Homotopy Type Theory MPIM-Bonn 2016

Dependent Type Theories

Lecture 2. Lawvere theories and C-systems.

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February, 2016

1. The trivial C-systems are the C-systems with only one object pt_{CC} and only the identity morphism from pt_{CC} to pt_{CC}.

Our choice of a one element set allows us to speak about *the* trivial C-system and we call it by the same name as the one element set.

2. The almost trivial C-system. It is the C-system whose underlying category is the category $\mathbf{N}_{triv}$ with the set of objects being the set $\mathbf{N}$ of natural numbers and the set of morphisms being the set $\mathbf{N} \times \mathbf{N}$ so that there is exactly one morphism between any two objects.

The length function is the identity. All other structures are uniquely determined by the axioms.

The second of the C-systems on the previous slide belongs to a class of C-systems that are called l-bijective C-systems.

Definition 1 *A l-bijective C-system is a C-system such that the length function* $Ob(CC) \to \mathbf{N}$ *is a bijection.*

Today we will define Lawvere theories, construct the category of Lawvere theories in U, construct the category of C-systems in U and then construct an isomorphism between the category of Lawvere theories in U and the full subcategory in the category of C-systems in U that consists of l-bijective C-systems.

For $m \in \mathbf{N}$ we choose as our standard set with m elements the set

$$stn(m) = \{i \in \mathbf{N} \,|\, 0 \leq i < m\}$$

Let

$$Ob(F) = \mathbf{N}$$

and

$$Mor(F) = \cup_{n,m \in \mathbf{N}} Fun(stn(n), stn(m))$$

where $Fun(X, Y)$ is the set of functions from X to Y.

We use the definition of a function given on p.81 of the "Theory of sets" by N.Bourbaki where a function f from X to Y is defined as a triple (X, Y, G) where G is a subset in $X \times Y$ satisfying the usual conditions.

Because of this choice of the definition every function has a well defined domain and codomain. This makes it possible to define a category F with the set of objects $Ob(F)$ and the set of morphisms $Mor(F)$ such that for each n and m the set

$$F(m,n) := \{f \in Mor(F) \,|\, dom(f) = m \,\text{and}\, codom(f) = n\}$$

equals to $Fun(stn(m), stn(n))$ and composition, when restricted to these subsets, is the composition of functions.

For $m, n \in \mathbf{N}$ let

$$ii_0^{m,n} : stn(m) \to stn(m+n)$$

and

$$ii_1^{m,n} : stn(n) \to stn(m+n)$$

be the injections of the initial segment of length m and the concluding segment of length n.

Definition 2 *A Lawvere theory structure on a category T is a functor $L : F \to T$ such that the following conditions hold:*

1. *L is a bijection on the sets of objects,*
2. *$L(0)$ is an initial object of T,*
3. *for any $m, n \in \mathbf{N}$ the square*

$$\begin{array}{ccc} L(0) & \longrightarrow & L(n) \\ \downarrow & & \downarrow L(ii_1^{m,n}) \\ L(m) & \xrightarrow{L(ii_0^{m,n})} & L(m+n) \end{array}$$

   is a push-out square.

A Lawvere theory is a pair (T, L) where T is a category and L is a Lawvere theory structure on T.

This definition is equivalent, in the strict sense, to the definition given by Lawvere in the 2004 “reprint” of his 1963 Ph.D. thesis where he called the objects that we have just defined “algebraic theories”.

Problem 3 *To construct the category $LW(U)$ of Lawvere theories in U.*

Construction 4 Following Lawvere we define a morphism from a Lawvere theory $\mathbf{T}_1 = (T_1, L_1)$ to a Lawvere theory $\mathbf{T}_2 = (T_2, L_2)$ as a functor $G : T_1 \to T_2$ such that $L_1 \circ G = L_2$.

Note that here one uses the equality rather than isomorphism of functors.

We let $Hom_{LW}(\mathbf{T}_1, \mathbf{T}_2)$ denote the subset in the set of functors from T_1 to T_2 that are morphisms of Lawvere theories.

The composition of morphisms is defined as composition of functors. The identity morphism is the identity functor. The associativity and the left and right unity axioms follow immediately from the corresponding properties of the composition of functors.

We let $Ob(LW(U))$ denote the set of Lawvere theories in U and let

$$Mor(LW(U)) = \coprod_{\mathbf{T}_1, \mathbf{T}_2 \in Ob(LW(U))} Hom_{LW}(\mathbf{T}_1, \mathbf{T}_2)$$

Together with the obvious domain, codomain, identity and composition functions the pair of sets $Ob(LW(U))$ and $Mor(LW(U))$ forms a category that we denote $LW(U)$ and call the category of Lawvere theories in U.

Now we have to deal with the main problem that I encountered trying to do categorical constructions carefully in the ZF.

Note that with our definition a morphism from $\mathbf{T}_1$ to $\mathbf{T}_2$ in the category of Lawvere theories *is not* a morphism of Lawvere theories but an iterated pair $((\mathbf{T}_1, \mathbf{T}_2), G)$ where G is a morphism of Lawvere theories.

In fact, it is easy to prove that there is no category whose set of objects is the set of Lawvere theories in U and such that for any $\mathbf{T}_1$, $\mathbf{T}_2$ the set of morphisms with the domain $\mathbf{T}_1$ and the codomain $\mathbf{T}_2$ in this category *equals* to the set $Hom_{LW}(\mathbf{T}_1, \mathbf{T}_2)$.

Indeed, consider the identity morphisms of Lawvere theories. Then for $\mathbf{T}_1$, $\mathbf{T}_2$ we have $Id_{\mathbf{T}_1} = Id_{\mathbf{T}_2}$ if $T_1 = T_2$. But there are non-equal Lawvere theories with the same underlying categories and therefore there is no domain function dom such that $dom(Id_{\mathbf{T}}) = \mathbf{T}$.

This problem is not particular to Lawvere theories and one encounters it almost always when one tries to construct a category from a given family of morphism sets.

The fact that it could be done for F and can be done for the category $Sets(U)$ of sets in U is due to our choice of the definition of a function between two sets. If we chose the definition where a function is the same as its graph we would not be able to have a category whose set of morphisms between two sets is the set of functions between these two sets.

Coming back to Lawvere theories consider the obvious bijection

$$Mor_{LW(U)}(\mathbf{T}_1, \mathbf{T}_2) \to Hom_{LW}(\mathbf{T}_1, \mathbf{T}_2)$$

We will use the functions in both directions given by this bijection as *coercions*, in the terminology of the computer proof assistants. That is, every time we have an expression which denotes an element of one of these sets in a position where an element of the other is expected it is assumed to be replaced by its image under the corresponding function.

This completes the construction of the category of Lawvere theories in U.

Next we will define homomorphisms of C-systems and the category of C-systems in U.

Definition 5 *Let CC_1, CC_2 be C-systems. A homomorphism F from CC_1 to CC_2 is a pair of functions $F_{Ob} : Ob(CC_1) \to Ob(CC_2)$, $F_{Mor} : Mor(CC_1) \to Mor(CC_2)$ such that:*

1. *F_{Ob} commutes with the length functions, i.e., for all $X \in Ob(CC_1)$ one has*
$$l(F_{Ob}(X)) = l(X),$$
2. *F_{Ob} commutes with the ft function, i.e., for all $X \in Ob(CC_1)$ one has*
$$ft(F_{Ob}(X)) = F_{Ob}(ft(X)),$$
3. *F is a functor,*

4. *F takes p-morphisms to p-morphisms, i.e., for all $X \in Ob(CC_1)$ one has*

$$p_{F_{Ob}(X)} = F_{Mor}(p_X),$$

5. *F takes q-morphisms to q-morphisms, i.e., for all $X, Y \in Ob(CC_1)$ such that $l(Y) > 0$ and all $f : X \to ft(Y)$ one has*

$$F_{Mor}(q(f, Y)) = q(F_{Mor}(f), F_{Ob}(Y)),$$

6. *F takes s-morphisms to s-morphisms, i.e., for all $X, Y \in Ob(CC_1)$ such that $l(Y) > 0$ and $f : X \to Y$ one has*

$$s_{F_{Mor}(f)} = F_{Mor}(s_f).$$

In what follows we will write F for both F_{Ob} and F_{Mor} since the choice of which one is meant is determined by the type of the argument. Note that the condition that F commutes with the domain function together with the q-morphism condition implies that for all $X, Y \in Ob(CC_1)$ such that $l(Y) > 0$ and all $f : X \to ft(Y)$ one has

$$F(f^*(Y)) = F(f)^*(F(Y)) \tag{1}$$

Lemma 6 *Let $F : CC_1 \to CC_2$ and $G : CC_2 \to CC_3$ be homomorphisms of C-systems. Then the compositions of functions $F_{Ob} \circ G_{Ob}$ and $F_{Mor} \circ G_{Mor}$ is a homomorphism of C-systems.*

The proof is straightforward but when written up it is a relatively long proof since many conditions need to be verified.

Lemma 7 *Let CC_1, CC_2, F_{ob} and F_{Mor} be as above. Assume further that these data satisfies all of the conditions of the definition except, possibly, the s-morphisms condition. Then it satisfies the s-morphisms condition and forms a homomorphism of C-systems.*

The proof can be found in the paper "A C-system defined by a universe category".

Since homomorphisms of C-systems are pairs of functions between sets satisfying certain conditions and the composition is given by composition of these functions, the associativity and unitality of this composition follows from the associativity and unitality of the composition of functions between sets.

Problem 8 *Let U be a universe. To construct a category $CS(U)$ of C-systems in U.*

Construction 9 The construction is similar to the construction of the category of Lawvere theories in U. We again encounter the same problem with the sets of morphisms and solve it at the level of notation by using the obvious bijections as coercions.

We let $Lw(T)$ denote the set of Lawvere theory structures on a category T.

We let $Cs_{\mathbf{N}}(CC)$ denote the set of l-bijective C-system structures on a category CC.

Problem 10 *For a category T to construct a function*

$$LwCs : Lw(T) \to Cs_{\mathbf{N}}(T^{op})$$

from the Lawvere theory structures on T to the l-bijective C-system structures on T^{op}.

Construction 11 Let $CC = T^{op}$. We need to construct a l-bijective C-system structure on CC. We set:

The length function $l = L^{-1}$.

The map $ft : Ob(CC) \to Ob(CC)$ maps $L(0)$ to $L(0)$ and any object X such that $l(X) > 0$ to $L(l(X) - 1)$.

The distinguished final object pt is $L(0)$.

For pt the morphism p_{pt} is the identity. For X such that $l(X) > 0$ the morphism $p_X : X \to ft(X)$ is $L(ii_0^{l(X)-1,1})$.

To define $q(f, X)$ observe first that for any X such that $l(X) > 0$ we have a pull-back square in CC of the form

$$\begin{array}{ccc} X & \xrightarrow{L(ii_1^{l(X)-1,1})} & L(1) \\ {\scriptstyle p_X}\downarrow & & \downarrow \\ ft(X) & \longrightarrow & L(0) \end{array} \tag{2}$$

Given $f : Y \to ft(X)$ we set

$$f^*(X) = L(l(Y) + 1)$$

Since (2) is a pull-back square and $L(0)$ is a final object there is a unique morphism $q(f, X) : f^*(X) \to X$ such that

$$q(f, X) \circ p_X = p_{f^*(X)} \circ f \quad q(f, X) \circ L(ii_1^{l(X)-1,1}) = L(ii_1^{l(Y),1}) \tag{3}$$

The verification of the axioms of a C-system can be found in "Lawvere theories and C-systems".

Lemma 12 *Let* $G : (T_1, L_1) \to (T_2, L_2)$ *be a morphism of Lawvere theories. Then the functor* G^{op} *is a homomorphism of C-systems* $(T_1^{op}, LwCs(L_1)) \to (T_2^{op}, LwCs(L_2))$.

For the proof see “Lawvere theories and C-systems”.

Problem 13 To construct a functor

$$LC : LW(U) \to CS_{\mathbf{N}}(U).$$

Construction 14 We set LC_{Ob} to be the function that takes a Lawvere theory to the opposite category of its underlying category with the C-system structure defined by Construction 11. We set LC_{Mor} to be the function that takes a functor G that is a morphism of Lawvere theories to G^{op}. It is well defined by Lemma 12. That the functions (LC_{Ob}, LC_{Mor}) form a functor, i.e., commute with the identity morphisms and compositions is straightforward.

Problem 15 *For a category CC to construct a function*

$$CsLw : Cs_{\mathbf{N}}(CC) \to Lw(CC^{op})$$

To perform a construction we will need a number of lemmas and intermediate constructions. Let us fix a category CC and a l-bijective C-system structure $cs = (l, pt, ft, p, q, s)$ on CC. We will often write CC both for the category and for the C-system (CC, cs).

Problem 16 *For* $m \in \mathbf{N}$ *and* $i = 0, \ldots, m-1$ *to construct a morphism* $\pi_i^m : l^{-1}(m) \to l^{-1}(1)$ *in* CC*.*

Construction 17 By induction on m.

For $m = 0$ there are no morphisms to construct.

For $m = 1$ we set $\pi_0^1 = Id_{l^{-1}(1)}$.

For the successor consider the canonical square:

$$\begin{array}{ccc} l^{-1}(m+1) & \xrightarrow{q(\pi,l^{-1}(1))} & l^{-1}(1) \\ p_{l^{-1}(m+1)}\downarrow & & \downarrow\ p_{l^{-1}(1)} \\ l^{-1}(m) & \xrightarrow{\pi} & l^{-1}(0) \end{array} \tag{4}$$

where we use π to denote the unique morphisms from objects of CC to the final object $l^{-1}(0)$. We set

$$\pi_i^{m+1} = \begin{cases} p_{l^{-1}(m+1)} \circ \pi_i^m & \text{for } i < m \\ q(\pi, l^{-1}(1)) & \text{for } i = m \end{cases}$$

Problem 18 *For any* $m, n \in \mathbf{N}$ *and a function* $f : stn(m) \to stn(n)$ *to construct a morphism* $L_f : l^{-1}(n) \to l^{-1}(m)$ *in* CC*.*

Construction 19 By induction on m.

For $m = 0$ we set $L_f = \pi$.

For $m = 1$ we set $L_f = \pi^n_{f(0)}$.

For the successor consider $f : stn(m+1) \to stn(n)$ and consider again the following square

$$
\begin{array}{ccc}
l^{-1}(m+1) & \xrightarrow{q(\pi,l^{-1}(1))} & l^{-1}(1) \\
p_{l^{-1}(m+1)}\downarrow & & \downarrow\ p_{l^{-1}(1)} \\
l^{-1}(m) & \xrightarrow{\pi} & l^{-1}(0)
\end{array}
\tag{5}
$$

We define L_f as the unique morphism such that:

$$
\begin{gathered}
L_f \circ p_{l^{-1}(m+1)} = L_{ii_0^{m,1} \circ f} \\
L_f \circ q(\pi, l^{-1}(1)) = L_{ii_1^{m,1} \circ f}
\end{gathered}
\tag{6}
$$

where, let us recall,

$$ii_0^{m,1} : stn(m) \to stn(m+1)$$

$$ii_1^{m,1} : stn(1) \to stn(m+1)$$

are the morphism that define the representation

$$stn(m+1) = stn(m) \amalg \{m+1\}$$

We can now complete the construction of the function

$$CsLw : Cs_{\mathbf{N}}(CC) \to Lw(CC^{op})$$

Construction 20 We need to construct a Lawvere theory structure on CC^{op}, i.e. a functor $L : F \to CC^{op}$ satisfying the conditions of Definition 2. We define the object part of L as l^{-1}. We define the morphism part of L as $L_{Mor}(f) = L_f$. For the proof that these two functions form a functor and that this functor is a Lawvere theory structure see "Lawvere theories and C-systems".

Lemma 21 *Let $G : (CC_1, cs_1) \to (CC_2, cs_2)$ be a homomorphism of C-systems. Then the functor $G^{op} : CC_1^{op} \to CC_2^{op}$ is a morphism of Lawvere theories $(CC_1^{op}, CsLw(cs_1)) \to (CC_2^{op}, CsLw(cs_2))$.*

For the proof see "Lawvere theories and C-systems".

Problem 22 *To construct a functor $CL : CS_{\mathbf{N}}(U) \to LW(U)$.*

Construction 23 Let (CC, cs) be a C-system. Then

$$CL_{Ob}(CC, cs) = (CC^{op}, CsLw(cs))$$

where $CsLw(cs)$ is defined by Construction 20.

The morphism component of CL takes a homomorphism

$$G : (CC_1, cs_1) \to (CC_2, cs_2)$$

to G^{op}. It is well defined by Lemma 21.

The identity and composition axioms are straightforward from the corresponding properties of functor composition and its compatibility with the function $G \mapsto G^{op}$.

Theorem 24 *For any universe U, Constructions 14 and 23 define mutually inverse isomorphisms between the categories of Lawvere theories in U and l-bijective C-systems in U.*

As a part of the proof of this theorem one proofs the following lemma.

Lemma 25 *For any category T the functions*

$$LwCs : Lw(T) \to Cs_{\mathbf{N}}(T^{op})$$

$$CsLw : Cs_{\mathbf{N}}(T^{op}) \to Lw(T)$$

are mutually inverse bijections.

For the complete proof see "Lawvere theories and C-systems".