GAUGE-THEORETIC OBSTRUCTIONS TO BOUNDING DEFINITE 4-MANIFOLDS

Notation: X = a 4-mfd
Y = a 3-mfd

① INTERSECTION FORM

$$Q_X : H_2(X)/_{\text{Tors}} \times H_2(X)/_{\text{Tors}} \longrightarrow \mathbb{Z}$$
$$(\alpha, \beta) \longmapsto \langle PD(\alpha) \cup PD(\beta), [X, \partial X] \rangle$$

Geometrically: if X smooth, $\alpha = [\Sigma_1]$, $\beta = [\Sigma_2]$, transverse,
$$Q_X(\alpha,\beta) = \#(\Sigma_1 \cap \Sigma_2)$$
signed count

EX 1: $X = \mathbb{CP}^2$, $H_2(X) \cong \mathbb{Z}^1$, gen'd by $\alpha = [\mathbb{CP}^1]$.
$$Q_X : \mathbb{Z}^1 \times \mathbb{Z}^1 \longrightarrow \mathbb{Z}$$
$$Q_X(\alpha,\alpha) = \#(\mathbb{CP}^1 \cap \mathbb{CP}^1) = 1 \rightsquigarrow \text{matrix } (1)$$
More generally, if $X = \#^n \mathbb{CP}^2$, then $H_2(X) \simeq \mathbb{Z}^n$ and Q_X is represented by the identity matrix $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$.

EX 2: $X = S^2 \times S^2$, $H_2(X) = \mathbb{Z}^2$
Q_X is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (Exercise)

RK: $X \xrightarrow[\text{orientation}]{\text{change}} \overline{X}$
Then $H_2(\overline{X}) \cong H_2(X)$ and $Q_{\overline{X}} = -Q_X$.
e.g. $Q_{\overline{\mathbb{CP}^2}}$ is represented by (-1).

Today's goal: the Poincaré sphere P does not bound a smooth 4-mfd with negative definite intersection form.

② LATTICES

Def: LATTICE $= (L, Q)$ where
L = fin. gen. free Abelian grp $(\cong \mathbb{Z}^n)$
Q = integral symm. bilin. form $L \times L \longrightarrow \mathbb{Z}$

Notation: $[Q]$ = repr. matrix for Q
Prototype: $(H_2(X)/_{\text{Tors}}, Q_X)$

EX 1: $\underline{\mathbb{Z}}^n = (\mathbb{Z}^n, \langle \cdot, \cdot \rangle_{std})$, repr. by $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

Positive diagonal lattice
Intersection lattice of $\#^n \mathbb{CP}^2$

EX 2: $H = (\mathbb{Z}^2, Q)$, w/ $[Q] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$Q(e_1, e_1) = 0 = Q(e_2, e_2)$
$Q(e_1, e_2) = 1$

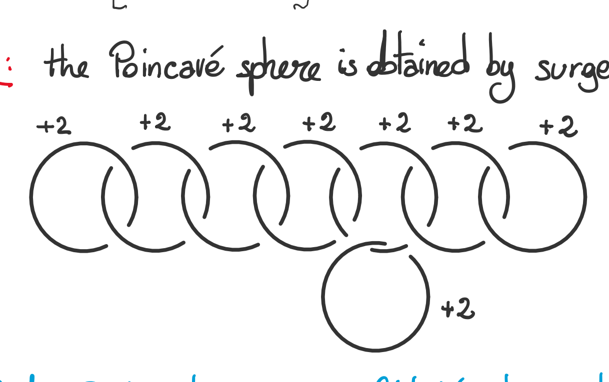
Hyperbolic lattice
Intersection lattice of $S^2 \times S^2$

RK: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ over $\mathbb{R}$ coefficients
but NOT over $\mathbb{Z}$ coefficients
$\Rightarrow H \not\cong \underline{\mathbb{Z}}^1 \oplus (-\underline{\mathbb{Z}}^1)$

EX 3: $E_8 = (\mathbb{Z}^8, Q_{E8})$ E_8 lattice

$$[Q_{E8}] = \begin{bmatrix} 2&1&0&0&0&0&0&0 \\ 1&2&1&0&0&0&0&0 \\ 0&1&2&1&0&0&0&0 \\ 0&0&1&2&1&0&0&0 \\ 0&0&0&1&2&1&0&1 \\ 0&0&0&0&1&2&1&0 \\ 0&0&0&0&0&1&2&1 \\ 0&0&0&0&1&0&1&2 \end{bmatrix} =: E_8$$

Last talk: the Poincaré sphere is obtained by surgery as:



Important fact: P bounds a 4-manifold X whose intersection form can be read off the figure $\rightsquigarrow Q_X = Q_{E_8}$.
$X :=$ the E_8 PLUMBING

More definitions on lattices

* UNIMODULAR if $\det[Q] = \pm 1$ (e.g. $\underline{\mathbb{Z}}^n, H, E_8$)
* $\xi \in L^*$ is CHARACTERISTIC if $\forall v \in L$
$$\xi(v) \equiv Q(v,v) \pmod 2$$
* EVEN if $0 \in L^*$ is characteristic, i.e. $\forall v \in L$
$$Q(v,v) \equiv 0 \pmod 2$$
(e.g. H, E_8)
* ODD if not even (e.g. $\underline{\mathbb{Z}}^n$)

$$\underset{H}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \qquad \underset{E_8}{\begin{pmatrix} 2&1&0&0&0&0&0&0 \\ 1&2&1&0&0&0&0&0 \\ 0&1&2&1&0&0&0&0 \\ 0&0&1&2&1&0&0&0 \\ 0&0&0&1&2&1&0&1 \\ 0&0&0&0&1&2&1&0 \\ 0&0&0&0&0&1&2&1 \\ 0&0&0&0&1&0&1&2 \end{pmatrix}} \qquad \underset{\underline{\mathbb{Z}}^n}{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}$$

* SIGNATURE ($\otimes \mathbb{R}$ + diagonalise)
e.g. $\sigma(\underline{\mathbb{Z}}^n) = n$, $\sigma(H) = 0$, $\sigma(E_8) = 8$
($\underline{\mathbb{Z}}^n$, E_8 are positive definite, H is indefinite)

Exercise: Pos. def. lattices $\longleftrightarrow$ Discrete subgroups of $(\mathbb{R}^n, \langle \cdot,\cdot \rangle_{std})$ on which $\langle \cdot,\cdot \rangle_{std}$ is integral
$$L \rightsquigarrow L \otimes_{\mathbb{Z}} \mathbb{R}$$

③ LATTICE OBSTRUCTIONS

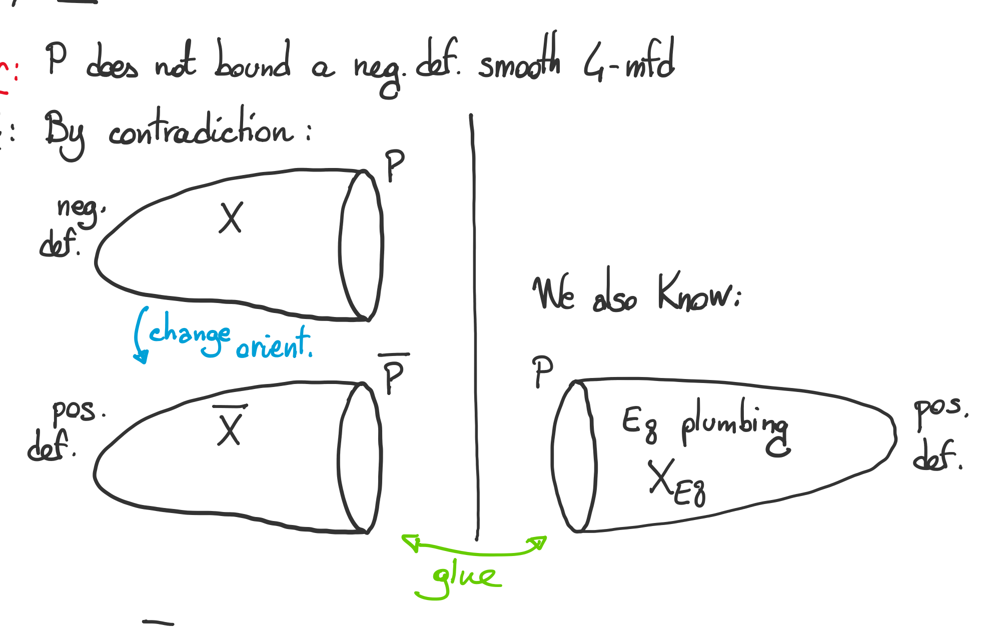
Thm: X^4 closed $\Rightarrow$ Q_X unimodular

Thm (Freedman): $\forall$ integral unimodular Q
$\exists$ topological closed 4-mfd X w/ $Q_X = Q$

e.g. $\underline{\mathbb{Z}}^n \rightsquigarrow X = \#^n \mathbb{CP}^2$
$H \rightsquigarrow X = S^2 \times S^2$
$E_8 \rightsquigarrow \exists$ topological 4-mfd M w/ $Q_M = E_8$.
However, M is NOT smooth.

Thm (Donaldson) X closed, smooth, pos. definite $\Rightarrow Q_X = \underline{\mathbb{Z}}^n$.
$E_8 \neq \underline{\mathbb{Z}}^n \Rightarrow M$ is not smooth.

Cor: P does not bound a neg. def. smooth 4-mfd
Pf: By contradiction:



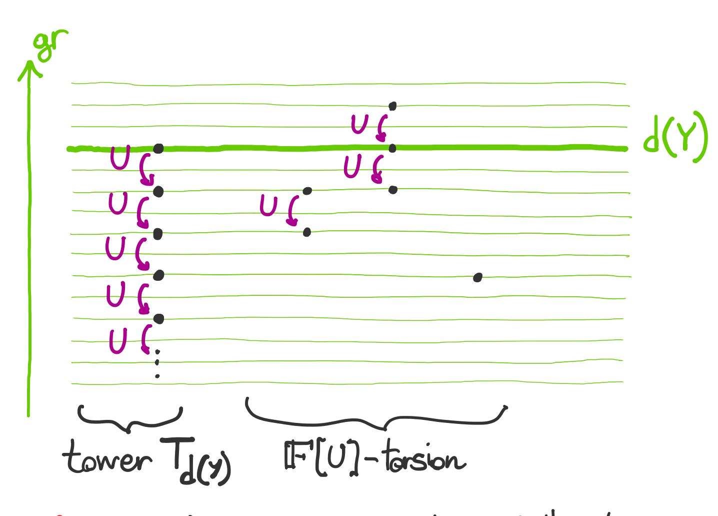
$$Z = \overline{X} \cup X_{E_8} \Rightarrow Q_Z = Q_{\overline{X}} \oplus E_8$$
$$\| \text{(Donaldson)}$$
$$\underline{\mathbb{Z}}^n$$

Contradiction: E_8 is not a summand of $\underline{\mathbb{Z}}^n$ (exercise) □

④ A MODERN APPROACH: HEEGAARD FLOER HOMOLOGY (Ozsváth-Szabó)

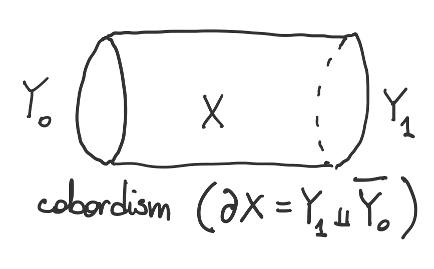
Y closed 3-mfd $\rightsquigarrow HF^-(Y)$
an $\mathbb{F}[U]$-module, where $\mathbb{F} = \mathbb{F}_2$

Property: if $H_*(Y) \cong H_*(S^3)$ (e.g. $Y = P$), then the rank of $HF^-(Y)$ over $\mathbb{F}[U]$ is 1.



* each dot is a generator / $\mathbb{F}$
* $\exists$ grading gr, with $gr(U) = -2$

Def: $d(Y) :=$ the top grading of the tower
FACT: $d(\overline{Y}) = -d(Y)$

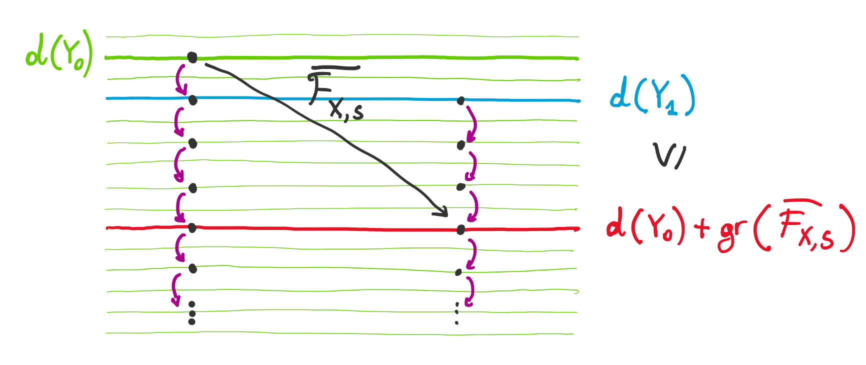


cobordism $(\partial X = Y_1 \sqcup \overline{Y_0})$
$+$ $\rightsquigarrow F_{X,s} : HF^-(Y_0) \longrightarrow HF^-(Y_1)$
$s \in Spin^c X$
defines $c_1(s) \in H^2(X)$, its "first Chern class"
$c_1(s)$ is characteristic for Q_X

Properties:
* $F_{X,s}$ is U-equivariant
* $gr(F_{X,s}) = \dfrac{Q_X(c_1(s), c_1(s)) - 2\chi(X) - 3\sigma(X)}{4}$

Euler characteristic; sign (Q_X)
* $F_{X,s}$ descends to $\overline{F_{X,s}} : T_{d(Y_0)} \longrightarrow T_{d(Y_1)}$

Key result (O-Sz): X neg. def. $\Rightarrow \overline{F_{X,s}} \neq 0$



$$d(Y_1) \geq d(Y_0) + \frac{Q_X(c_1(s), c_1(s)) - 2\chi(X) - 3\sigma(X)}{4}$$

Cor 1: Let X^4 neg. def. w/ $\partial X = Y$. Then
$$d(Y) \geq \max_{\xi \text{ char}} \frac{Q_X(\xi,\xi) + b_2(X)}{4}$$

Pf: $X \setminus Int(B^4)$ is a cob. from $Y_0 = S^3$ to $Y_1 = Y$.
Every ξ characteristic is realised as $c_1(s)$. □

Cor 2 (Donaldson's thm): X closed, smooth, pos. def. $\Rightarrow Q_X \cong \underline{\mathbb{Z}}^n$.
Pf: $W = \overline{X} \setminus Int(B^4)$ is neg. def. w/ $\partial W = S^3$.
By Cor 1: $\underbrace{d(S^3)}_{0} \geq \max_{\xi \text{ char}} \dfrac{Q_W(\xi,\xi) + b_2(W)}{4}$

> **Lemma (Elkies)**
> $\forall$ neg. def. lattice (L, Q), $\max_{\xi \text{ char}} \dfrac{Q(\xi,\xi) + rk L}{4} \geq 0$
> and it is $=$ iff $(L, Q) \cong -\underline{\mathbb{Z}}^n$

$\Rightarrow$ Thus, max $= 0$ and $Q_W = -\underline{\mathbb{Z}}^n \Rightarrow Q_X = \underline{\mathbb{Z}}^n$. □

Cor 3: P does not bound a neg def. smooth 4-mfd.
Pf: By contradiction, suppose $P = \partial X$ neg. def. Then:
$$d(P) \underset{\text{Cor 1}}{\geq} \max_{\xi \text{ char}} \frac{Q_X(\xi,\xi) + b_2(X)}{4} \underset{\text{Elkies}}{\geq} 0$$
However, $d(P) = -2$. ↯ □

⑤ $d(P) = -2$

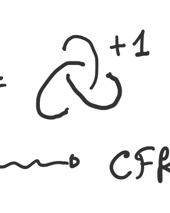
Method 1: plumbings
Recall: $P = \partial X$, with $Q_X = E_8$
$\Rightarrow \overline{P} = \partial \overline{X}$, with $Q_{\overline{X}} = -E_8$ (neg definite)
By Cor 1: $d(\overline{P}) \geq \max_{\xi \text{ char}} \dfrac{Q_{\overline{X}}(\xi,\xi) + b_2(\overline{X})}{4}$
$=$ for plumbings (O-Sz)

$$d(\overline{P}) = \max_{\xi \text{ char}} \frac{Q_{-E8}(\xi,\xi) + 8}{4} = \frac{0+8}{4} = 2$$

0 is charact.
All other ξ char give $Q(\xi,\xi) < 0$ (neg. def.)

$\Rightarrow d(P) = -d(\overline{P}) = -2$. □

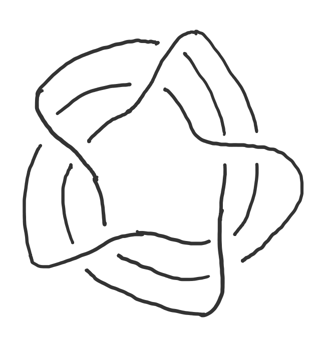
Method 2: surgery
Recall (Ray's talk): P =



[O-Sz] $K \subseteq S^3$ Knot $\rightsquigarrow CFK(K)$
Thm (O-Sz): If Y = surgery on K, then $HF(Y)$ can be recovered from $CFK(K)$.
$\rightsquigarrow$ Can compute $d(P)$.

Method 3: branched covers
$P = \Sigma_2(T_{5,-3})$ = the double cover of S^3 branched over $T_{5,-3}$.

$T_{5,-3}$ =



Thm (R, M, G-W, H-R-R-W)
If $\Sigma_2(K)$ is a Seifert fibered space w/ finite π_1, then
$$d(\Sigma_2(K)) = 2\tau(\Sigma_2(K)) + \frac{1}{6} \cdot \frac{V_K'(-1)}{V_K(-1)} - \frac{\sigma_K}{4}$$
where $\tau(\Sigma_2(K))$ = Reidemeister torsion
$V_K(q)$ = Jones polynomial
σ_K = signature of K

e.g. $P = \Sigma_2(T_{5,-3})$.
$\tau(P) = 0$
$V_{T_{5,-3}}(q) = -q^{-10} + q^{-6} + q^{-4} \rightsquigarrow$ $V_{T_{5,-3}}(-1) = 1$, $V'_{T_{5,-3}}(-1) = 0$
$\sigma_{T_{5,-3}} = 8$

$$\Rightarrow d(P) = 2 \cdot 0 + \frac{1}{6} \cdot \frac{0}{1} - \frac{8}{4} = -2$$