Homotopy Type Theory MPIM-Bonn 2016

Dependent Type Theories

Lecture 1. General introduction and C-systems.

By Vladimir Voevodsky

from Institute for Advanced Study in Princeton, NJ.

February, 2016

Let me first tell you some things about what to expect from the coming lectures:

1. These are lectures in mathematics.
2. Most of the material will be pure mathematics. Applications to the real world objects, type theories, will appear only at the end when a sufficiently rich mathematical language has been developed.
3. Most of the mathematics that I will talk about can be found in my papers posted on the arXiv.

At the center of my approach to the mathematical theory of type theories is a class of mathematical objects that are called C-systems. They were originally defined by John Cartmell in his 1978 Ph.D. thesis under the name contextual categories.

A C-system equipped with additional operations corresponding to the inference rules of a type theory is called a model or a C-system model of this system of rules or of this type theory.

There are other classes of objects on which one can define operations corresponding to inference rules of type theories most importantly categories with families or CwFs. They lead to other classes of models.

In order to provide a mathematical representation (semantics) for a type theory one constructs two C-systems.

- One C-system, that we will call the term C-system of a type theory, is constructed from the formulas of type theory.

  To explain how to do it in sufficient generality and at the same time with mathematical rigor is the first and main goal of these lectures.

- The second C-system is constructed from the category of abstract mathematical objects.

  To explain how to do this construction is the second goal of the lectures.

Both C-systems are then equipped with additional operations corresponding to the inference rules of the type theory making them into models of type theory. The model whose underlying C-system is the term C-system is called the term model.

In my papers I have described how to construct abstract models of two systems of inference rules - the rules for dependent products and the rules for Martin-Lof identity types. How to construct models for the rules for dependent sums will be described in a forthcoming paper.

How to construct models of these systems of rules on the term C-systems is, at the moment, one of the missing pieces of the general theory but this particular piece should be easy to fill in.

A homomorphism from the term model to another, usually abstract, model of a type theory is called a representation of this type theory.

More generally, any functor from the category underlying the term C-system of the type theory to another category may be called a representation of the type theory in that category.

Since objects and morphisms of term models are built from formulas of the type theory and objects and morphisms of abstract C-systems are built from mathematical objects such as sets or homotopy types and the corresponding functions, such representations provide mathematical meaning to formulas of type theory.

A crucial component of this approach is the expected result that for a particular class of the inference rules the term model is an initial object in the category of models. This is known as the *Initiality Conjecture*.

In the case of the pure Calculus of Constructions with a decorated application operation this conjecture was proved in 1988 by Thomas Streicher in his Ph.D. thesis.

The problem of finding an appropriate formulation of the general version of this conjecture and of proving this general version is the key problem of this theory. If we have time I will try to demonstrate its complexity and to suggest a partial answer to the first question that arise there - how to give a mathematical definition of a general system of inference rules.

For systems of inference rules for which the initiality conjecture holds there is a unique homomorphism from the term model to any other model.

Only if we know that the initiality conjecture holds for a given system of inference rules can we claim that a model defines a representation.

A similar problem also arises in the predicate logic but there, since one considers only one fixed system of syntax and inference rules, it can and had been solved once without the development of a general theory.

The term models and representations for a class of type theories can be obtained by considering slices of the term model of the type theory called Logical Framework (LF), but unfortunately it is unclear how to extend this approach to type theories that have more substitutional (definitional) equalities than LF itself.

Finally let me make a remark about another class of type theories.

For the type theories that I have been talking about the class of valid sentences is defined as the class of derivable sentences.

There are other type theories where the relationship between valid sentences and the inference rules is less direct. Much of the theory that I will be speaking about applies to these type theories as well. However, the initiality conjecture is unlikely to hold for them and therefore not every model of the rules defines a representation.

What are the additional requirements that models have to satisfy to define representations for these type theories is, to me at least, a completely open question.

At this let me end the introduction and start the mathematical part of the lectures.

Since we want the constructions that I will be talking about to be used for certification of type theories and eventually of proof assistants for consistency and for their applicability as tools in developing pure mathematics these constructions themselves need to be done in a formal system that has a high level of trust in the mathematical community.

Today such a system is the Zermelo-Fraenkel set theory and theories that are close to it.

Therefore all of the mathematics that I will be explaining will be done with an eye to future formalization in the ZF or ZF with a Grothendieck universe.

While we will work in the ZF we will work constructively.

What it means to me is that everything in these lectures and in my recent papers can be formalized both in the ZF and in the UniMath without using any additional axioms.

In particular it means that we will not use the axiom of the excluded middle or the axiom of choice.

We fix a universe U. In the ZF a universe is simply a set that satisfies some conditions. We do not make precise here what conditions are required from U. It will be always sufficient to require that U satisfies the conditions that define a Grothendieck universe but for most of our work much weaker conditions, weak enough to be able to construct such a universe inside ZF, suffice.

By a category C we mean a pair of sets $Mor(C)$ and $Ob(C)$ with four maps

$$\partial_0, \partial_1 : Mor(C) \to Ob(C)$$

$$Id : Ob(C) \to Mor(C)$$

and

$$\circ : Mor(C)_{\partial_1} \times_{\partial_0} Mor(C) \to Mor(C)$$

which satisfy the well known conditions of unity and associativity.

Important note: we write composition of morphisms in the diagrammatic order, that is we write $f \circ g$ or fg for $f : X \to Y$ and $g : Y \to Z$.

Definition 1 *A C0-system is a pre-category CC with additional structure of the form*

1. *a function* $l : Ob(CC) \to \mathbf{N}$,
2. *a map* $ft : Ob(CC) \to Ob(CC)$ *such that if* $l(X) > 0$ *then* $l(ft(X)) = l(X) - 1$ *and if* $l(X) = 0$ *then* $ft(X) = X$,
3. *for each* $X \in Ob(CC)$ *a morphism* $p_X : X \to ft(X)$,
4. *for each* $X \in Ob(CC)$ *such that* $l(X) > 0$ *and each morphism* $f : Y \to ft(X)$ *an object* f^*X *and a morphism*

$$q(f, X) : f^*X \to X$$

such that the following additional conditions are satisfied:

1. *for* $X \in Ob(CC)$ *such that* $l(X) > 0$ *and* $f : Y \to ft(X)$ *one has* $l(f^*(X)) > 0$ *and* $ft(f^*X) = Y$ *and the square*

$$\begin{array}{ccc} f^*X & \stackrel{q(f,X)}{\longrightarrow} & X \\ p_{f^*X}\downarrow & & \downarrow p_X \\ Y & \stackrel{f}{\longrightarrow} & ft(X) \end{array} \qquad (1)$$

   commutes,

2. *for* $X \in Ob(CC)$ *such that* $l(X) > 0$ *one has* $id^*_{ft(X)}(X) = X$ *and* $q(id_{ft(X)}, X) = id_X$*,*

3. *for* $X \in Ob(CC)$ *such that* $l(X) > 0$*,* $g : Z \to Y$ *and* $f : Y \to ft(X)$ *one has* $(gf)^*(X) = g^*(f^*(X))$ *and*

$$q(gf, X) = q(g, f^*X)q(f, X),$$

4. $l^{-1}(0)$ *is a one element set and its element* pt_{CC} *is a final object of* CC*.*

For $f : Y \to X$ in CC we let $ft(f) : Y \to ft(X)$ denote the composition $f \circ p_X$.

Definition 2 *A C-system is a C0-system together with an operation $f \mapsto s_f$ defined for all $f : Y \to X$ such that $l(X) > 0$ and such that*

1. $s_f : Y \to (ft(f))^*(X)$,
2. $s_f \circ p_{(ft(f))^*(X)} = Id_Y$,
3. $f = s_f \circ q(ft(f), X)$,
4. *if* $X = g^*(U)$ *where* $g : ft(X) \to ft(U)$ *then* $s_f = s_{f \circ q(g,U)}$.

Proposition 3 *Let CC be a C0-system. Then the following are equivalent:*

1. *the canonical squares (1) of CC are pull-back squares,*
2. *there is given a structure of a C-system on CC.*

The proof can be found in the paper "Subsystems and regular quotients of C-systems".

Note: we fix a distinguished one element set in U and call it $unit$. It's only element is called tt. This strange naming is taken from the naming used in the proof assistant Coq.

The choice of a one element set allows us to speak about things such as *the* one point category.

Examples

1. The trivial C-systems are the C-systems with only one object pt_{CC} and only the identity morphism from pt_{CC} to pt_{CC}. Note that as we are working in ZF there are very many such C-systems - the set of trivial C-systems in the universe U is in a bijective correspondence with the set of pairs of one element sets in U and therefore in a bijective correspondence with $U \times U$. All trivial C-systems do not form a collection that can be defined as a set in the ZF.

Our choice of a one element set allows us to speak about *the* trivial C-system and we call it by the same name as the one element set.

2. The almost trivial C-system. It is the C-system whose underlying category is the category $\mathbf{N}_{triv}$ with the set of objects being the set $\mathbf{N}$ of natural numbers and the set of morphisms being the set $\mathbf{N}\times\mathbf{N}$ so that there is exactly one morphism between any two objects.

The length function is the identity. All other structures are uniquely determined by the axioms.

3. **Important**: Consider the category $stn(2)_{triv}$ with two objects 0 and 1 and the set of morphisms $\{0,1\} \times \{0,1\}$ so that there is exactly one morphism between any two objects. Then there is no C-system structure on this category.

Indeed, the axiom that $l^{-1}(0) = \{pt\}$ implies that for the other object X we have $l(X) > 0$. Then $p_X^*(X)$ is defined and $l(p_X^*(X)) = l(X)+1$ and therefore $p_X^*(X) \neq pt$ and $p_X^*(X) \neq X$. However, there is no third object in the category.

On the other hand it is easy to construct an equivalence between the category $\mathbf{N}_{triv}$ and the category $stn(2)_{triv}$. This shows that the C-system structure on categories can not be transported along the equivalences of categories.