ON THE MORDELL-GRUBER SPECTRUM

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This note is a summery of the talk I gave at the Dynamical Numbers conference on July 2014 at MPI Bonn based on a joint work with Barak Weiss bearing the same title. My aim in this note is to describe what I perceive as the main result in [SW14] and along the way illustrate some arguments dealing with simplifications.

Let $X_n \stackrel{\text{def}}{=} G/\Gamma$ denote the space of n-dimensional unimodular lattices in $\mathbb{R}^n$ where $G \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{R}), \Gamma \stackrel{\text{def}}{=} \mathrm{SL}_n(\mathbb{Z})$. Define a function $\kappa : X_n \to [0,1]$ by

$$\kappa(x) \stackrel{\text{def}}{=} \sup\left\{2^{-n}\operatorname{vol}(\mathcal{B}) : \begin{array}{l}\text{$\mathcal{B}$ is a symmetric x-admissible box with}\\ \text{faces parallel to the hyperplanes of the axis}\end{array}\right\}.$$

Here a set $\mathcal{B}$ is x-admissible if $x \cap \mathcal{B} = \{0\}$. We refer to $\kappa(x)$ as the Mordell constant of x and to the image of κ as the Mordell-Gruber spectrum. Note that the upper bound $\kappa(x) \le 1$ is a consequence of Minkowski's convex body theorem and that indeed there are lattices (such as $\mathbb{Z}^n$) with Mordell constant 1.

The dynamical interpretation of the Mordell constant is as follows. Let us denote by $\|\cdot\|$ the ∞-norm on $\mathbb{R}^n$ and for $\epsilon > 0$ let $X_n(\epsilon) \stackrel{\text{def}}{=} \{x \in X_n : x \cap B_{\epsilon^{1/n}} = \{0\}\}$, where $B_\epsilon = \{v \in \mathbb{R}^n : \|v\| < \epsilon\}$. Then, by the Mahler compactness criterion, $X_n(\epsilon)$ is an exhaustion of X_n by compact sets as $\epsilon \to 0$. Let $A < G$ denote the group of diagonal matrices with positive diagonal entries. As A acts transitively on all symmetric boxes with faces parallel to the axis of a given volume, we see by chasing the definitions that

$$\kappa(x) = \sup\left\{\epsilon : \overline{Ax} \cap X_n(\epsilon) \neq \varnothing\right\}. \tag{0.1}$$

Observe the following:

(1) κ is A-invariant.
(2) κ is semi-continuous in the sense that if $x_n \to x$ then $\kappa(x) \ge \limsup \kappa(x_n)$.

As a consequence we deduce that for any $x \in X_n$ we have that

$$\kappa(x) = \max\left\{\kappa(x') : x' \in \overline{Ax}\right\}. \tag{0.2}$$

The following measure theoretical analogue of the above topological statement will be more suitable for our discussion: Given an A-invariant and ergodic Radon measure μ on X_n, it is natural to define κ_μ as the μ-almost sure value of the A-invariant function κ. By (0.2)

$$\kappa_\mu = \max\{\kappa(x) : x \in \mathrm{supp}(\mu)\}.$$

It is clear that when $\mathrm{supp}\,\mu_1 \subset \mathrm{supp}\,\mu_2$ then $\kappa_{\mu_1} \leq \kappa_{\mu_2}$. Our discussion deals with trying to understand when does a strict inequality $\kappa_{\mu_1} < \kappa_{\mu_2}$ holds. Note that if $\mu = \mathbf{m}_{X_n}$ is the G-invariant probability measure on X_n then $\kappa_\mu = 1$. The following observation was the starting point of our study.

Proposition 0.1. Let μ be an A-invariant measure supported on a compact set. Then $\kappa_\mu < 1$.

Proof. The following short argument relies on a rather heavy tool, namely Hajós Theorem [Haj49]. In our context this theorem (which settled a conjecture of Minkowski) asserts the equality

$$X_n(1) = \sqcup_\sigma \sigma U \mathbb{Z}^n,$$

where $U < G$ is the subgroup of upper triangular unipotent matrices and σ is a permutation matrix. A straightforward check shows the inclusion $\supset$ in the above equation and the content of Hajós' theorem is the inclusion $\subset$. What we need to take out of this theorem is the fact that any lattice in $X_n(1)$ contains a non-trivial vector on one of the axis. In particular, by (0.2), if $\kappa_\mu = 1$ then in $\mathrm{supp}\,\mu$ there exists a lattice having a non-trivial vector on one of the axis. This vector could then be made as short as we wish by acting upon with a suitable element of A in contradiction to the compactness of $\mathrm{supp}\,\mu$. $\square$

The following is a first approximation of the main result I wish to describe here. It generalizes the above proposition when μ is assumed to be homogeneous.

Theorem 0.2. *Let $H_1 < H_2$ be a strict containment between two connected closed subgroups of G containing A. Let μ_i be an H_i-invariant Radon measure supported on an H_i-orbit (i.e. a homogeneous measure). Suppose that $\mathrm{supp}\,\mu_1 \subset \mathrm{supp}\,\mu_2$ and that $\mu_1(X_n) < \infty$. Then*[1] $\kappa_{\mu_1} < \kappa_{\mu_2}$.

Note that the assumption $\mu_1(X_n) < \infty$ is needed as is shown by considering the orbit containment $A\mathbb{Z}^n \subset G\mathbb{Z}^n$ each of which supports an A-invariant and ergodic Radon measures having generic Mordell constants equal to 1.

[1] One can show that in this case μ_i are A-ergodic.

The main theorem that we prove in [SW14] is stronger than Theorem 0.2 but is more elaborate to state. We attach to each A-invariant and ergodic homogeneous Radon measure μ an algebraic invariant; namely a certain finite dimensional $\mathbb{Q}$-algebra $\mathcal{A}_\mu$ in such a way that if μ_i are such measures and $\operatorname{supp}\mu_1 \subset \operatorname{supp}\mu_2$ then there is a reversed inclusion $\mathcal{A}_{\mu_2} \hookrightarrow \mathcal{A}_{\mu_1}$. The associated algebras $\mathcal{A}_\mu$ that arise in this way are of the form $\oplus_1^r \mathbb{F}_i$ where $\mathbb{F}_i$ are number fields. We say that an inclusion $\mathcal{A}_{\mu_2} \hookrightarrow \mathcal{A}_{\mu_1}$ is *essential* if it is onto when post-composing with the projections onto the number field components of $\mathcal{A}_{\mu_1}$. Otherwise this inclusion is said to be *non-essential*.

Theorem 0.3. *Let μ_i be two measures as in Theorem 0.2 but without the finiteness assumption $\mu_1(X_n) < \infty$. Then, if the containment of the associated algebras $\mathcal{A}_{\mu_2} \hookrightarrow \mathcal{A}_{\mu_1}$ is non-essential, then there is a strict inequality $\kappa_{\mu_1} < \kappa_{\mu_2}$.*

We end noting two things:

(1) In the notation of the Theorem, if $\mu_1(X_n) < \infty$ then the containment $\mathcal{A}_{\mu_2} \hookrightarrow \mathcal{A}_{\mu_1}$ is automatically non-essential and so Theomre 0.3 indeed implies Theorem 0.2.
(2) In the example of orbit-inclusion $A\mathbb{Z}^n \subset G\mathbb{Z}^n$ discussed above, giving rise to an equality between the generic Mordell constants, the associated algebras turn to be $\mathcal{A}_{\mu_1} = \oplus_1^n \mathbb{Q}$, and $\mathcal{A}_{\mu_2} = \mathbb{Q}$. So, the (diagonal) inclusion $\mathbb{Q} \hookrightarrow \oplus_1^n \mathbb{Q}$ is essential and so this fits with Theorem 0.3.

References

[Haj49] G. Hajós, *Sur la factorisation des groupes abéliens*, Časopis Pěst. Mat. Fys. **74** (1949), 157–162 (1950). MR0045727 (13,623a)

[SW14] U. Shapira and B. Weiss, *On the mordellgruber spectrum*, International Mathematics Research Notices (2014), available at http://imrn.oxfordjournals.org/content/early/2014/06/23/imrn.rnu099.full.pdf+html.