A structure theorem for multiplicative functions and applications

(joint work with Bernard Host)

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Three interconnected topics

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1. Structure theorem for multiplicative functions on the integers:

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$$9x^2 + 16y^2 = \lambda^2.$$

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2. Partition regularity of quadratic equations:

$$9x^2 + 16y^2 = \lambda^2.$$

3. Problems related to Chowla's conjecture:

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{1\leq m,n\leq N} \lambda(P(m,n)) = 0.$$

Partition regularity

Definition

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Problem

- $x^2 + y^2 = z^2$, *(Erdös, 70's).*
- $x^2 + y^2 = 2z^2$, *(Green, Gyarmati, Rusza).*
- $ax^2 + by^2 = cz^2$, *iff* $a = c$, *or* $b = c$, *or* $a + b = c$.

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is w.p.r. if the following numbers are non-zero squares

$$e^2 - 4ac, \quad f^2 - 4bc, \quad (e + f)^2 - 4c(a + b + d).$$

We can also deal with some higher degree equations.

Idea of proof for $9x^2 + 16y^2 = \lambda^2$

- Solutions of $9x^2 + 16y^2 = \lambda^2$ in parametric form:

$$x = km(m + 3n),\ y = k(m + n)(m - 3n),$$

$$\lambda = k(5m^2 + 9n^2 + 6mn).$$

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- **Density regularity:** If $d_{mult}(E) > 0$, then $\exists\, k, m, n \in \mathbb{N}$ s.t.

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- **Ergodic reformulation:** $(X, \mathcal{X}, \mu, T_n)$, $T_{mn} = T_m \circ T_n$, $\mu(A) > 0$, then $\exists\, m, n \in \mathbb{N}$ s.t.

$$\mu\big(T^{-1}_{m(m+3n)}A \cap T^{-1}_{(m+n)(m-3n)}A\big) > 0.$$

Idea of proof for $9x^2 + 16y^2 = \lambda^2$

- **Herglotz's theorem on $\mathbb{Q}$:** There exists a positive measure ν on

$$\mathcal{M} = \{\chi\colon \mathbb{N} \to \mathbb{T}\colon \chi(mn) = \chi(m)\chi(n) \text{ for every } m, n \in \mathbb{N}\}$$

such that for every $r, s \in \mathbb{N}$

$$\mu(T_r^{-1}A \cap T_s^{-1}A) = \int_{\mathcal{M}} \chi(r) \cdot \overline{\chi}(s)\, d\nu(\chi).$$

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- **Analytic reformulation:** Under some assumptions on ν we have

$$\liminf_{N\to\infty} \int_{\mathcal{M}} A_N(\chi)\, d\nu(\chi) > 0, \quad \text{where}$$

$$A_N(\chi) = \frac{1}{N^2} \sum_{1 \le m,n \le N} \chi(m) \cdot \chi(m+3n) \cdot \overline{\chi}(m+n) \cdot \overline{\chi}(m-3n).$$

- **Key tool:** A structural result for multiplicative functions.

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- Dirichlet characters (periodic).
- $\chi(2) = -1$, $\chi(p) = 1$ for $p \neq 2$ (non-uniform and non-periodic).

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For every $\varepsilon > 0$, $s \in \mathbb{N}$, *there exist* $q \in \mathbb{N}$, $C > 0$, *such that for every* $\chi \in \mathcal{M}$ *and* $N \in \mathbb{N}$, *there exist* χ_{st}, χ_{un} *bounded by* 2 *such that*

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- $\chi_{st} = \chi * \psi$ where ψ is a kernel with close to "rational" spectrum.

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A multiplicative function is aperiodic if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(an+b) = 0 \ \text{ for every } a, b \in \mathbb{N}.$$

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For a bounded multiplicative function the following are equivalent:

- χ *is aperiodic;*
- $\lim_{N\to\infty} \|\chi\|_{U^2(\mathbb{Z}_N)} = 0$;
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Proof of structure theorem: Main steps

1. **Inverse theorem (Green, Tao, Ziegler):** If $\|\chi\|_{U^s(\mathbb{Z}_N)} \geq \varepsilon$, then

$$\Big|\frac{1}{N}\sum_{n=1}^{N}\chi(n)f(n)\Big| \geq \delta(\varepsilon, s), \tag{1}$$

where $(f(n))$ is a $(s-1)$-step nilsequence of bdd complexity.

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3. ϕ kernel with spectrum supported on "bad" frequences (2), then
$$\|\chi-\chi_{st}\|_{U^s(\mathbb{Z}_N)} \leq \varepsilon, \quad \text{where } \chi_{st} = \chi * \phi;$$
$$\chi = \chi_{st} + \chi_{un}, \quad \text{where } \chi_{un} = \chi - \chi_{st}.$$

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  $X = G/\Gamma$ nilmanifold, $a \in G$ totally ergodic, $\int_X \Phi \, dm_X = 0$, then

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- **Kátai's orthogonality criterion (1984):** For $\chi \in \mathcal{M}$ we have

$$\frac{1}{N}\sum_{n=1}^{N} f(pn)\bar{f}(qn) \to 0, \ \forall p \neq q \in \mathbb{N} \implies \frac{1}{N}\sum_{n=1}^{N} \chi(n)f(n) \to 0.$$

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$$(\Phi \otimes \overline{\Phi})(v \cdot y) = c \cdot (\Phi \otimes \overline{\Phi})(y),\ c \neq 1 \implies \int_Y (\Phi \otimes \overline{\Phi})\, dm_Y = 0.$$

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- **Idea:** Use total ergodicity of a to show that if $Y = H/\Delta$, then

$$(g^p, g^q) \in H \cdot (G_2 \times G_2), \quad \text{for every } g \in G.$$

Then take iterated commutators $(s-1)$-times.

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- If $(a^n b^{n^2})$ totally equidistributed in X and $\int_X \Phi \, dm_X = 0$, then

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- **Can show:** If $Y = H/\Delta$, then $\exists G^1, G^2 \lhd H$ s.t. $G = G^1 \cdot G^2$ and
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- Taking iterated commutators $(s-1)$-times we get

$$U = \{u \in G_s\colon (u^{p^j}, u^{q^j}) \in H \text{ for some } j \in \mathbb{N}\} \ \text{ generates } \ G_s.$$

Idea of proof of Step 2 (infinitary world-quadratic case)

- If $(a^n b^{n^2})$ totally equidistributed in X and $\int_X \Phi \, dm_X = 0$, then

$$\frac{1}{N}\sum_{n=1}^{N} \chi(n)\Phi(a^n b^{n^2}\Gamma) \to 0 \qquad \forall \chi \in \mathcal{M}.$$

- **Can show:** If $Y = H/\Delta$, then $\exists G^1, G^2 \triangleleft H$ s.t. $G = G^1 \cdot G^2$ and

$$\{(g_1^p g_2^{p^2}, g_1^q g_2^{q^2})\colon g_1 \in G^1, g_2 \in G^2\} \subset H \cdot (G_2 \times G_2).$$

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- This again suffices to show that

$$\int_Y (\Phi \otimes \overline{\Phi}) \, dm_Y = 0.$$

Chowla conjecture

Problem (Chowla's Conjecture)

If $P \in \mathbb{Z}[x, y]$ homogeneous, $P \neq cQ^2$ and $\lambda =$Liouville, then

$$\frac{1}{N^2} \sum_{1 \leq m, n \leq N} \lambda(P(m, n)) \to 0.$$

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Theorem (F., Host 2014)

If χ averages to 0 on every infinite AP (for ex. the Liouville) and

$$P(m,n) = (m^2+n^2)^r \prod_{i=1}^{s} L_i(m,n), \quad r \geq 0,\ s \in \mathbb{N},$$

where L_i are pairwise independent linear forms, then

$$\frac{1}{N^2} \sum_{1 \leq m,n \leq N} \chi(P(m,n)) \to 0.$$

Chowla conjecture: Idea of proof

- **Idea:** $m + in \mapsto \chi(m^2 + n^2) = \chi(\mathcal{N}(m + in))$ is multiplicative, so we can apply Kátai's criterion for the Gaussian integers.

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- Such averages are bounded by a multiple of $\|\chi\|_{U^{2s-1}(\mathbb{Z}_N)}$.
- Assumption + U^{2s-1}-structure theorem $\implies \|\chi\|_{U^{2s-1}(\mathbb{Z}_N)} \to 0$.

Further directions

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Extend the U^s-structure theorem to more general number fields.

Example

Prove a U^s-structure theorem for $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{2}]$.

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Problem

Develop tools suitable for proving multiple recurrence for mps with multiplicative structure.

Example

$(X, \mathcal{X}, \mu, T_n)$ mps, $T_{mn} = T_m \circ T_n$, $\mu(A) > 0$. Show $\exists m, n \in \mathbb{N}$ s.t.

$$\mu\big(T^{-1}_{m(m+n)}A \cap T^{-1}_{(m+2n)(m+3n)}A \cap T^{-1}_{(m+4n)(m+5n)}A\big) > 0.$$

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THANK YOU!