EFFECTIVE RATNER EQUIDISTRIBUTION FOR $\mathrm{SL}(2,\mathbb{R}) \ltimes (\mathbb{R}^2)^{\oplus k}$ AND APPLICATIONS TO QUADRATIC FORMS

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Let $G = \mathrm{SL}(2,\mathbb{R}) \ltimes (\mathbb{R}^2)^{\oplus k}$, Γ be a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}^2)^{\oplus k}$, and let $X = \Gamma\backslash G$. Let

$$U(x) = \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \right), \text{ and } a(y) = \left(\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}; \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \right), \ (x \in \mathbb{R},\, y > 0),$$

and let $\boldsymbol{\xi} = \left(\begin{smallmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{smallmatrix}\right) \in \mathbb{R}^{2k}$ be an "irrational" vector. We consider a family of one-dimensional unipotent orbits $\Gamma(I_2;\boldsymbol{\xi})a(y)U(\mathbb{R})$ in X, where I_2 denotes the identity matrix. A result of Shah [1] implies the ineffective equidistribution result for the above unipotent orbits, as y tends to 0:

Theorem 1. *Let $\boldsymbol{\xi} \in \mathbb{R}^{2k}\backslash\mathbb{Q}^{2k}$ is such that for any $\boldsymbol{m} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$, either $\boldsymbol{m}\boldsymbol{\xi}_1$ or $\boldsymbol{m}\boldsymbol{\xi}_2$ is not in $\mathbb{Z}$, and if $f \in C_c(X)$, $\nu \in \mathrm{L}^1(\mathbb{R})$, then*

$$\lim_{y\to 0} \int_{\mathbb{R}} f(\Gamma(I_2;\boldsymbol{\xi})a(y)U(x))\nu(x)\,dx = \int_X f\,d\mu.$$

Shah's result however depends crucially on the Ratner's measure classification theorem, which fails to give an effective statement in this setting. We instead use spectral methods. In particular, we use the Fourier expansion in the torus component along with bounds for certain exponential sums to get an effective Ratner equidistribution result for these class of unipotent trajectories. More specifically, we prove that

Theorem 2. *Let $\boldsymbol{\xi} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\xi}_2 \end{pmatrix}$ or $\begin{pmatrix} \boldsymbol{\xi}_1 \\ \mathbf{0} \end{pmatrix}$, where $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, are of "certain Diophantine type" in $\mathbb{R}^k$, then for $f \in C_c^{2k+6}(X)$, $\nu \in C_c^2(\mathbb{R})$, there exists $\delta > 0$, such that for any $y \leq 1/2$,*

$$\int_{\mathbb{R}} f(\Gamma(I_2;\boldsymbol{\xi})a(y)U(x))\nu(x)dx = \int_X f d\mu \int \nu dx + O(y^\delta).$$

The value of δ depends on the Diophantine nature of $\boldsymbol{\xi}$. For a generic $\boldsymbol{\xi}$ however, δ can be replaced by $1/4$. We thus generalize a result of Strombergsson [3] in the general k setting, where $\boldsymbol{\xi}$ of type $\begin{pmatrix} \mathbf{0} \\ \boldsymbol{\xi}_2 \end{pmatrix}$ or $\begin{pmatrix} \boldsymbol{\xi}_1 \\ \mathbf{0} \end{pmatrix}$.

We apply this result to obtain an effective Oppenheim type result for a class of indefinate irrational quadratic forms. Given $\alpha, \beta \in \mathbb{R}$, let $Q_{\alpha,\beta}(\boldsymbol{x}) = (x_1-\alpha)^2 + (x_2-\beta)^2 - (x_3-\alpha)^2 - (x_4-\beta)^2$ denote a family of quadratic forms. Given $a < b$, and a positive real T, let

$$N_{a,b}^{\alpha,\beta}(T) = \#\{|\boldsymbol{x}| \leq T : \boldsymbol{x} \in \mathbb{Z}^4, a < Q_{\alpha,\beta}(\boldsymbol{x}) < b\}.$$

Using an idea of Marklof [2] we have,

$$N_{a,b}^{\alpha,\beta}(T) = \int_{\mathbb{R}} \left| \Theta\Big(\Gamma(I_2;\boldsymbol{\xi})a(1/T)U(x)\Big) \right|^2 \nu(x)\,dx,$$

where $\boldsymbol{\xi} = {}^{\mathrm{t}}(0,0,\alpha,\beta)$. Here Θ denotes a certain *theta* function, which belongs to $C^\infty(\Gamma\backslash G)$, where $G = \mathrm{SL}(2,\mathbb{R}) \ltimes (\mathbb{R}^2)^{\oplus 2}$, and Γ is a certain congruence subgroup. We apply theorem 2, along with a lattice point counting argument to prove the following effective version of a result of Marklof [2]:

Theorem 3. *Let α, β of certain Diophantine type and let $a < b$, be arbitrary real numbers. Then there exists $\delta > 0$ such that*

$$N_{a,b}^{\alpha,\beta}(T) = C(b-a)T^2 + O(T^{2-\delta}),$$

where C is an absolute constant.

As before, the value of δ depends on the Diophantine properties of α and β.

References

[1] N. Shah, Limit distributions of expanding translates of certain orbits on homogeneous spaces, Proc. Indian Acad. Sci. (Math. Sci.) **106** (1996), 105–125.

[2] J. Marklof, Pair correlation densities of inhomogeneous quadratic forms, Annals of Math. **158** (2003), 419-471.

[3] A. Strömbergsson, Effective Ratner equidistribution for $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$, to appear in Duke Math. J.