EVERY FLAT SURFACE IS BIRKHOFF AND OSCELEDETS GENERIC IN ALMOST EVERY DIRECTION

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1. Introduction

Flat surfaces and strata. Suppose $g \geq 1$, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a partition of $2g-2$, and let $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials, i.e. the space of pairs (M, ω) where M is a Riemann surface and ω is a holomorphic 1-form on M whose zeroes have multiplicities $\alpha_1 \ldots \alpha_n$. The form ω defines a canonical flat metric on M with conical singularities at the zeros of ω. Thus we refer to points of $\mathcal{H}(\alpha)$ as *flat surfaces* or *translation surfaces*. For an introduction to this subject, see the survey [Zo].

Affine measures and manifolds. Let $\mathcal{H}_1(\alpha) \subset \mathcal{H}(\alpha)$ denote the subset of surfaces of (flat) area 1. An affine invariant manifold is a closed subset of $\mathcal{H}_1(\alpha)$ which is invariant under the $SL(2,\mathbb{R})$ action and which in *period coordinates* (see [Zo, Chapter 3]) looks like an affine subspace. Each affine invariant manifold $\mathcal{M}$ is the support of an ergodic $SL(2,\mathbb{R})$ invariant probability measure $\nu_{\mathcal{M}}$. Locally, in period coordinates, this measure is (up to normalization) the restriction of Lebesgue measure to the subspace $\mathcal{M}$, see [EM] for the precise definitions. It is proved in [EMM] that the closure of any $SL(2,\mathbb{R})$ orbit is an affine invariant manifold.

The most importatant case of an affine invariant manifold is a connected component a stratum $\mathcal{H}_1(\alpha)$. In this case, the associated affine measure is called the Masur-Veech or Lebesgue measure [Mas], [Ve].

The Teichmüller geodesic flow. Let

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \qquad r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The element $r_\theta \in SL(2,\mathbb{R})$ acts by $(M,\omega) \to (M, e^{i\theta}\omega)$. This has the effect of rotating the flat surface by the angle θ. The action of g_t is called the *Teichmüller geodesic flow*. The orbits of $SL(2,\mathbb{R})$ are called *Teichmüller disks*.

A variant of the Birkhoff ergodic theorem. We use the notation $C_c(X)$ to denote the space of continuous compactly supported functions on a space X.

One of our main results is the following:

Theorem 1.1. *Suppose $x \in \mathcal{H}_1(\alpha)$. Let $\mathcal{M} = \overline{SL(2,\mathbb{R})x}$ be the smallest affine invariant manifold containing x. Then, for any $\phi \in C_c(\mathcal{H}_1(\alpha))$, for almost all $\theta \in [0, 2\pi)$, we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(g_t r_\theta x)\, dt = \int_{\mathcal{M}} \phi\, d\nu_{\mathcal{M}}, \tag{1.1}$$

where $\nu_{\mathcal{M}}$ is the affine measure whose support is $\mathcal{M}$.

Remark. The fact that (1.1) holds for almost all x with respect to the Masur-Veech measure is an immediate consequence of the Birkhoff ergodic theorem and the ergodicity of the Teichmüller geodesic flow [Mas], [Ve]. The main point of Theorem 1.1 is that it gives a statement for every flat surface x. This is important e.g. for applications to billiards in rational polygons (since the set of flat surfaces one obtains from unfolding rational polygons has Masur-Veech measure 0).

The proof of Theorem 1.1 is based on the following:

Proposition 1.2. *Fix $x \in \mathcal{M}$. For almost every $\theta \in [0, 2\pi]$, if ν_θ is any weak-star limit point (as $T \to \infty$) of $\eta_{T,\theta} * \delta_x$, then ν_θ is invariant under P, where $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset SL(2, \mathbb{R})$.*

The proof of Proposition 1.2 is based on the strong law of large numbers. In fact, Proposition 1.2 holds for arbitrary measure-preserving $SL(2, \mathbb{R})$ actions.

In addition to Proposition 1.2, the proof of Theorem 1.1 is based on the results of [EM] and [EMM]. One complication is controlling the visits to neigborhoods of smaller affine submanifolds, which we do using the techniques of [EMM], [A], [EMa] and which were originally introduced by Margulis in [EMaMo].

In addition to Theorem 1.1 we prove an analogous version of the Osceledets multiplicative ergodic theorem for the Kontsevich-Zorich cocycle, which has application e.g. to the wind-tree model. This proof requires additional ingredients, in particular a result of Filip [Fi].

References

[A] J. Athreya. "Quantitative recurrence and large deviations for Teichmuller geodesic flow." *Geom. Dedicata* **119** (2006).

[EMa] A. Eskin, H. Masur. "Asymptotic formulas on flat surfaces." *Ergodic Theory and Dynamical Systems,* **21** (2) (2001), 443–478.

[EMaMo] A. Eskin, G. Margulis, and S. Mozes. "Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture." *Ann. of Math. (2)* **147** (1998), no. 1, 93–141.

[EM] A. Eskin, M. Mirzakhani. "Invariant and stationary measures for the $SL(2, \mathbb{R})$ action on moduli space." `arXiv:1302.3320 [math.DS]` (2013).

[EMM] A. Eskin, M. Mirzakhani, A. Mohammadi. "Isolation, equidistribution and orbit closures for the $SL(2\mathbb{R})$ action on moduli space." `arxiv:1305.3015[math.DS]` (2013).

[Fi] S. Filip. "Semisimplicity and rigidity of the Kontsevich-Zorich cocycle." `arXiv:1307.7314[math.DS]` (2013).

[Mas] H. Masur. Interval exchange transformations and measured foliations. *Ann. of Math.* (2) **115** (1982), no. 1, 169–200.

[Ve] W. Veech, Gauss measures for transformations on the space of interval exchange maps, *Ann. of Math.*, **15** (1982), 201–242.

[Zo] A. Zorich, A. Flat Surfaces. *Frontiers in Number Theory, Physics, and Geometry. I. Berlin: Springer, 2006.* 437–583.