The Artin-Hasse isomorphism of perfectoid open unit disks and a Fourier-type theory for continuous functions on $\mathbb{Q}_p$.

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Let

$$F(T) = \exp \sum_{i=0}^{\infty} T^{p^i}/p^i \in \mathbb{Z}_{(p)}[[T]]$$

be the Artin-Hasse series. In [1] we identified the (p,T)-adic completion $\mathscr{D}$ of the ring $\mathbb{Z}_{(p)}[T^{1/p^\infty}]$ with a topological Hopf algebra of $\mathbb{Z}_p$-valued measures on the uniformly open subsets of $\mathbb{Q}_p$, equipped with the topology of uniform convergence on the families of balls of equal radius, in such a way that, for any $i \in \mathbb{Z}$,

$$\lim_{n \to +\infty} F(T^{p^{i-n}})^{p^n} = \Delta_{p^i}$$

is the Dirac mass at p^i. So, for the inverse series E of F, let

$$\mu_{\mathrm{can}} := \lim_{n \to +\infty} E(\Delta_{p^{-n}} - \Delta_0)^{p^n}$$

be the $\mathbb{Z}_p$-valued measure on $\mathbb{Q}_p$ corresponding to T. We let $\mathbb{D} := \mathrm{Spa}\,(\mathscr{D}, \mathscr{D})$ be the "formal perfectoid" unit open disk over $\mathbb{Z}_p$ [6] and, for any perfectoid extension $K/\mathbb{Q}_p$, consider the isomorphism

$$\overline{F} : \mathbb{D}_{K^\flat}(0) \xrightarrow{\sim} \mathbb{D}_{K^\flat}(1)$$

of formal perfectoid unit open disks over $K^\flat$, centered at 0 (resp. at 1), obtained from F. We use Barsotti's algorithms [5] and the magic of the entire function $\Psi_p \in \mathbb{Z}[[x]] \cap \mathbb{Q}_p\{x\}$ defined by the functional equation [2]

$$x = \sum_{j=0}^{\infty} p^{-j} \Psi_p(p^j x)^{p^j}$$

which satisfies $\Psi_p(\mathbb{Q}_p) \subset \mathbb{Z}_p$, to describe the analytic properties of the untilted form

$$(\overline{F})^\sharp : \mathbb{D}_K(0) \xrightarrow{\sim} \mathbb{D}_K(1)$$

of $\overline{F}$.

Our main result is the construction of a topological basis $\{G_q\}_{q \in S}$, where $S = \mathbb{Z}[1/p] \cap \mathbb{R}_{\geq 0} = S' \dot{\cup} \{0\}$, of the Fréchet space of continuous functions $\mathbb{Q}_p \to \mathbb{Q}_p$, consisting of entire functions $G_q : \mathbb{C}_p \to \mathbb{C}_p$ defined over $\mathbb{Z}_{(p)}$ and taking integral values all over $\mathbb{Q}_p$. It satisfies $G_0(x) = 1$ and

$$G_q(x+y) = \sum_{q_1+q_2=q} G_{q_1}(x) G_{q_2}(y) \;\text{ and }\; G_{pq}(px) = G_q(x)\;, \;\; \forall q \in S\;.$$

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Then, for any $a \in \mathbb{Q}_p$, the Dirac mass Δ_a at a we have

$$\Delta_a = \sum_{q \in S} G_q(a) \mu^q_{\text{can}}$$

where the sum converges in $\mathscr{D}$, along the filter $\mathcal{F}(S)$ of cofinite subsets of S. It follows that any continuous function $f : \mathbb{Q}_p \to K$ admits the following generalized Amice-Fourier expansion

$$f(-) = \sum_{q \in S} (\int_{\mathbb{Q}_p} f \, \mu^q_{\text{can}}) G_q(-) \ ,$$

the convergence being uniform on compact subsets of $\mathbb{Q}_p$ along $\mathcal{F}(S)$. We point out that $a \mapsto \Delta_a$ is Colmez' "second" p-adic analog [3, V.1] of the complex-analytic function $x \mapsto e^{2i\pi x}$ (it is the one of [4, IV.1]).

As an application we show that, for any K-valued point $\chi : (S', +) \to (K^{\circ\circ}, \cdot)$ of $\mathbb{D}_K(0)$, the series of entire functions

$$1 + \sum_{q \in S'} \chi(q) \, G_q(x)$$

converges along $\mathcal{F}(S)$ uniformly on a neighborhood of 0 in $\mathbb{C}_p$ to the function $\exp \pi(\chi) x$, where

$$\pi(\chi) = \sum_{i \in \mathbb{Z}} \chi(p^i) p^{-i} \in K \ ,$$

and, moreover, converges uniformly on $\mathbb{Q}_p$ to a continuous character

$$\psi : (\mathbb{Q}_p, +) \to (1 + K^{\circ\circ}, \cdot) \ ,$$

which induces a K-rational point ψ of $\mathbb{D}_K(1)$. Then

$$(\overline{F})^\sharp(\chi) = \psi \ .$$

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