The Calculus of Functors

&

The Derivatives of the Identity

Plan

(I) Categorifying Calculus

(II) The identity functor

(III) Other versions of functor calculus.

(I) Categorifying Calculus

$f: A \to B$ real-valued function

(cts, differentiable, ...)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

TAYLOR'S THM

$$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(0)\, x^n}{n!} \quad \longleftrightarrow \quad f(x) = \lim_{n \to \infty} p_n(x)$$

where

$$p_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)\, x^i}{i!}$$

CONSTANT APPROXIMATIONS

$f : \mathbb{R} \to \mathbb{R}$ constant if $f(x) = a \quad \forall x$.

$\rightsquigarrow \quad p_0(x) = f(a)$ where $a \in \mathbb{R}$ is the expansion point.

(weak) homotopy equiv.

$F : Top_* \longrightarrow Top_*$ CONSTANT if $F(X) \simeq F(Y)$

DEF $\quad P_0 F(V) = F(*)$

LINEAR FUNCTORS

* $f : \mathbb{R} \to \mathbb{R}$ LINEAR if

$$f(x+y) - f(x) - f(y) + f(0) = 0.$$

Motivation

① $\tilde{H}_* : \mathrm{Top}_* \longrightarrow {}_g\mathrm{Ab}$ ← graded Abelian groups.

"+"

$$\tilde{H}_*(X \vee Y) \cong \tilde{H}_*(X) \oplus \tilde{H}_*(Y)$$

More generally, $\tilde{H}_*$ has a Mayer-Vietoris sequence.

① $\tilde{H}_* : \mathrm{Top}_* \longrightarrow {}_g\mathrm{Ab}$

More generally, $\tilde{H}_*$ has a Mayer-Vietoris sequence:

$$\cdots \longrightarrow \tilde{H}_*(U \cap V) \longrightarrow \tilde{H}_*(U) \oplus \tilde{H}_*(V) \longrightarrow \tilde{H}_*(U \cup V) \longrightarrow \cdots$$

* Can think of $\tilde{H}_*$ as a linear approximation to π_*:

$$\pi_*(X) \xrightarrow{h} \tilde{H}_*(X)$$

which, if X k-connected, is iso. $* \leq k+1$

surj $* = k+2$.

② Stable homotopy: $\pi_*^s : Top_* \longrightarrow gAb.$

DEF A SPECTRUM E is a sequence of spaces $\{E_n\}_{n\in\mathbb{N}}$ together with maps

$$E_n \longrightarrow \Omega E_{n+1}, \quad \forall n\in\mathbb{N}.$$

Every space gives a spectrum and vice versa:

$$\Sigma^\infty : Top_* \rightleftarrows Sp : \Omega^\infty$$

"$E \longmapsto E_0$"

DEF $\pi_*^s(X) = \pi_*(\Omega^\infty \Sigma^\infty X).$

② Stable homotopy: $\boxed{\pi_*^s : Top_* \longrightarrow gAb.}$

* π_*^s generalised homology theory and has Mayer-Vietoris sequence.

* $\pi_*(X) \xrightarrow{s} \pi_*^s(X)$

X k-connected $\Rightarrow$ iso $* \leq 2k$

surj $* = 2k+1$.

$\boxed{\text{Functor calculus is STABLE DATA converging to UNSTABLE DATA.}}$

REMARK Both $\tilde{H}_*(-)$ & $\pi_*^s(-)$ are obtained by applying $\pi_*(-)$ to functors $\mathrm{Top}_* \longrightarrow \mathrm{Top}_*$.

e.g. $$\tilde{H}(X) \cong \pi_*(\Omega^\infty(H\mathbb{Z} \wedge X)),$$

$$\pi_*^s(X) = \pi_*(\Omega^\infty \Sigma^\infty X)$$

DEF A functor F is "linear" **1-EXCISIVE** if for any

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] \arrow[ur, phantom, "\ulcorner h" near start, pos=0.9] & X_{12} \end{tikzcd}$$

the square

$$\begin{tikzcd} F(X_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "h\lrcorner" near start] & F(X_1) \arrow[d] \\ F(X_2) \arrow[r] & F(X_{12}) \end{tikzcd}$$

.

* F 1-excisive $\Rightarrow \pi_* F$ has Mayer-Vietoris sequences.

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \cup V \end{tikzcd}$$

π_n

Example

$$\tilde{H}_*(X) \cong \pi_* \Omega^\infty(H\mathbb{Z} \wedge X):$$

$\Omega^\infty(H\mathbb{Z} \wedge -)$ is 1-excisive.

In fact, $\Omega^\infty(E \wedge -)$ is 1-excisive for any spectrum E.

Linear Approximation

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] \arrow[ur, phantom, "\ulcorner_h"] & \Sigma X \end{tikzcd} \xrightarrow{\quad F \quad} \begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(CX) \arrow[d] \\ F(CX) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

* F 1-excisive :

$$F(X) \xrightarrow{\sim} F(CX) \times^h_{F(\Sigma X)} F(CX) =: T_1 F(X)$$

IDEA T_1F is closer to linear than F.

$T_1(T_1F)$ is closer to linear than T_1F.

$\vdots$

DEF $P_1F := \operatorname{hocolim}(F \longrightarrow T_1F \longrightarrow T_1^2F \to \cdots)$

is the LINEAR APPROXIMATION of F.

THEOREM [Goodwillie]

* $P_1 F$ is 1-excisive.
* F 1-excisive $\Rightarrow F \simeq P_1 F$
* $P_1 F$ is UNIVERSAL :

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d, "\wr"] \\ P_1 F \arrow[r] & P_1 G \end{tikzcd}$$

linear

EXAMPLE $\mathrm{Id} : \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$

* is NOT LINEAR.

(every pushout is not a pullback.)

* has an interesting calculus.

HIGHER POLYNOMIALS

* $f : \mathbb{R} \to \mathbb{R}$ QUADRATIC if

$$f(x+y+z) - f(x+y) - \cdots = 0.$$

"polynomial of degree $\leq n$"

DEF F is n-EXCISIVE if for any strongly homotopy cocartesian $(n+1)$-cube $\mathcal{X}$, $F(\mathcal{X})$ is homotopy cartesian.

A picture for $n=2$.

$$\begin{tikzcd}
X_0 \arrow[rr] \arrow[dr] \arrow[dd] & & X_3 \arrow[dr] \arrow[dd] & \\
& X_2 \arrow[rr] \arrow[dd] & & X_{23} \arrow[dd] \\
X_1 \arrow[rr] \arrow[dr] & & X_{13} \arrow[dr] & \\
& X_{12} \arrow[rr] & & X_{123}
\end{tikzcd}$$

$$\rightsquigarrow$$

$$\begin{tikzcd}
F(X_0) \arrow[rr] \arrow[dr] \arrow[dd] & & F(X_3) \arrow[dr] \arrow[dd] & \\
& F(X_2) \arrow[rr] \arrow[dd] & & F(X_{23}) \arrow[dd] \\
F(X_1) \arrow[rr] \arrow[dr] & & F(X_{13}) \arrow[dr] & \\
& F(X_{12}) \arrow[rr] & & F(X_{123})
\end{tikzcd}$$

HIGHER APPROXIMATIONS

* Can play the same game with n-excisive functors and define $T_n F$ and $P_n F$ for all $n \in \mathbb{N}$.

THEOREM [Goodwillie]

* $P_n F$ is n-excisive.
* F n-excisive $\Rightarrow$ $F \simeq P_n F$
* $P_n F$ is UNIVERSAL :

n-excisive

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & G \arrow[d, "\simeq"] \\
P_n F \arrow[r] & P_n G
\end{tikzcd}$$

"Error" Terms

* Classically,

$$P_n(x) - P_{n-1}(x) = \frac{f^{(n)}(0)\, x^n}{n!}$$

* Homotopically,

$$D_nF \longrightarrow P_nF \longrightarrow P_{n-1}F.$$

homotopy fibre

A Formula for "Errors"

n=1 $\quad D_1F \longrightarrow P_1F \longrightarrow F(*).$

$$\underset{P_0F(X)}{\overset{\shortparallel}{F(*)}}$$

THEOREM [Goodwillie]

$F : Top_* \longrightarrow Top_*$ 1-excisive, reduced and finitary. Then

Spectrum

$$F(X) \simeq \Omega^\infty(E \wedge X)$$

1st derivative.

looks like : $f(x) = cx$ for c a constant

THEOREM [Goodwillie]

$F : Top_* \longrightarrow Top_*$ 1-excisive, reduced and finitary. Then

$$F(X) \simeq \Omega^\infty(E \wedge X)$$

Rmk

* F Linear $\Rightarrow \pi_* F$ homology theory & the classification theorem is a version of Brown representability.

THEOREM [Goodwillie]

$P_{n-1}F \simeq *$

F is n-excisive, n-reduced and finitary.

Then,

$$F(X) \simeq \Omega^\infty\left(\partial_n F \wedge X^{\wedge n}\right)_{h\Sigma_n}$$

$\partial_n F$: n-th derivative.

looks like: $\dfrac{f^{(n)}(0)\,x^n}{n!}$

E.g. $D_n F$ is n-excisive, n-reduced and finitary.

Taylor Series ⇝ Taylor Tower

$$
\begin{tikzcd}
 & & & F \arrow[dll, bend right] \arrow[dl, bend right] \arrow[dr, bend left] \arrow[drr, bend left] & & \\
\cdots \arrow[r] & P_n F \arrow[r] & P_{n-1} F \arrow[r] & \cdots \arrow[r] & P_1 F \arrow[r] & P_0 F \\
 & D_n F \arrow[u] & D_{n-1} F \arrow[u] & & D_1 F \arrow[u] &
\end{tikzcd}
$$

However, $\lim_n P_n F \not\simeq F$.

(II) The identity functor

The Linear Approximation

* $T_1 \mathrm{Id}(X) = CX \underset{\Sigma X}{\overset{h}{\times}} CX \simeq * \underset{\Sigma X}{\overset{h}{\times}} * \simeq \Omega \Sigma X$

* $P_1 \mathrm{Id}(X) = \operatorname{hocolim}\left(X \longrightarrow \Omega\Sigma X \longrightarrow \Omega^2 \Sigma^2 X \rightarrow \right)$

$$\simeq \Omega^\infty \Sigma^\infty X$$
$$=: QX.$$

* $\pi_*\left(\mathrm{Id}(X) \longrightarrow P_1 \mathrm{Id}(X) \right)$ is $\pi_*(X) \xrightarrow{S} \pi_*^s(X)$.

* In fact, $\mathrm{Id} : \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$ is not n-excisive for any n.

* X 1-connected $\Longrightarrow$ $X \simeq \lim\limits_{n} P_n \mathrm{Id}(X)$.

* $P_n \mathrm{Id}(X)$ is very complicated.

THEOREM [Arone-Kankaanrinta]

$$P_n \mathrm{Id}(X) \simeq \mathrm{Tot}\left(P_n Q X \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}} P_n Q^2 X \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}} \cdots \right)$$

$$P_n Q^2 X \simeq \prod_{m=1}^{n} D_m Q^2 X$$

THE DERIVATIVES

Recall: $D_n F(X) \simeq \Omega^\infty (\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$.

THEOREM [Johnson, Arone-Mahowald]

There is a finite CW-complex K_n, such that

$$\partial_n \mathrm{Id} \simeq \mathrm{Map}_*(K_n, \mathbb{S})$$

$$\Rightarrow D_n \mathrm{Id}(X) \simeq \Omega^\infty \mathrm{Map}_*(K_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$$

Rmk $K_n \simeq \bigvee_{(n-1)!} S^{n-1}$.

If $X = S^{2n-1}$ $(n \in \mathbb{N})$:

THEOREM [Serre]

$X \longrightarrow \Omega^\infty \Sigma^\infty X$ is a rational equivalence.

THEOREM [Arone-Mahowald]

The tower at X is rationally constant.

THEOREM [Ching]

The (symmetric) sequence

$$\partial_*(\mathrm{Id}) = \{\partial_n(\mathrm{Id})\}_{n\in\mathbb{N}} \qquad \partial_n \mathrm{Id} \circlearrowleft \Sigma_n$$

forms an operad in Spectra.

In particular, $\partial_*(\mathrm{Id}) \simeq \mathrm{Comm}^\vee$.

* Chain rule: $\partial_*(G\circ F) \simeq \partial_*(G) \circ_{\partial_*(\mathrm{Id})} \partial_*(F)$

(III) Other versions of functor calculus.

1. Orthogonal calculus [Weiss]

$$\mathrm{Vect}_{\mathbb{R}} \longrightarrow \mathrm{Top}_*$$

2. Manifold/Embedding calculus: [Goodwillie-Weiss]

M a manifold, $\quad \mathcal{O}(M)^{op} \longrightarrow \mathrm{Top}_*$

3. Abstract homotopy theories: [Kuhn, Lurie]

$$\mathcal{C} \longrightarrow \mathcal{D}$$

with $\mathcal{C}, \mathcal{D}$ model categories/ ∞-categories.

④ Equivariant: [Dotto]

$$\mathrm{Top}_*^G \longrightarrow \mathrm{Top}_*^G$$

⑤ Algebraic versions: [Johnson-McCarthy, Bauer-Johnson-McCarthy]

$$\mathcal{A} \longrightarrow \mathcal{B} \quad , \quad \mathcal{A}, \mathcal{B} \text{ abelian.}$$