Homotopy Type Theory MPIM-Bonn 2016

Dependent Type Theories

Lecture 5.
C-systems defined by universe categories.
C-systems and categories with families.

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In the previous lecture we arrived at the following theorem.

Let $\mathbf{RR}$ be a monad of expressions.

Let $B(\mathbf{RR}, \mathbf{RR})$ be the set whose elements are pairs $(n, (T_0, \ldots, T_{n-1}))$ where T_i is an expression with free variables from the set $\{x_0, \ldots, x_{i-1}\}$. Such a pair can also be written as

$$(T_0, \ldots, T_{n-1})\ Ok$$

Let $\widetilde{B}(\mathbf{RR}, \mathbf{RR})$ be the set of pairs $(n, (T_0, \ldots, T_n, r))$ where T_i is an expression with free variables from the set $\{x_0, \ldots, x_{i-1}\}$ and r is an expression with free variables from the set $\{x_0, \ldots, x_{n-1}\}$. Such a pair can also be written as

$$(T_0, \ldots, T_{n-1} \rhd r : T_n)$$

Theorem 1 *The mapping that we have constructed is a bijection between the C-subsystems of the C-system* $CC(\mathbf{RR})[\mathbf{RR}]$ *and pairs of subsets* $(B, \widetilde{B})$ *in the sets* $B(\mathbf{RR}, \mathbf{RR})$, $\widetilde{B}(\mathbf{RR}, \mathbf{RR})$ *that are closed under the action of the B-system operations* $ft, \partial, T, \widetilde{T}, S, \widetilde{S}, \delta$ *that we have described.*

This theorem generalizes to the case of a general module of expressions **LM** over the monad of expressions **RR**. Such pairs can be described by two sorted binding signatures where the sorts correspond to the type expressions and element expressions.

We have considered the case when no distinction on the level of the raw syntax is made between types and their elements - the case that is somewhat orthogonal to the case of simple type theories where the type expressions can not depend on variables and so can not contain element expressions as subexpressions.

Let me now remind a slide from the first lecture:

"In order to provide a mathematical representation (semantics) for a type theory one constructs two C-systems.

- One C-system, that we will call the term C-system of a type theory, is constructed from the formulas of type theory.

  To explain how to do it in sufficient generality and at the same time with mathematical rigor is the first and main goal of these lectures.

- The second C-system is constructed from the category of abstract mathematical objects.

  To explain how to do this construction is the second goal of the lectures."

In Theorem 1 and pertaining to it constructions we have accomplished the first of these goals. Now we move to the second one.

Definition 2 *Let $\mathcal{C}$ be a category. A universe structure on a morphism $p : \widetilde{U} \to U$ in $\mathcal{C}$ is a mapping that assigns to any morphism $f : X \to U$ in $\mathcal{C}$ a pull-back square*

$$\begin{array}{ccc} (X; f) & \xrightarrow{Q(f)} & \widetilde{U} \\ {\scriptstyle p_{X,f}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & U \end{array}$$

A universe in $\mathcal{C}$ is a morphism p together with a universe structure on it.

In what follows we may write $(X; f_1, \ldots, f_n)$ for $(\ldots((X; f_1); f_2) \ldots; f_n)$.

Definition 3 *A universe category is a triple* $(\mathcal{C}, p, pt)$ *where* $\mathcal{C}$ *is a category,* $p : \widetilde{U} \to U$ *is a morphism in* $\mathcal{C}$ *with a universe structure on it and* pt *is a final object in* $\mathcal{C}$.

We will often denote a universe category by a pair $(\mathcal{C}, p)$.

Let $(\mathcal{C}, p)$ be a universe category.

Define by induction on n pairs $(Ob_n(\mathcal{C}, p), int_n)$ where

$$Ob_n = Ob_n(\mathcal{C}, p)$$

are sets and

$$int_n : Ob_n \to Ob(\mathcal{C})$$

are functions, as follows:

1. $Ob_0 = unit$ where $unit$ is our distinguished one element set with the only element tt and $int_0(tt) = pt$.
2. $Ob_{n+1} = \amalg_{A \in Ob_n} Hom_{\mathcal{C}}(int_n(A), U)$ and

$$int_{n+1}(A, F) = (int_n(A); F).$$

We will often write int instead of int_n because n can usually be inferred.

Problem 4 *For a universe category* $(\mathcal{C}, p)$ *to define a C0-system* $CC(\mathcal{C}, p)$*.*

Construction 5 We set

$$Ob(CC(\mathcal{C}, p)) = \amalg_{n \geq 0} Ob_n(\mathcal{C}, p)$$

where $Ob_n = Ob_n(\mathcal{C}, p)$ are the sets introduced above. Let

$$int_{Ob} : Ob(CC(\mathcal{C}, p)) \to \mathcal{C}$$

be the sum of the functions int_n. Let

$$Mor(CC(\mathcal{C}, p)) = \amalg_{\Gamma, \Gamma' \in Ob(CC(\mathcal{C}, p))} Mor_{\mathcal{C}}(int_{Ob}(\Gamma), int_{Ob}(\Gamma'))$$

Define the function

$$int_{Mor} : Mor(CC(\mathcal{C}, p)) \to Mor(\mathcal{C})$$

by the formula

$$int_{Mor}((\Gamma, \Gamma'), a) = a$$

We will often write simply int for int_{Ob} and int_{Mor}.

The identity morphisms and the composition of morphisms are defined as in $\mathcal{C}$. The proofs of the axioms of a category are straightforward.

This completes the construction of a category $CC(\mathcal{C}, p)$.

The length function is defined by $l(n, A) = n$.

Define for each n the function

$$ft_{n+1} : Ob_{n+1} \to Ob_n$$

by the formula $ft_{n+1}(A, F) = A$ and define ft_0 as the identity function of Ob_0. The function $ft : Ob(CC) \to Ob(CC)$ is defined as the sum of functions ft_n.

For $(n + 1, B) = (n + 1, (ft(B), F))$ define

$$p_{(n+1,B)} : int(B) \to int(ft(B))$$

as $p_{int(ft(B)),F}$. For $(0, tt)$ define $p_{(0,tt)}$ as $Id_{(0,tt)}$.

For (m, A), $(n+1, B) = (n+1, (ft(B), F))$ and

$$f : (m, A) \to (n, ft(B))$$

define $f^*(n+1, B)$ as

$$f^*(n+1, B) = (m+1, (A, int(f) \circ F)) \tag{1}$$

To define $q(f, (n+1, B))$ recall that

$$\begin{array}{ccc} int(n+1, B) & \xrightarrow{Q(F)} & \widetilde{U} \\ p_{int(ft(B)),F}\downarrow & & \downarrow p \\ int(n, ft(B)) & \xrightarrow{F} & U \end{array}$$

is a pull-back square.

We define

$$q(f,(n+1,B)) : f^*(n+1,B) \to (n+1,B)$$

by the condition that the diagram

$$\begin{array}{ccccc}
int(A, int(f) \circ F) & \xrightarrow{int(q(f,(n+1,B)))} & int(ft(B), F) & \xrightarrow{Q(F)} & \widetilde{U} \\
{\scriptstyle p_{A,int(f)\circ F}}\downarrow & & \downarrow{\scriptstyle p_{ft(B),F}} & & \downarrow{\scriptstyle p} \\
int(A) & \xrightarrow{int(f)} & int(ft(B)) & \xrightarrow{F} & U
\end{array}$$

commutes and $int(q(f,(n+1,B))) \circ Q(F) = Q(int(f) \circ F)$.

Define pt as the unique element $(0, tt)$ of length zero.

For the proof of the fact that these data satisfies the C0-system axioms see "C-system defined by a universe category".

Lemma 6 *The pair of functions* (int_{Ob}, int_{Mor}) *is a fully faithful functor*

$$CC(\mathcal{C}, p) \to \mathcal{C}$$

Proof: Immediate from definitions.

Theorem 7 *The C0-system* $CC(\mathcal{C}, p)$ *is a C-system.*

Proof: By Proposition 3 of Lecture 1 it is sufficient to show that the canonical squares of $CC(\mathcal{C}, p)$ are pull-back squares.

For (m, A),

$$(n+1, B) = (n+1, (ft(B), F))$$

and $f : (m, A) \to (n, ft(B))$ we have the diagram

$$\begin{array}{ccccc}
int(A, int(f)\circ F) & \xrightarrow{int(q(f,(n+1,B)))} & int(ft(B), F) & \xrightarrow{Q(F)} & \widetilde{U} \\
p_{A,int(f)\circ F}\downarrow & & \downarrow p_{ft(B),F} & & \downarrow p \\
int(A) & \xrightarrow{int(f)} & int(ft(B)) & \xrightarrow{F} & U
\end{array}$$

where the left hand side square is the int of the canonical square for f and $(n+1, B)$.

The external square of this diagram is the square of the universe structure on p for $int(f)\circ F$ and in particular it is a pull-back square.

The right hand side square is pull-back as the square of the universe structure on p for F.

We conclude, by the standard lemma, that the left hand side square is pull-back.

Since int is fully faithful we conclude that the canonical square for f and $(n+1, B)$ is a pull-back square in $CC(\mathcal{C}, p)$.

This completes the proof of the lemma.

The construction that we have just described is fundamental to the theory of C-systems and to its use in the theory of type theories.

In particular, the C-system of the simplicial univalent model is obtained using this construction.

An important feature of this construction is that it transforms equivalences of universe categories into isomorphisms of C-systems.

Definition 8 *Let $(\mathcal{C}, p)$ and $(\mathcal{C}', p')$ be universe categories. A functor of universe categories from $(\mathcal{C}, p)$ to $(\mathcal{C}', p')$ is a triple $(\Phi, \phi, \widetilde{\phi})$ where $\Phi : \mathcal{C} \to \mathcal{C}'$ is a functor and $\phi : \Phi(U) \to U'$, $\widetilde{\phi} : \Phi(\widetilde{U}) \to \widetilde{U}'$ are morphisms such that:*

1. *Φ takes the canonical pull-back squares based on p to pull-back squares,*
2. *Φ takes pt to a final object of $\mathcal{C}'$,*
3. *the square*

$$\begin{array}{ccc} \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

   is a pull-back square.

Problem 9 *Let*

$$(\Phi, \phi, \widetilde{\phi}) : (\mathcal{C}, p, pt) \to (\mathcal{C}', p', pt')$$

be a functor of universes categories. To define a homomorphism $H = H(\Phi, \phi, \widetilde{\phi})$ *from* $CC(\mathcal{C}, p)$ *to* $CC(\mathcal{C}', p')$.

Construction 10 See “A C-system defined by a universe category”.

Construction 10 most likely extends to a functor from a naturally defined 2-category of universe categories to the 1-category of C-systems.

Lemma 11 *Let $(\Phi, \phi, \widetilde{\phi})$ be as in Problem 9 and let H be the solution given by Construction 10. Then one has:*

1. *If Φ is a faithful functor and ϕ is a monomorphism then H is an injection of C-systems.*
2. *If Φ is a fully faithful functor and ϕ is an isomorphism then H is an isomorphism.*

Proof: See “A C-system defined by a universe category”.

An important case of Lemma 11(2) is the identity functor Φ and identity morphisms ϕ and $\widetilde{\phi}$ as a universe category functor between universe categories $(\mathcal{C}, p)$ and $(\mathcal{C}, p)'$ with the same $\mathcal{C}$ and p but different choices of pull-back squares and final objects.

Our next result shows that any C-system is isomorphic to a C-system of the form $CC(\mathcal{C}, p)$.

Problem 12 *Let* CC *be a C-system. Construct a universe category* $(\mathcal{C}, p)$ *and an isomorphism* $CC \cong CC(\mathcal{C}, p)$.

For three different constructions to this problem see "A C-system defined by a universe category".

One can also use our main construction to provide, in a fully constructive fashion and without any use of the axiom of choice, for any category $\mathcal{C}$, with a given final object and fiber products, a C-system and an equivalence between the underlying category of this C-system and $\mathcal{C}$. More precisely one has:

Problem 13 *Let C be a precategory with a given final object pt and fiber products. To construct a C-system CC and an equivalence of categories $J_* : CC \to C$, $J^* : C \to CC$.*

Construction 14 *See "A C-system defined by a universe category".*

Finally let me show that universe categories can be considered as categories with families with a special additional structure.

First we need to give a definition of a category with families that uses familiar mathematical components. Such a definition was first devised, to the best of our knowledge, by Marcelo Fiore and explicitly appeared in his talk at ICALP 2012. We continue to call the object so defined "a category with families" because the definition is equivalent in a very strict sense to the original definition of Peter Dybjer.

For an object X of $\mathcal{C}$ and a presheaf F on $\mathcal{C}$ we let $v(X)$ denote the usual bijection

$$v(X) : F(X) \to Mor_{PreShv(C)}(Yo(X), F)$$

where Yo is the Yoneda embedding. We often abbreviate $v(X)$ to v.

Definition 15 *A category with families is a collection of data of the following form:*

1. *A category* $\mathcal{C}$,
2. *Two presheaves* Tm *and* Ty *on* $\mathcal{C}$ *and a morphism* $\pi : Tm \to Ty$,
3. *For any object* $X \in \mathcal{C}$ *and an element* $A \in Ty(X)$ *a collection of data of the form:*
   (a) *an object* $X.A$ *of* $\mathcal{C}$,
   (b) *an element* $Q(X, A)$ *of* $Tm(X.A)$,
   (c) *a morphism* $p_{X,A} : X.A \to X$,

These data should satisfy the following condition: for any object $X \in \mathcal{C}$ and an element $A \in Ty(X)$ the square in $PreShv(\mathcal{C})$

$$\begin{array}{ccc} Yo(X.A) & \xrightarrow{v(Q(X,A))} & Tm \\ {\scriptstyle Yo(p_{X,A})}\downarrow & & \downarrow{\scriptstyle \pi} \\ Yo(X) & \xrightarrow{v(A)} & Ty \end{array}$$

is a pull-back square.

So defined, categories with families are also closely related to the natural models of Steve Awodey (Awodey, 2014). The major difference is that in his definition Awodey requires the existence of the comprehension structure, i.e., of $X.A$, $p_{X,A}$ and $Q(X, A)$ satisfying the pull-back condition, while in a category with families we are given a particular choice of this structure.

This difference is substantial since different choices of the comprehension structure lead to non-isomorphic categories with families.

One can show that

Universe categories are almost the same as categories with families equipped with representations of the presheaves Ty and Tm.

"Almost the same" here means that one can construct an equivalence of the corresponding 1-categories. That is, when we go back and forth along the functors in opposite directions we obtain objects that are not only equivalent but isomorphic to each other.

The construction of a C-system from a universe category extends easily to a construction of a C-system from a category with families.

Given a category with families $(\mathcal{C}, \pi, c)$ where $c = (X.A, p_{X,A}, Q(X, A))$ is a comprehension structure one constructs a C-system $CC(\mathcal{C}, \pi, c)$ together with a fully faithful functor $CC(\mathcal{C}, \pi, c) \to \mathcal{C}$.

In the opposite direction, any C-system CC defines a category with families $(\mathcal{C}, \pi, c)(CC)$ with the same underlying category as CC.

These two constructions are mutually inverse for categories with families for which the functor $CC(\mathcal{C}, \pi, c) \to \mathcal{C}$ is a bijection on objects.

This allows one to consider C-systems as categories with families having a particular property. *However this property transports only along isomorphisms of categories with families but not along their equivalences.*

Another property of a category with families that is likely to be transportable along isomorphisms but not along equivalences is the property of being a free object generated by some system of operations corresponding to the inference rules of type theories.

This is the main reason why I consider the theory C-systems, which appear to be more difficult to work with than categories with families, to be an absolutely necessary part of the general theory of syntax and semantics of type theories.