

THE CALCULUS OF FUNCTORS

&

THE DERIVATIVES OF THE IDENTITY

Plan

(I) Categorifying Calculus

(II) The identity functor

(III) Other versions of functor calculus.

(I) Categorifying Calculus

$f: A \rightarrow B$ real-valued function
(cts, differentiable, ...)

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

TAYLOR'S THM

$$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(0) x^n}{n!}$$

$$\longleftrightarrow f(x) = \lim_{n \rightarrow \infty} p_n(x)$$

where

$$p_n(x) = \sum_{i=0}^n \frac{f^{(i)}(0) x^i}{i!}$$

CONSTANT APPROXIMATIONS

$f: \mathbb{R} \rightarrow \mathbb{R}$ constant if $f(x) = a \quad \forall x$.

$\leadsto p_0(x) = f(a)$ where $a \in \mathbb{R}$ is the expansion point.

$F: \text{Top}_* \rightarrow \text{Top}_*$ CONSTANT if $F(x) \cong F(y)$

(weak)
homotopy equiv.

DEF

$$p_0 F(v) = F(*)$$

LINEAR FUNCTORS

* $f: \mathbb{R} \rightarrow \mathbb{R}$ **LINEAR** if
 $f(x+y) = f(x) + f(y) + f(0) = 0$.

Motivation

① $\tilde{H}_*: \text{Top}_* \longrightarrow \text{gAb}$ ← **graded Abelian groups.**

$$\tilde{H}_*(X \overset{+}{\vee} Y) \cong \tilde{H}_*(X) \oplus \tilde{H}_*(Y)$$

More generally, $\tilde{H}_*$ has a Mayer-Vietoris sequence.

$$\textcircled{1} \quad \tilde{H}_* : \text{Top}_* \longrightarrow \text{gAb}$$

More generally, $\tilde{H}_*$ has a Mayer-Vietoris sequence:

$$\cdots \longrightarrow \tilde{H}_*(U \cap V) \longrightarrow \tilde{H}_*(U) \oplus \tilde{H}_*(V) \longrightarrow \tilde{H}_*(U \cup V) \longrightarrow \cdots$$

* Can think of $\tilde{H}_*$ as a linear approximation to π_* :

$$\pi_*(X) \xrightarrow{h} \tilde{H}_*(X)$$

which, if X k -connected, is

iso.	$*$ $\leq k+1$
surj	$*$ $= k+2$.

② Stable homotopy: $\pi_*^s : \text{Top}_* \longrightarrow \text{gAb}$.

DEF A SPECTRUM E is a sequence of spaces $\{E_n\}_{n \in \mathbb{N}}$ together with maps $E_n \longrightarrow \Omega E_{n+1}$, $\forall n \in \mathbb{N}$.

Every space gives a spectrum and vice versa:

$$\Sigma^\infty : \text{Top}_* \rightleftarrows \text{Sp} : \Omega^\infty$$

↑ " $E \mapsto E_0$ "

DEF $\pi_*^s(X) = \pi_*(\Omega^\infty \Sigma^\infty X)$.

② Stable homotopy: $\pi_*^s : \text{Top}_* \longrightarrow \text{gAb.}$

* π_*^s generalised homology theory and has Mayer-Vietoris sequence.

* $\pi_*(X) \xrightarrow{s} \pi_*^s(X)$

X k -connected $\Rightarrow$ iso $*$ $\leq 2k$

surj $*$ $= 2k+1$.

Functor calculus is STABLE DATA
converging to UNSTABLE DATA.

REMARK Both $\tilde{H}_*(-)$ & $\pi_*^S(-)$ are obtained by applying $\pi_*(-)$ to functors $Top_* \longrightarrow Top_*$.

e.g. $\tilde{H}(X) \cong \pi_*(\Omega^\infty(H\mathbb{Z} \wedge X))$,

$$\pi_*^S(X) = \pi_*(\Omega^\infty \Sigma^\infty X)$$

DEF A functor F is ^{"linear"} 1-EXCISIVE if for any

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & \Gamma_h & \downarrow \\ X_2 & \longrightarrow & X_{12} \end{array}$$

the square

$$\begin{array}{ccc} F(X_0) & \longrightarrow & F(X_1) \\ \downarrow \quad h \lrcorner & & \downarrow \\ F(X_2) & \longrightarrow & F(X_{12}). \end{array}$$

* F 1-excursive $\Rightarrow \pi_* F$ has Mayer-Vietoris sequences.

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \downarrow & \searrow \pi_h & \downarrow \\ U & \longrightarrow & U \cup V \end{array}$$

Example

$$\tilde{H}_*(X) \cong \pi_* \Omega^\infty(H\mathbb{Z} \wedge X) :$$

$\Omega^\infty(H\mathbb{Z} \wedge -)$ is 1-excisive.

In fact, $\Omega^\infty(E \wedge -)$ is 1-excisive for any spectrum E .

LINEAR APPROXIMATION

$$\begin{array}{ccc}
 X & \longrightarrow & CX \\
 \downarrow & & \downarrow \\
 CX & \longrightarrow & \Sigma X
 \end{array}
 \xrightarrow{F}
 \begin{array}{ccc}
 F(X) & \longrightarrow & F(CX) \\
 \downarrow & & \downarrow \\
 F(CX) & \longrightarrow & F(\Sigma X)
 \end{array}$$

τ_h

* F 1-excisive :

$$F(X) \xrightarrow{\sim} F(CX) \underset{F(\Sigma X)}{\overset{h}{\times}} F(CX) =: T_1 F(X)$$

IDEA

$T_1 F$ is closer to linear than F .

$T_1(T_1 F)$ is closer to linear than $T_1 F$.

$\vdots$

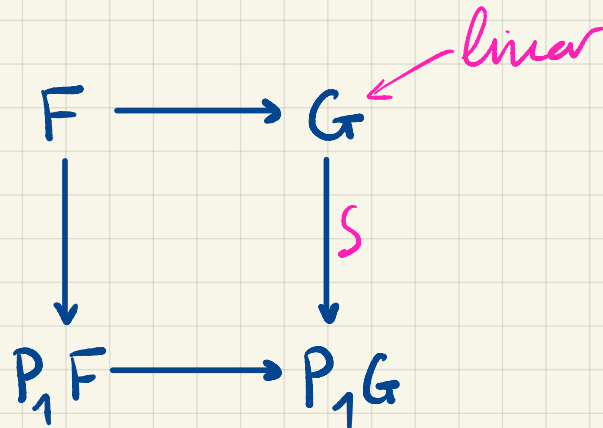
DEF

$P_1 F := \text{hocolim}(F \rightarrow T_1 F \rightarrow T_1^2 F \rightarrow \dots)$

is the LINEAR APPROXIMATION of F .

THEOREM [Goodwillie]

- * $P_1 F$ is 1-excise.
- * F 1-excise $\Rightarrow F \simeq P_1 F$
- * $P_1 F$ is UNIVERSAL :



EXAMPLE $\text{Id}: \text{Top}_* \longrightarrow \text{Top}_*$

* is NOT LINEAR.

(every pushout is not a pullback.)

* has an interesting calculus.

HIGHER POLYNOMIALS

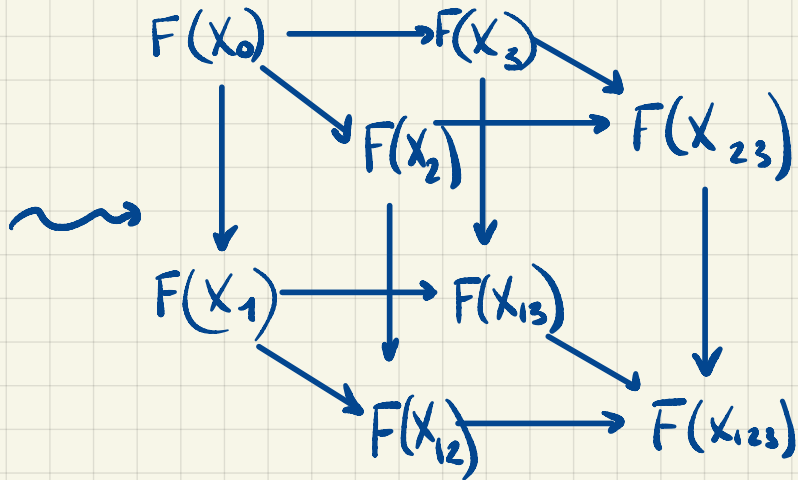
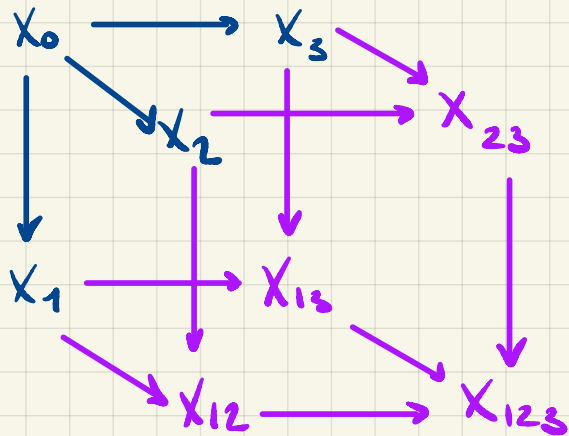
* $f: \mathbb{R} \rightarrow \mathbb{R}$ QUADRATIC if

$$f(x+y+z) - f(x+y) - \dots = 0.$$

DEF F is n -EXCISIVE if for any strongly homotopy cocartesian $(n+1)$ -cube $\mathcal{X}$, $F(\mathcal{X})$ is homotopy cartesian.

→ "polynomial of degree $\leq n$ "

A picture for $n=2$.

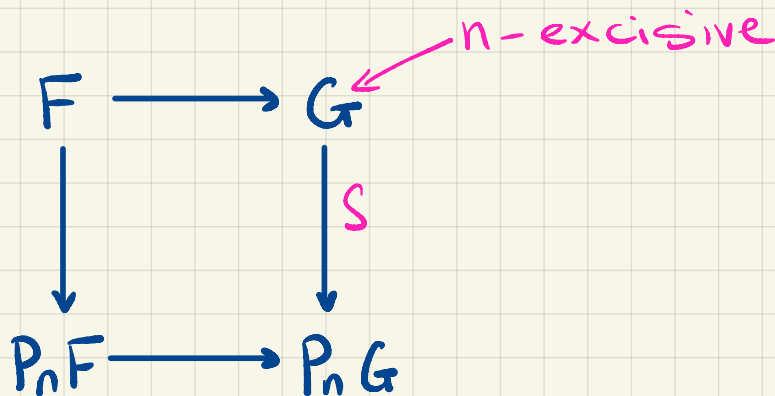


HIGHER APPROXIMATIONS

- * Can play the same game with n -excisive functors and define $T_n F$ and $P_n F$ for all $n \in \mathbb{N}$.

THEOREM [Goodwillie]

- * $P_n F$ is n -excisive.
- * F n -excisive $\Rightarrow F \simeq P_n F$
- * $P_n F$ is UNIVERSAL :



"ERROR" TERMS

* Classically,
$$p_n(x) - p_{n-1}(x) = \frac{f^{(n)}(0) x^n}{n!}$$

* Homotopically,

$$D_n F \longrightarrow P_n F \longrightarrow P_{n-1} F.$$

↑ homotopy fibre

A FORMULA FOR "ERRORS"

$$n=1 \quad D_1 F \longrightarrow P_1 F \longrightarrow F(*).$$

$\parallel$
 $P_0 F(X)$

THEOREM [Goodwillie]

$F: \text{Top}_* \longrightarrow \text{Top}_*$ 1-excursive, reduced and finitary. Then

$$F(X) \cong \Omega^\infty(E \wedge X)$$

$\nwarrow$ Spectrum
 $\uparrow$ 1st derivative.

looks like: $f(x) = cx$ for
 c a constant

THEOREM [Goodwillie]

$F: \text{Top}_* \longrightarrow \text{Top}_*$ 1-excisive, reduced and finitary. Then

$$F(X) \simeq \Omega^\infty(E \wedge X)$$

Rmk

* F Linear $\Rightarrow \pi_* F$ homology theory
& the classification theorem is a version of Brown representability.

THEOREM [Goodwillie]

F is n -excisive, n -reduced and finitary.
Then,

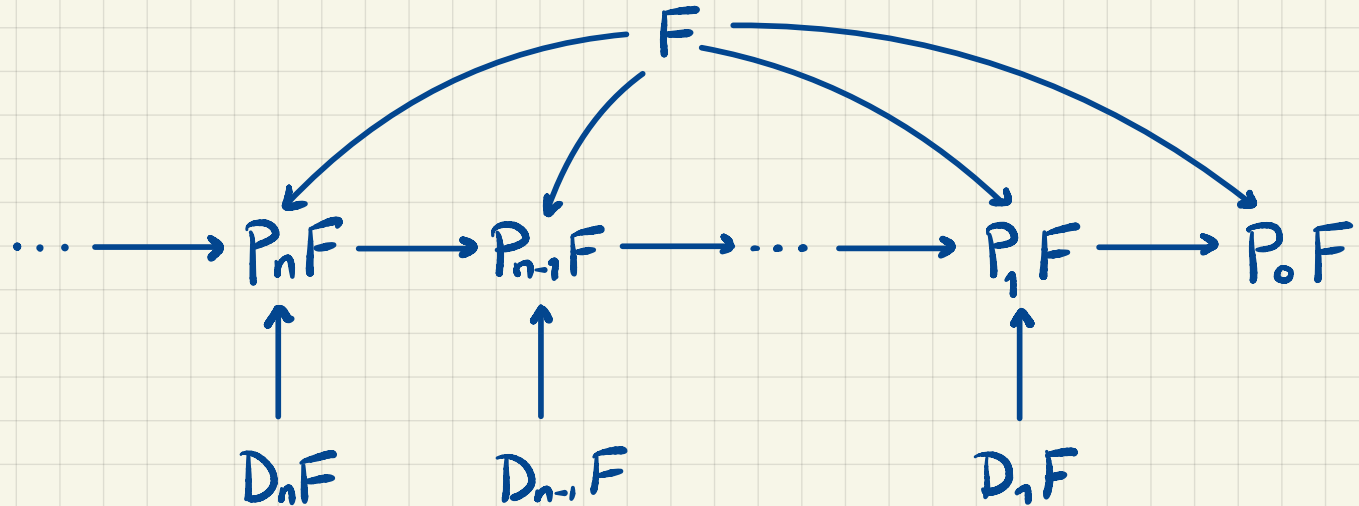
$$F(X) \simeq \Omega^\infty \left(\partial_n F \wedge X^{\wedge n} \right)_{h\mathbb{Z}_n}$$

$\uparrow$ n -th derivative.

looks like: $\frac{f^{(n)}(0) x^n}{n!}$

E.g. $D_n F$ is n -excisive, n -reduced and finitary.

TAYLOR SERIES $\rightsquigarrow$ TAYLOR TOWER



HOWEVER, $\lim_n P_n F \neq F.$

(II) The identity functor

THE LINEAR APPROXIMATION

$$* \quad T_1 \text{Id}(X) = \underset{\Sigma X}{CX} \overset{h}{X} \underset{\Sigma X}{CX} \simeq * \underset{\Sigma X}{X}^h * \simeq \Omega \Sigma X$$

$$\begin{aligned} * \quad P_1 \text{Id}(X) &= \text{hocolim} (X \longrightarrow \Omega \Sigma X \longrightarrow \Omega^2 \Sigma^2 X \longrightarrow) \\ &\simeq \Omega^\infty \Sigma^\infty X \\ &=: QX. \end{aligned}$$

$$* \quad \pi_* (\text{Id}(X) \longrightarrow P_1 \text{Id}(X)) \text{ is } \pi_*(X) \xrightarrow{S} \pi_*^S(X).$$

* In fact, $\text{Id}: \text{Top}_* \longrightarrow \text{Top}_*$ is not n -excisive for any n .

* X 1-connected $\Rightarrow X \cong \lim_n P_n \text{Id}(X)$.

* $P_n \text{Id}(X)$ is very complicated.

THEOREM [Arone-Kankaanrinta]

$$P_n \text{Id}(X) \cong_{\text{Tot}} \left(P_n QX \begin{matrix} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{matrix} P_n Q^2 X \begin{matrix} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{matrix} \dots \right)$$

$$\overset{\text{IS}}{\prod_{m=1}^n} D_m Q^2 X$$

THE DERIVATIVES

Recall: $D_n F(X) \approx \Omega^\infty (\partial_n F \wedge X^{\wedge n})_{h\mathbb{Z}_n}$.

THEOREM [Johnson, Arone-Mahowald]

There is a finite CW-complex K_n , such that

$$\partial_n \text{Id} \approx \text{Map}_*(K_n, \mathbb{S})$$

$$\Rightarrow D_n \text{Id}(X) \approx \Omega^\infty \text{Map}_*(K_n, \Sigma^\infty X^{\wedge n})_{h\mathbb{Z}_n}$$

$\text{Rnk } K_n \approx \sum_{(n-i)!} S^{n-1}.$

If $X = S^{2n-1}$ ($n \in \mathbb{N}$):

THEOREM [Serre]

$X \longrightarrow \Omega^\infty \Sigma^\infty X$ is a rational equivalence.

THEOREM [Arone-Mahowald]

The tower at X is rationally constant.

THEOREM [Ching]

The (symmetric) sequence

$$\partial_*(Id) = \{\partial_n(Id)\}_{n \in \mathbb{N}}$$

$\partial_n Id \downarrow \Sigma_n$

forms an operad in Spectra.

In particular, $\partial_*(Id) \simeq \text{Comm}^\vee$.

* Chain rule: $\partial_*(G \circ F) \simeq \partial_*(G) \circ_{\partial_*(Id)} \partial_*(F)$

(III) Other versions of functor calculus.

① Orthogonal calculus [Weiss]

$$\text{Vect}_{\mathbb{R}} \longrightarrow \text{Top}_*$$

② Manifold/Embedding calculus: [Goodwillie-Weiss]

$$M \text{ a manifold, } \mathcal{O}(M)^{\text{op}} \longrightarrow \text{Top}_*$$

③ Abstract homotopy theories: [Kuhn, Lurie]

$$\mathcal{L} \longrightarrow \mathcal{D}$$

with $\mathcal{L}, \mathcal{D}$ model categories / ω -categories.

④ Equivariant: [Dotto]

$$\mathrm{Top}_*^G \longrightarrow \mathrm{Top}_*^G$$

⑤ Algebraic versions: [Johnson-McCarthy,
Bauer-Johnson-McCarthy]

$$A \longrightarrow B, \quad A, B \text{ abelian.}$$