EFFECTIVE EQUIDISTRIBUTION ON ADELIC QUOTIENTS

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In this talk we presented the joint work [4] with G. Margulis, A. Mohammadi, and A. Venkatesh concerning the equidistribution of semisimple orbits on adelic quotients. The motivation to studying these orbits come from the fact that number theoretical problems often relate to orbits of subgroups (also called periods) and so can be attacked by dynamical methods.

1. Statement of main theorem

To be more specific let us recall the following terminology. Let $X = \Gamma\backslash G$ be a homogeneous space defined by a lattice $\Gamma < G$ in a locally compact group G. Note that any subgroup $H < G$ acts naturally by right multiplication on X, sending $h \in H$ to the map $x \in X \mapsto xh^{-1}$. We will refer to H as the acting subgroup. A *homogeneous (probability) measure* on X is, by definition, a probability measure μ that is supported on a single closed orbit $Y = \Gamma g H_Y$ of its stabilizer $H_Y = \mathrm{Stab}(\mu)$.

Ratner's celebrated measure classification theorem [12] and the so called linearization techniques (cf. [3] and [10]) imply in the case where G is a real Lie group that, given a sequence of homogeneous probability measures $\{\mu_i\}$ with the property that $H_i = \mathrm{Stab}(\mu_i)$ contain "enough" unipotents, any weak* limit of $\{\mu_i\}$ is also homogeneous, where often the stabilizer of the weak* limit has bigger dimension than H_i for every i. This has been extended also to quotients of S-algebraic groups (see [13], [9], [6, App. A] and [8, Sect. 6]) for any finite set S of places (containing the infinite place and finitely many primes). We note that the latter allow similar corollaries (see [8]) for adelic quotients *if* the acting groups H_i contain unipotents at one and the same place for all i – let us refer to this as a *splitting condition.* These theorems have found many applications in number theory (see e.g. [7], [6], and [8] to name a few examples), but are (in most cases) ineffective.

In [4] one instance of an adelic result is presented which dispenses with the splitting condition and is effective in terms of the volume of the orbit. A simplified version of this theorem is given by the following statement.

Theorem 1 (Equidistribution of adelic periods)**.** *Let* $\mathbf{G}$ *be a semisimple simply connected algebraic group defined over* $\mathbb{Q}$ *and define the adelic homogeneous space* $X = \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ *with Haar measure* m_X *normalized to be a probability measure. Also assume that* $\mathbf{H} < \mathbf{G}$ *is a semisimple simply connected algebraic subgroup which is a maximal subgroup*[1] *of* $\mathbf{G}$*. Let* $Y = \mathbf{G}(\mathbb{Q})\mathbf{H}(\mathbb{A})g$ *be the corresponding adelic orbit (pushed by some* $g \in \mathbf{G}(\mathbb{A})$*) and let* μ_Y *denote the normalized Haar*

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[1]Here maximality is understood in the collection of connected algebraic subgroups over the algebraic closure of $\mathbb{Q}$.

measure on the orbit Y. Then

$$\left|\int f \,\mathrm{d}\mu_Y - \int f \,\mathrm{d}m_X\right| \ll \mathrm{vol}(Y)^{-\kappa_0}\mathcal{S}(f) \quad \text{for all } f \in C_c^\infty(X),$$

where $\mathcal{S}(f)$ denotes a certain adelic Sobolev norm, and κ_0 is a positive constant which depends only on $\dim \mathbf{G}$.

We note that the Sobolev norm $\mathcal{S}(f)$ of a smooth function $f \in C_c^\infty(X)$ takes into account the support, the smoothness properties in the real directions as well as the smoothness properties (level) of f in the p-adic directions.

Let us highlight two features of this theorem. Our method relies on Clozel's property (τ). However, it also allows us to give an independent proof of property (τ) except for groups of type A_1 – i.e., if we only suppose property (τ) for groups of type A_1, we can deduce property (τ) in all other cases as well as our theorem. The theorem also allows $\mathbf{H}$ to vary without any splitting condition.

2. Volume of orbits

To make sense of the theorem another question needs to be answered: What is the volume for a homogeneous set?

If $H = \mathrm{Stab}(\mu)$ is fixed, then one may define the volume of an H-orbit xH using a fixed Haar measure on H. However, as we will allow the acting group H to vary we give another reasonably intrinsic way of measuring this.

Let Y be an algebraic homogeneous set with corresponding probability measure μ_Y and associated group $H_Y = g^{-1}\mathbf{H}(\mathbb{A})g$. We shall always consider H_Y as equipped with that measure, denoted by m_Y, which projects to μ_Y under the orbit map.

Fix an open subset $\Omega_0 \subset \mathbf{G}(\mathbb{A})$ that contains the identity and has compact closure. Set

$$\mathrm{vol}(Y) := m_Y(H_Y \cap \Omega_0)^{-1}, \tag{2.1}$$

this should be regarded as a measure of the "volume" of Y. It depends on Ω_0, but the notions arising from two different choices of Ω_0 are comparable to each other, in the sense that their ratio is bounded above and below. Consequently, we do not explicate the choice of Ω_0 in the notation.

The above notion of the volume of an adelic orbit is strongly related to the discriminant of the orbit. The theorem could also be phrased using this notion of arithmetic height or complexity instead of the volume.

3. An overview of the argument

The dynamical argument is similar to the one from [5], where a splitting condition is made at the infinite place. Here we wish to use dynamics at a p-adic component instead of the real component. To overcome the absence of a splitting condition we make crucial use of Prasad's volume formula in [11] to find a small prime where the acting group has good properties.

The dynamical argument uses unipotent flows. Assuming that the volume is large, we find by a pigeon-hole principle nearby points that have equidistributing orbits. Using polynomial divergence of the unipotent flow we obtain almost invariance under a transverse direction. By maximality and spectral gap on the ambient space we conclude the equidistribution.

The first difficulty in this outline is that the notion of nearby depends here on the volume of the orbit, the "distance" will be a negative power $V^{-\kappa}$ of the volume V (for some $\kappa > 0$). If we take just any prime where the acting group is split, the argument might fail as the prime might even be larger than V^{κ}. Using [11] we establish a logarithmic bound for the first useful ("good") prime in terms of the volume. We also need to use [1] if **H** is not simply connected.

The second difficulty is that we also need to know that there are many points for which the orbit effectively equidistributes with respect to the measure in question. This effectivity also relies on spectral gap, but as the measure μ_Y (and so its L^2-space) varies we need uniformity for this spectral gap. This is Clozel's property (τ), see [2].

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