode with tonic D. The progression starts with a fundament D (measure 1), which is labelled italic-T (tonic). Omitting the embellishing harmonies in measures 2 and 4 we find in measure 6 a fundament G, labelled italic-S (subdominant), measure 7 a fundament A, labelled italic-D (dominant), followed in measure 8 by a fundament D, labelled italic-T. In the first mode the interval between T and S is a perfect fourth, the interval between subdominant and dominant is a major second and the interval between dominant and (upper octave of the) tonic is again a perfect fourth.

Tetractys Modes as Functional Modes: Examples

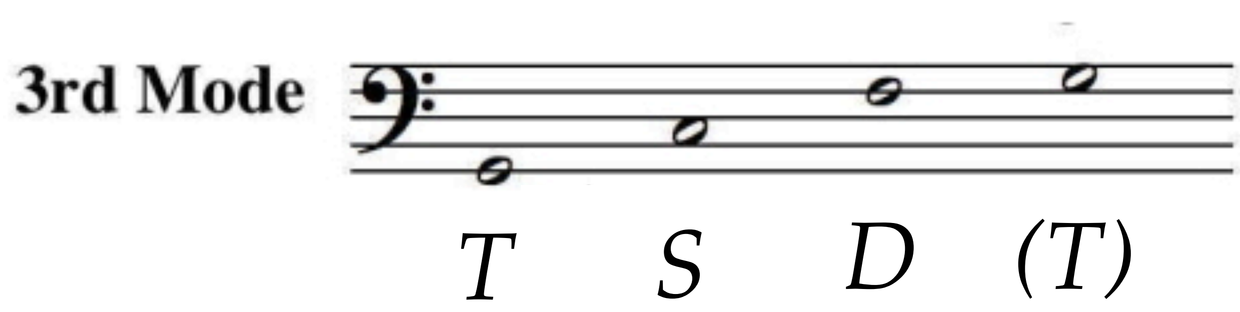


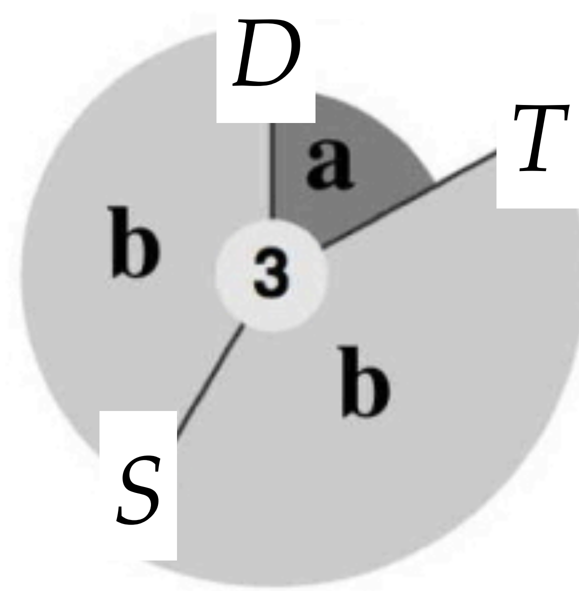
Handel's Aria *Lascia ch'io pianga*



Here is a 2nd mode example. Its tonic (see measure 1) is associated with the fundament F (labeled italic-T). But unlike in the first mode, the interval between tonic degree and subdominant degree is now a major second (see the fundament G in measure 2, labeled italic-S). The fundaments C and F in measures 3 and 4 are labeled italic-D and italic-T, respectively.

Tetractys Modes as Functional Modes: Examples

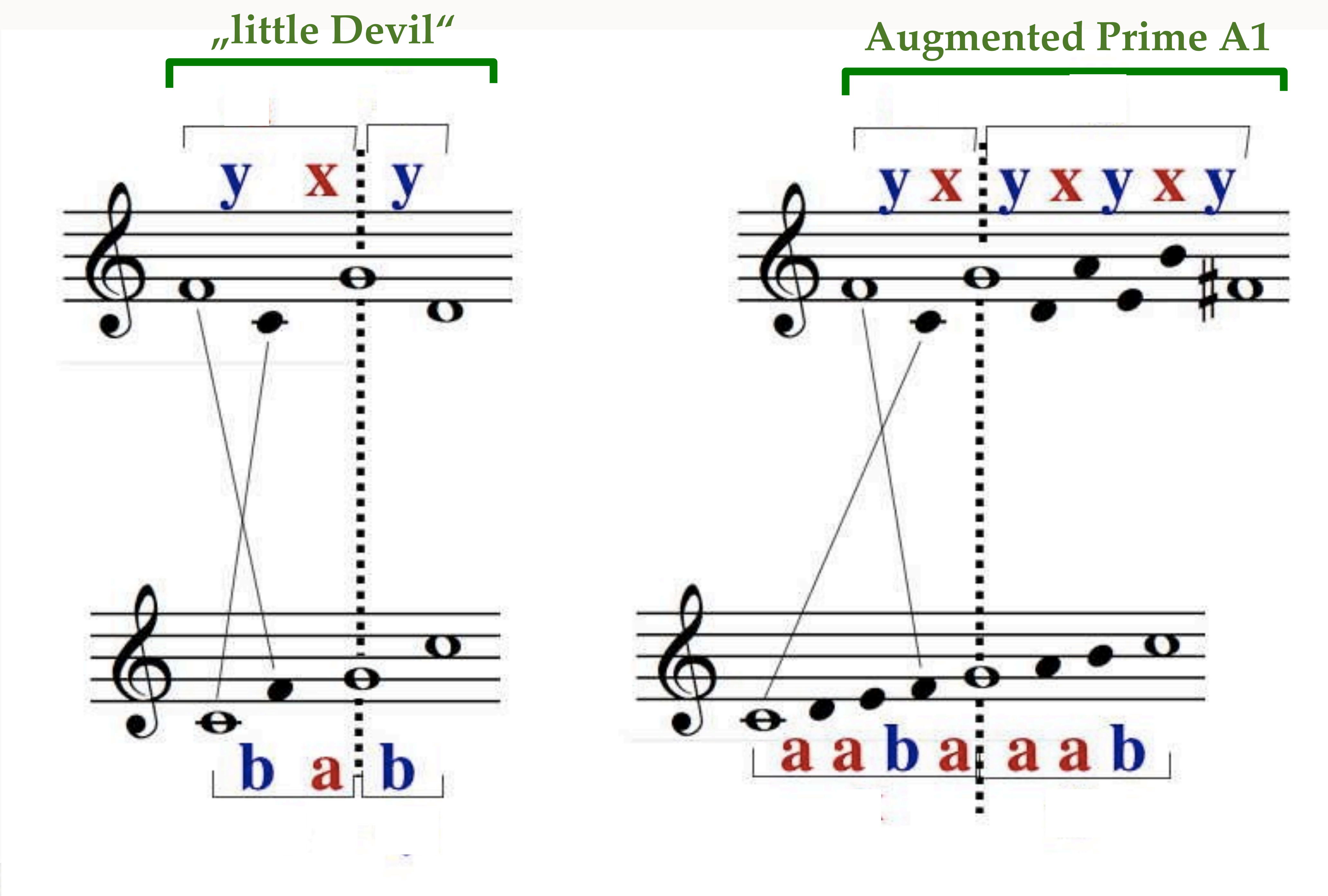




Sting: *An Englishman in New York*

The 3rd functional mode is the bad conjugate (analogous to the Locrian Mode in the diatonic scale). Interestingly this mode is very rare in classical music and jazz. It appears in Pop music, though. The example by Sting is here analysed as a third mode with tonic B. Here the major-second interval occurs between dominant and tonic (measure 3, 3rd beat and measure 4, 1st beat).

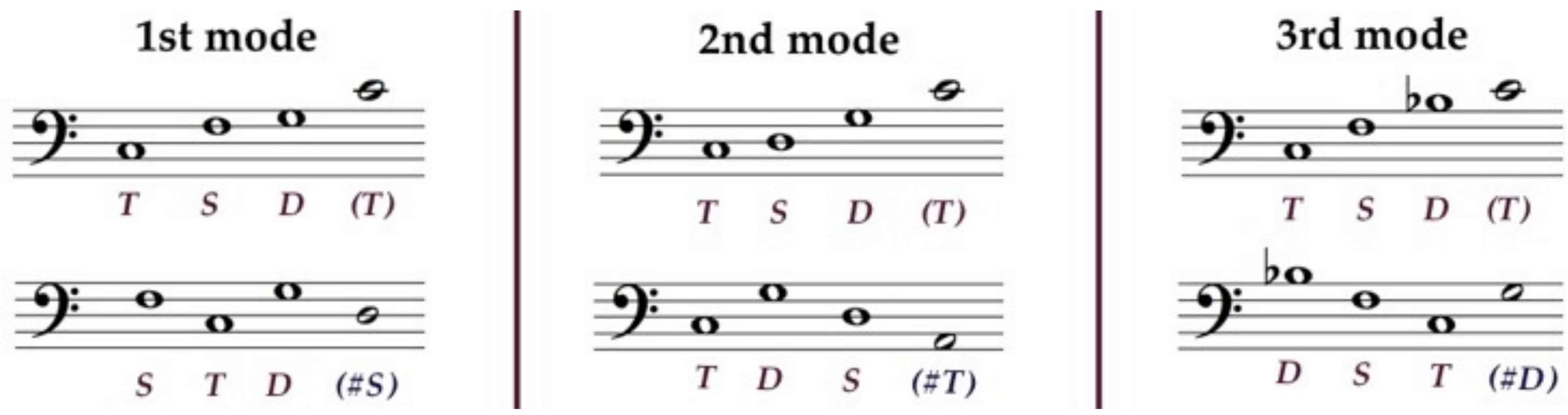
Structural Comparison between Diatonic and Tetractys Modes



So far I referred to the scale as a step pattern. But Karst and I think, that often the associated foldings are literally manifest in the music. On the left side of this slide you see the ascending step pattern of the 1st Tetractys mode (bottom) and the associated sharpward folding (top). This includes the folding of the “little devil”, the minor third. On the right side you see the analogous data for the Ionian mode.

Algebraic Combinatorics of the Tetractys Modes

Tetractys modes



The position of the minor third distinguishes the three common finalis modes

Here you see all three modes with the same tonic note c. Each mode is characterised by a different instance of the minor third (little devil). We denote it with the help of sharp signs: S – #S (first mode), T – #T (second mode), D – #D (third mode). These sharp signs are not meant to be augmented primes in the diatonic world: They are little devils: chromatic intervals in the tetractys world. In accordance with the theory they should play a prominent role in the constitution of fundamental bass melodies. Karst de Jong and I dedicated a detailed investigation to this question. And the results confirm our hypothesis.
Here are illustrative examples:

Aspects of Harmonic Tonality

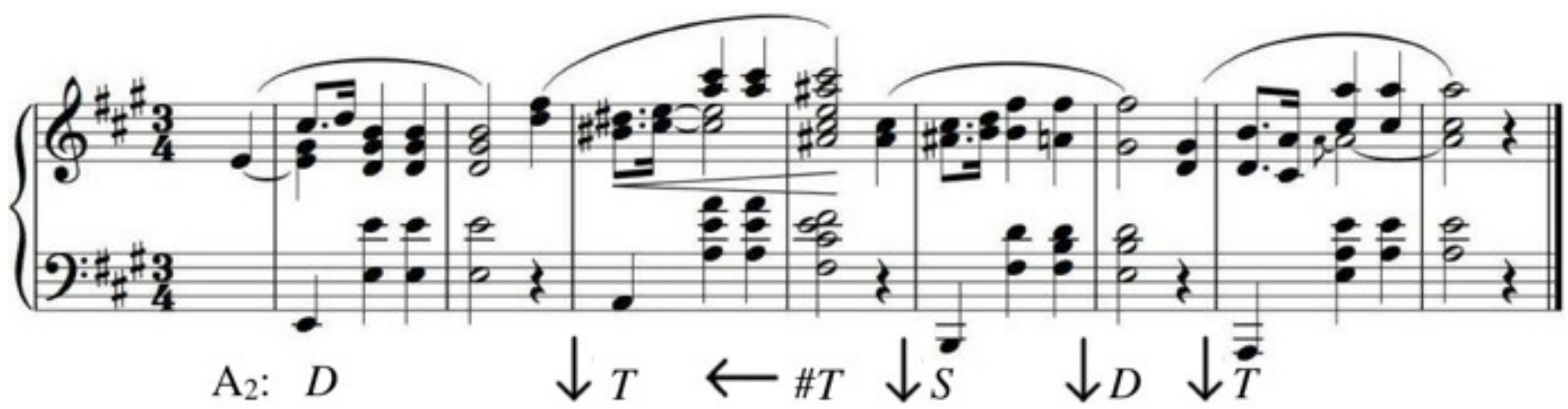
Chromatically Extended Functional Modes

Errol Garner

Turn Around (in Major) and Minor: *Frantonality*

Frederic Chopin

A „Single Turn Around“ Piece: *Prelude Op. 28, No 7*



For the analytical anntotation of the fundamental bass we use arrows. The down arrow stands for a flatward fifth or fourth and the left arrow stands dor a sharpward (= downward) minor third. A quite typical manifestation of the “little devil” is the so called “Turn Around”. This is a 2nd functional mode with an inserted fundament a minor third below the tonic, which is then followed by three flatward fifths/fourths. It is the sharpward folding pattern. [NB: In this simple 3-note-mode the plain adjoint of the second mode does not produce a conjugation defect]

The Errol Garner example is in A-flat minor and the minor third below the tonic is F. The Chopin example is in A-Major and the minor third below the tonic is F-sharp.

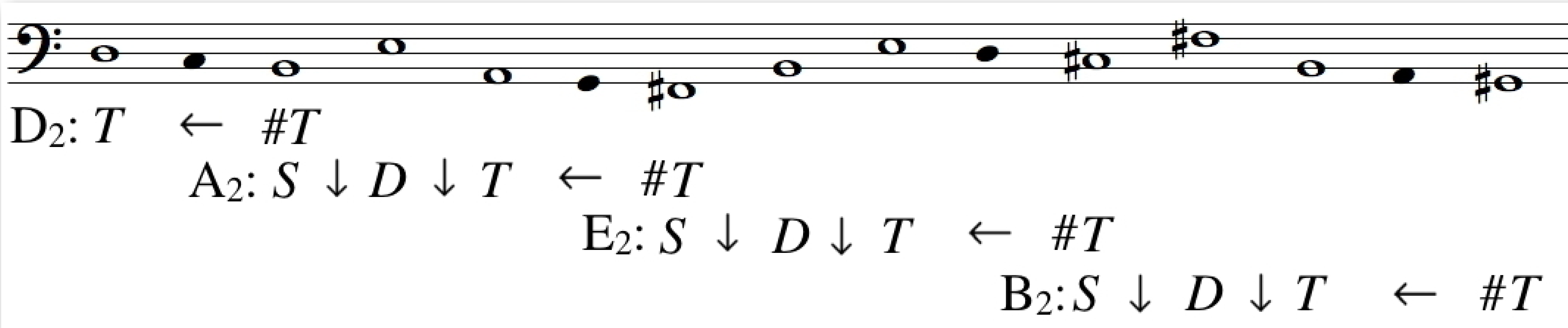
Non-Diatonic Voice-Leading over a Functional Mode

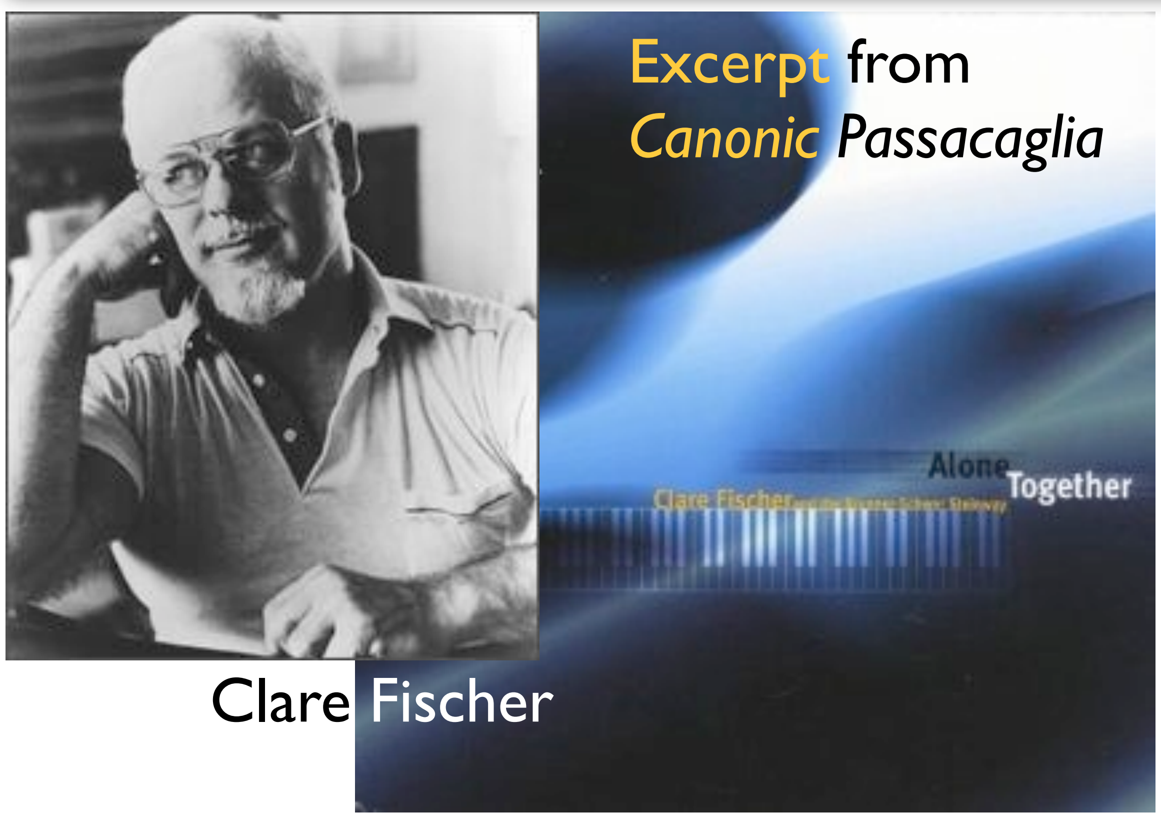
Maurice Ravel

Forlane (Le tombeau de Couperin)

In the 1st mode the little devil is inserted after the subdominant. This is quite typical for classical and romantic music in progressions [for musicians: such as IV - II or IV - V/V] where the involved chords and voice-leadings tend to be diatonic. Here we have a rather exceptional variant, where the fundament exemplifies a functional mode (with an inserted little devil) and where the upper structures are nevertheless quite chromatic.

Karst de Jong and Thomas Noll: **Fundamental Passacaglia**
Harmonic Functions and the Modes of the Musical Tetractys





Functional Mode Analysis of the Fundamental Bass

Thank you for your attention !

Karst de Jong and I presented this application of Mathematical music theory since 2009 at several conferences and we got positive feedback. In the mean time Karst develops, test and soon publishes pedagogical material on the basis of our work. So I'm optimistic that this research is not restricted for a small circle of experts.

Let me conclude with a very suggestive example:
The augmented prime in the diatonic word often occurs as an indicator and motor for modulation. In a world of functional modes it is the little devil, which may play an analogous role. The jazz-composition "Excerpt from Canonic Passacaglia" by Clare Fischer is a brilliant example for such a process. It modulates continously through all 2nd functional modes. The little devil (minor or third below the tonic) is re-interpreted as the subdominant degree of the new mode, which is a fifth higher (sharper) than the pervious one.

Before we listen together to the first few modulations from this piece I wish to thank for your attention.