$GL(2,\mathbb{Z})$ $Out(F_2)$

From Flat Doughnuts to Outer Space: A Classifying Space Odyssey

- Jean Pierre Mutanguha

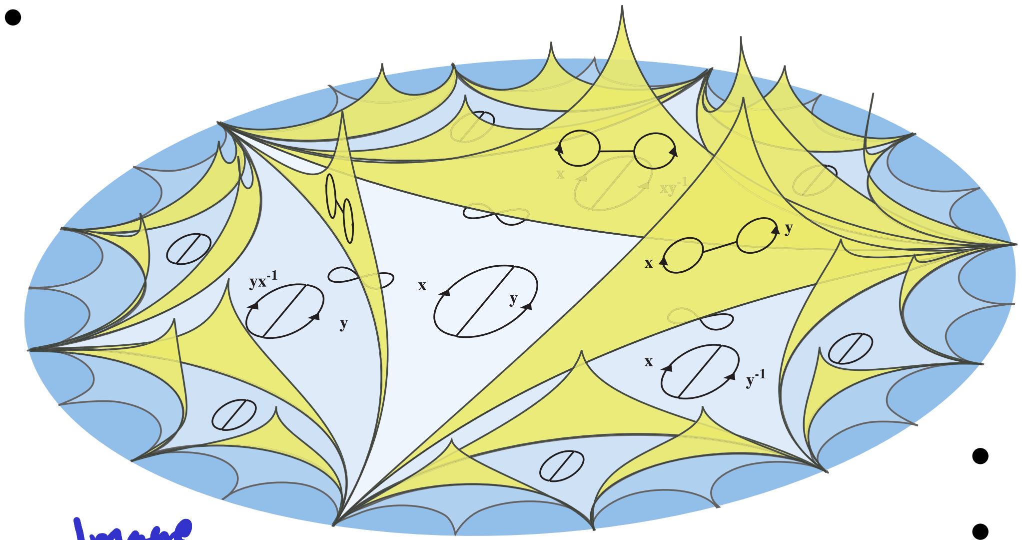


Image Credit: Karen Vogtmann "What is... Outer Space"

RAAG Definition

Γ simplicial fin. graph

$A_\Gamma = \langle s \in V(\Gamma) \mid [s,t] \in E(\Gamma) \rangle$

Sal_Γ fin. $K(A_\Gamma, 1)$ cube complex

Extreme Examples

① Γ complete $\rightsquigarrow$ $A_\Gamma = \mathbb{Z}^n$, $Sal_\Gamma =$ n-torus $\mathbb{T}^n$

② Γ edgeless $\rightsquigarrow$ $A_\Gamma = F_n$, $Sal_\Gamma =$ n-rose R_n

Today's special: $n=2$, $\mathbb{Z}^2$ & F_2

Outer Automorphism Group of A_Γ

Algebraic Def$^{\underline{n}}$: $\mathrm{Out}(A_\Gamma) = \mathrm{Aut}(A_\Gamma)/\mathrm{Inn}(A_\Gamma)$

Topological Def$^{\underline{n}}$: $\mathrm{Out}(A_\Gamma) = \{ \mathrm{Sal}_\Gamma \xrightarrow{h.e.} \mathrm{Sal}_\Gamma \}/\simeq$

"Homotopy MCG(Sal_Γ)"

Extreme E.g.

① $\mathrm{Out}(\mathbb{Z}^n) = \mathrm{Aut}(\mathbb{Z}^n) = GL(n, \mathbb{Z})$

② $\mathrm{Out}(F_n)$

⊗ **accidental isomorphism:** $GL(2, \mathbb{Z}) \cong \mathrm{Out}(F_2)$

Today's Objectives:

① Define fin.-dim. "$K(GL(n,\mathbb{Z}),1)$" & "$K(Out(F_n),1)$" spaces

$$\boxed{\begin{array}{l}\text{Unfortunately: } GL(n,\mathbb{Z}) \text{ \& } Out(F_n) \text{ have torsion} \\ \qquad \Rightarrow \text{ no fin.-dim } K(\pi,1) \text{ space!} \\ \text{Fortunately: i, Define fin-dim } \underline{E}\pi \text{ space} \ \ddot\smile \\ \qquad \text{ii, Both groups are virtually torsion-free!}\end{array}}$$

② Deduce virtual cohomological dimensions, vcd

③ Classify finite subgroups of $GL(2,\mathbb{Z}) \mid Out(F_2)$

Next Talk's Objectives:

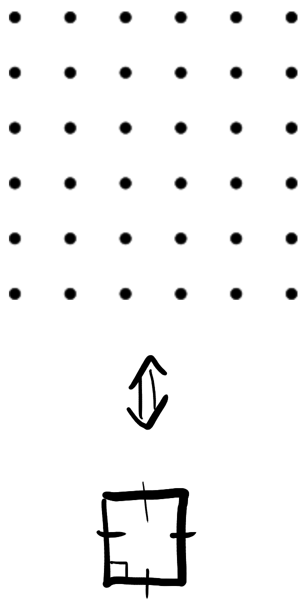
⊗ Define a fin.-dim. "$K(Out(A_\Gamma),1)$" space

0a) Define fin-dim. "$K(GL(2,\mathbb{Z}),1)$" space $M(\mathbb{Z}^2)$

- Moduli space of $\mathbb{Z}^2$-lattices in $\mathbb{E}^2$, up to similarity
- Moduli space of flat metrics of $\mathbb{T}^2$, up to similarity

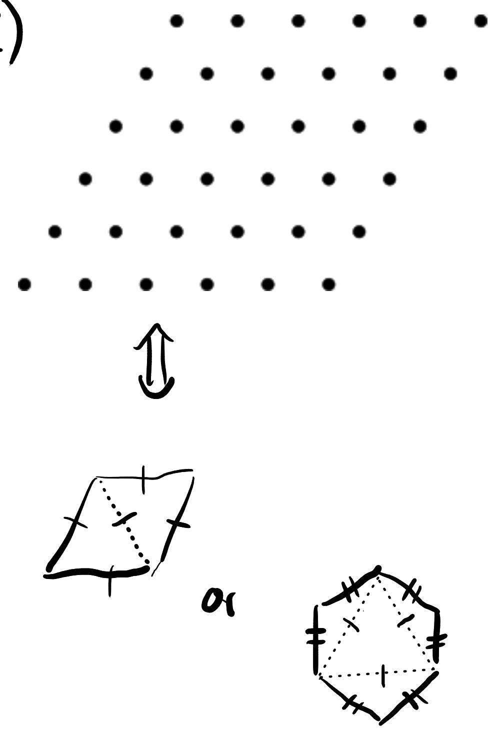
E.g.

I)



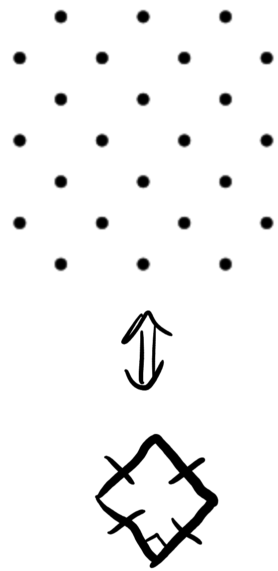
Square torus

II)



hex. torus

III)



Sq. torus

GGT Theme

$$G \overset{\text{nicely}}{\curvearrowright} \tilde{X}$$

$$\Downarrow$$

- $\tilde{X}$ conn. $\rightsquigarrow$ G f.g.
- $\tilde{X}$ simply conn. $\rightsquigarrow$ G f.p.

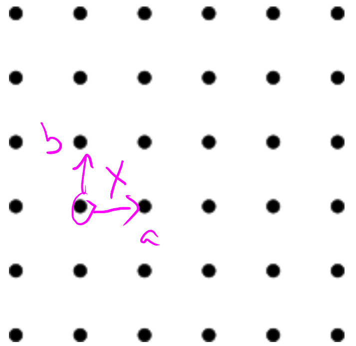
$\vdots$

- $\tilde{X}$ contractible $\rightsquigarrow$ G "finite type"

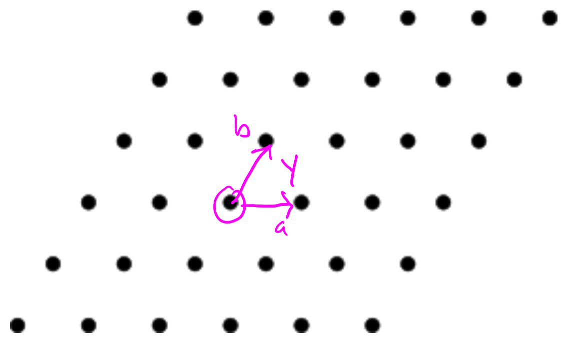
$$\widetilde{\mathcal{M}}(\mathbb{Z}^2) = \{\underline{\text{marked}}\ \mathbb{Z}^2\text{-lattice in } \mathbb{E}^2\}/\text{marked sim}$$
$$= \{\underline{\text{marked}}\ \text{flat metric on } \mathbb{T}^2\}/\sim$$

E.g. Fix a basis $\{a, b\}$ of $\mathbb{Z}^2$

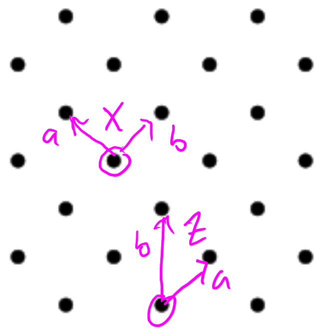
I)



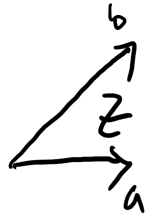
II)



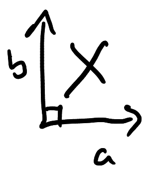
III)



The right $GL(2,\mathbb{Z})$-action, by example:



$=$



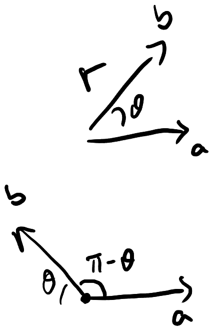
$$\cdot \begin{array}{cc} a & b \\ \end{array} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\widetilde{\mathcal{M}}(\mathbb{Z}^2)/GL(2,\mathbb{Z}) = \mathcal{M}(\mathbb{Z}^2)$$

A parametrization of $\tilde{M}(\mathbb{Z}^2)$

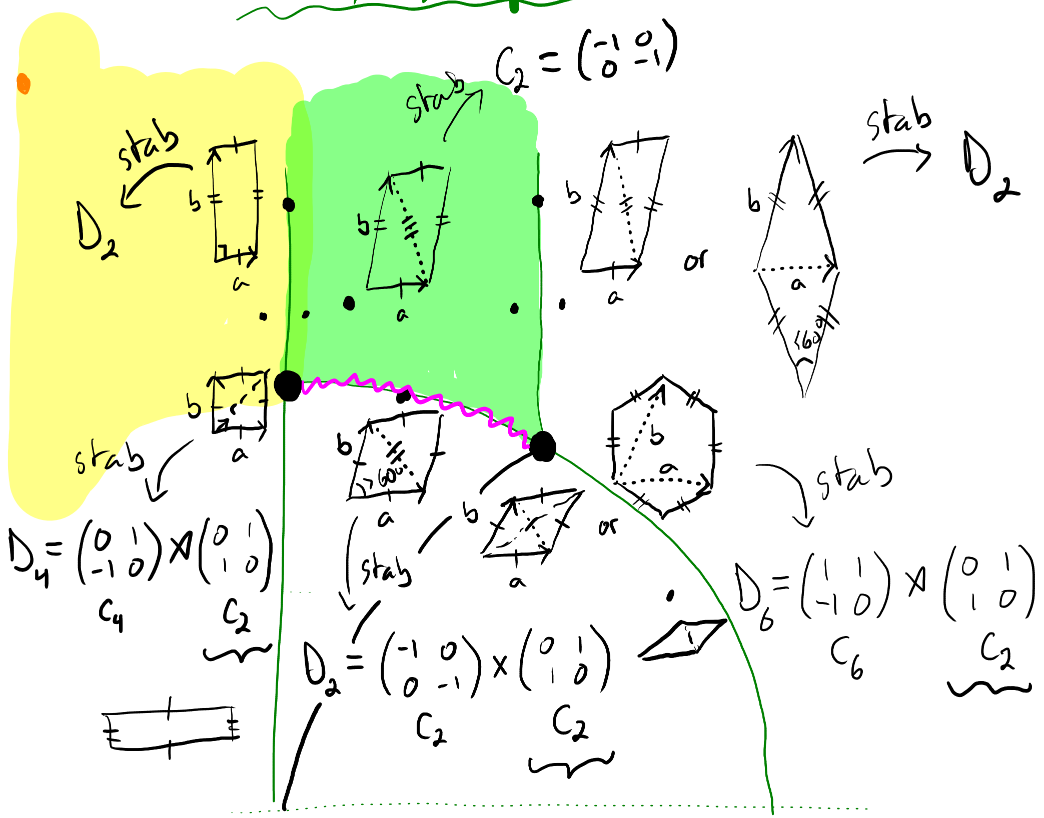
$$\tilde{M}(\mathbb{Z}^2) = \{ \ r, \theta, 1 \ : \ r>0, \ 0<\theta<\pi \}$$

$= $ upper half-plane H^2.



$\therefore$ $\tilde{M}(\mathbb{Z}^2)$ is contractible!

- $GL(2,\mathbb{Z})$ fundamental domain:



$\Rightarrow$ $\tilde{M}(\mathbb{Z}^2) \curvearrowleft GL(2,\mathbb{Z})$

proper discontinuous

GGT Classic

3a Play Ping-Pong

$\Rightarrow GL(2,\mathbb{Z}) \cong D_4 *_{D_2} D_6$

$\cong \underbrace{(C_4 *_{C_2} C_6)}_{SL(2,\mathbb{Z})} \rtimes C_2$

$\Rightarrow GL(2,\mathbb{Z})$ is virt. free!

(1a) $\widetilde{M}(\mathbb{Z}^n) = \{\text{marked flat } n\text{-torus}\}/\sim$

$= \boxed{\text{symmetric space } O(n)\backslash GL(n,\mathbb{R})}$

$\underset{\text{isometry}}{\not\cong} \mathbb{H}^2 \times \cdots \times \mathbb{H}^n$, $n > 2$

$\underset{\text{homeo.}}{\cong} H^2 \times H^3 \times \cdots \times H^n$

(2a) $\widetilde{M}^{WR}(\mathbb{Z}^n) = \{\text{well-rounded marked flat } n\text{-torus}\}/\sim$

- equivariant deformation retract
- CW complex, $\dim = \binom{n}{2}$
- compact quotient

$\Rightarrow \boxed{vcd = \binom{n}{2}}$

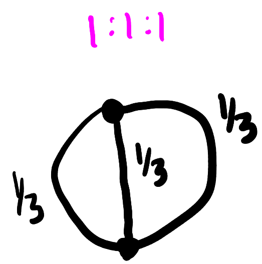
$\cancel{\mathbb{Z}^{\binom{n}{2}} \cong} \underbrace{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{cd = \binom{n}{2}} \leq GL(n,\mathbb{Z})$

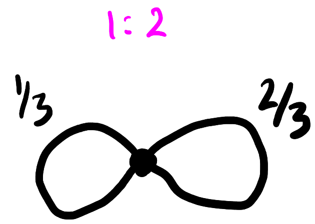
(0b) Define fin.-dim. "$K(\mathrm{Out}(F_2),1)$" space $\mathcal{M}(F_2)$

- Moduli space of metric rank-2 graphs, up to similarity

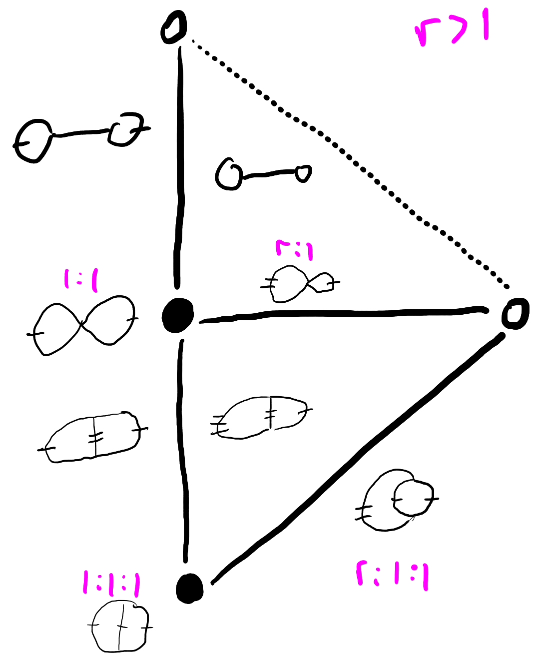
E.g.

I) 

II) 

III) 

Picture of $\mathcal{M}(F_2)$:



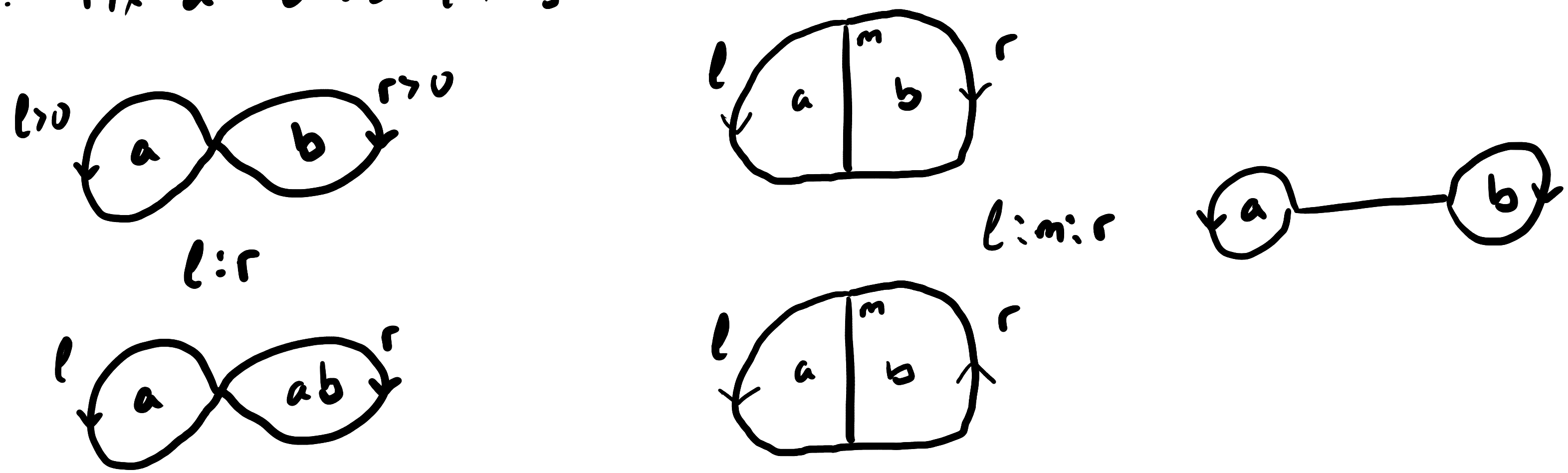
$\tilde{\mathcal{M}}(F_2) = \{$ marked metric rank-2 graph $\}/$ marked sim.

$=$ $\boxed{\text{Outer Space}}$ CV_2 or $CV(F_2)$ [Culler-Vogtmann '86]

Points of $\tilde{\mathcal{M}}(F_2)$: pair of $[\Gamma] \in \mathcal{M}(F_2)$

$+ \, [\alpha : R_2 \xrightarrow[h.e.]{} \Gamma]$

E.g. Fix a basis $\{a, b\}$ of F_2



⊛ Compare the tuple $(\|a\|, \|b\|, \|ab\|, \|a^{-1}b\|)$ in each $[\Gamma, \alpha]$.

$(\ell : \cancel{b}\, r : \ell + r : \ell + r)$

$(\ell : \ell + r : r : 2\ell + r)$

Picture of $\tilde{M}(F_2)$

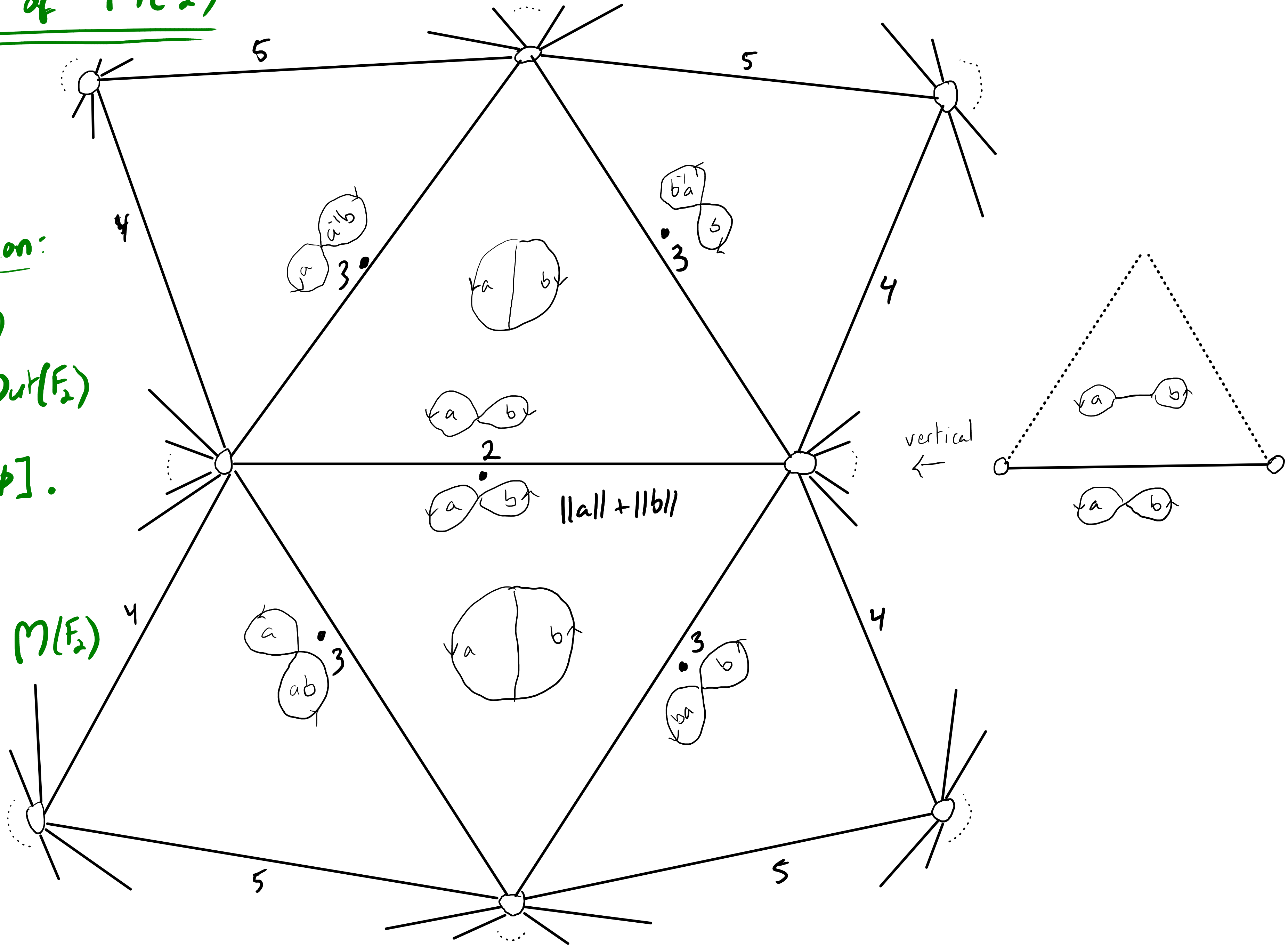
right $Out(F_2)$-action:

- $[\Gamma, \alpha] \in \tilde{M}(F_2)$
- $[\phi : R_2 \xrightarrow{h.e.} R_2] \in Out(F_2)$

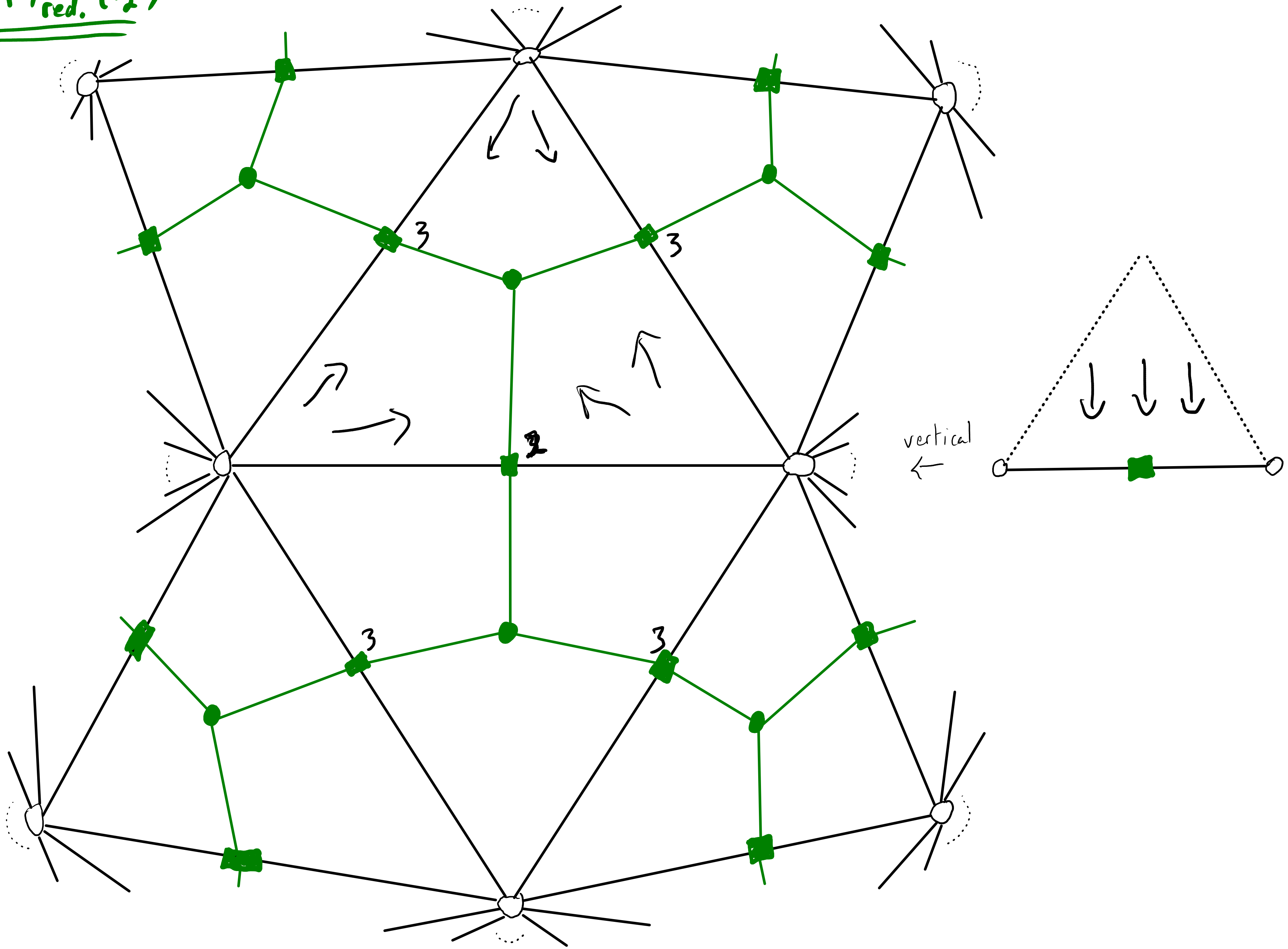
Then $[\Gamma, \alpha] \cdot [\phi] = [\Gamma, \alpha\phi]$.

By definition,

$$\tilde{M}(F_2)/Out(F_2) = M(F_2)$$

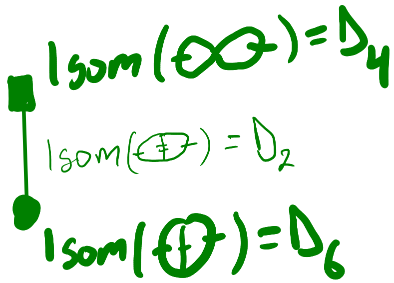


Picture of $\tilde{M}^{spine}_{red.}(F_2)$



- $\tilde{M}^{spine}_{red.}(F_2)$ graph
- $\tilde{M}^{spine}_{red.}(F_2) \curvearrowleft Out(F_2)$ simplicial

<u>Fundamental Domain</u>:



$\Rightarrow$ action is proper & cocompact!

GGT Classics

$\tilde{M}^{spine}_{red.}(F_2)$	$Out(F_2)$	method
conn.	f.g.	folding
simply conn. $\Updownarrow$ tree	f.p. $\Updownarrow$ virt. free	peak-reduction
	3b $\cong D_4 *_{D_2} D_6$	Bass-Serre Theory

<u>Stallings '70s</u>

<u>Rapaport '58</u>

<u>'70s</u>

(1b) $\widetilde{\mathcal{M}}(F_n) = \{\text{marked metric rank-}n\text{ graphs}\}/\sim$

$= \boxed{\text{Outer space}}$, CV_n or $CV(F_n)$

- $\widetilde{\mathcal{M}}(F_n) \curvearrowleft Out(F_n)$ proper
- $\widetilde{\mathcal{M}}^{spine}_{red.} \curvearrowleft Out(F_n)$ proper & cocompact
  - ↳ Equivariant deformation retract
  - ↳ simplicial complex, $\dim = 2n-3$

Folding $\rightsquigarrow$ $\widetilde{\mathcal{M}}^{spine}_{red.}(F_n)$ is connected

"Super" peak-reduction $\rightsquigarrow$ $\boxed{\widetilde{\mathcal{M}}^{spine}_{red.}(F_n) \text{ is contractible!!!}}$ ~ Culler-Vogtmann '86

(2b) $\Rightarrow$ $\boxed{vcd = 2n-3}$

Let $F_n = \langle a, b_1, \ldots, b_{n-1} \rangle$.

Define $r_i : \begin{array}{l} a \mapsto a \\ b_i \mapsto b_i a \\ b_j \mapsto b_j \quad (j \neq i) \end{array}$ & $l_i : \begin{array}{l} a \mapsto a \\ b_i \mapsto a^{-1} b_i \\ b_j \mapsto b_j \end{array} \quad (j \neq i)$

Then $\mathbb{Z}^{2n-2} \cong \langle r_1, l_1, \ldots, r_{n-1}, l_{n-1} \rangle \leq Aut(F_n)$

& $\underbrace{\mathbb{Z}^{2n-3}}_{cd = 2n-3} \cong \langle \bar{r}_1, \bar{l}_1, \ldots, \bar{r}_{n-1}, \bar{l}_{n-1} \rangle \leq Out(F_n)$ as $\overline{r_1 l_1 \cdots r_{n-1} l_{n-1}} = \bar{1}$

Post-talk discussion of the Ping-Pong Lemma

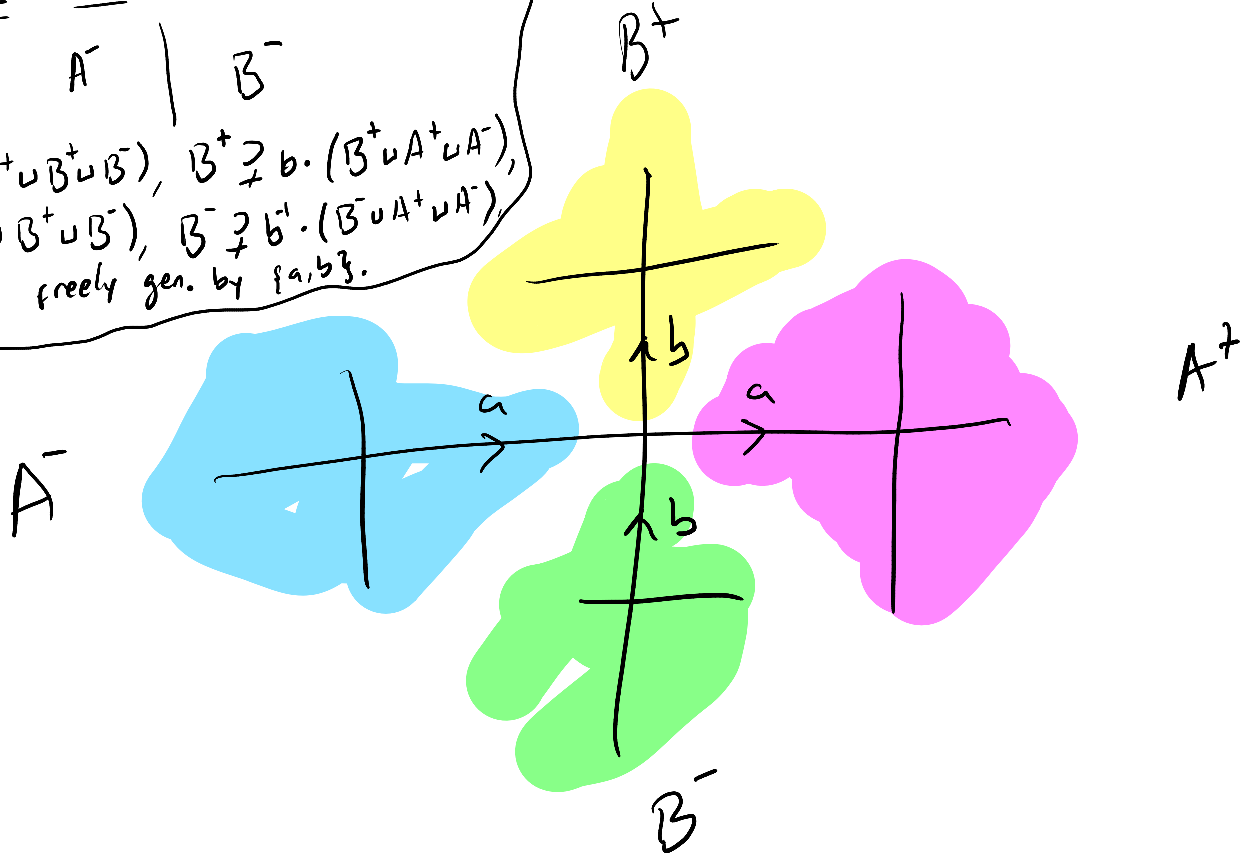
Ping-Pong Lemma

$H \leq G$ gen. by $\{a,b\}$; $G \curvearrowright$ space $X = A^+ \sqcup A^- \sqcup B^+ \sqcup B^-$

$$X = \begin{array}{c|c} B^+ & A^+ \\ \hline A^- & B^- \end{array}$$

If $A^+ \supsetneq a\cdot(A^+ \sqcup B^+ \sqcup B^-)$, $B^+ \supsetneq b\cdot(B^+ \sqcup A^+ \sqcup A^-)$,
$A^- \supsetneq a^{-1}\cdot(A^- \sqcup B^+ \sqcup B^-)$, $B^- \supsetneq b^{-1}\cdot(B^- \sqcup A^+ \sqcup A^-)$,
then H is freely gen. by $\{a,b\}$.

E.g. $H = G = F_2$
$X = F_2 \setminus \{1\}$



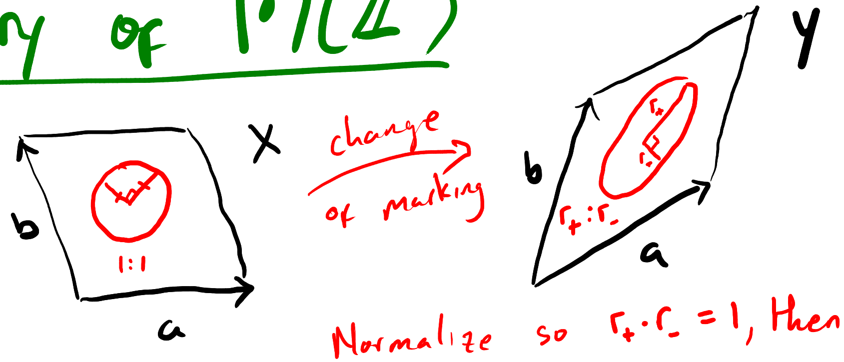
$$F_2 \cong \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \leq GL(2,\mathbb{Z})$$

Soulé "Group cohomology of $SL(3,\mathbb{Z})$"

Geometry of $\tilde{M}(\mathbb{Z}^2)$

Outtake #1

metric:



Normalize so $r_+ \cdot r_- = 1$, then

$$d(X, Y) = \|(\ln r_+, \ln r_-)\|_2 .$$

$GL(2,\mathbb{Z})$ acts by isometry

(*) accidental isometry: $\left(\tilde{M}(\mathbb{Z}^2), \frac{\sqrt{2}}{2}\cdot d\right) = \mathbb{H}^2$!

$GL(2,\mathbb{Z})$-action given by obvious representation:

$$GL(2,\mathbb{Z}) \to PGL(2,\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$$

Classification of $A \in GL(2,\mathbb{Z})$

(3a)

- A elliptic $\Leftrightarrow$ fin. order; conj. into D_4 or D_6 (or both)
- A loxodromic $\Leftrightarrow$ anosov; conj. in $GL(2,\mathbb{R})$ to $\begin{pmatrix} \lambda & 0 \\ 0 & \pm 1/\lambda \end{pmatrix}$, $\lambda > 1$
- A parabolic $\Leftrightarrow$ (rotated) shear; conj. to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}$, $n \neq 0$

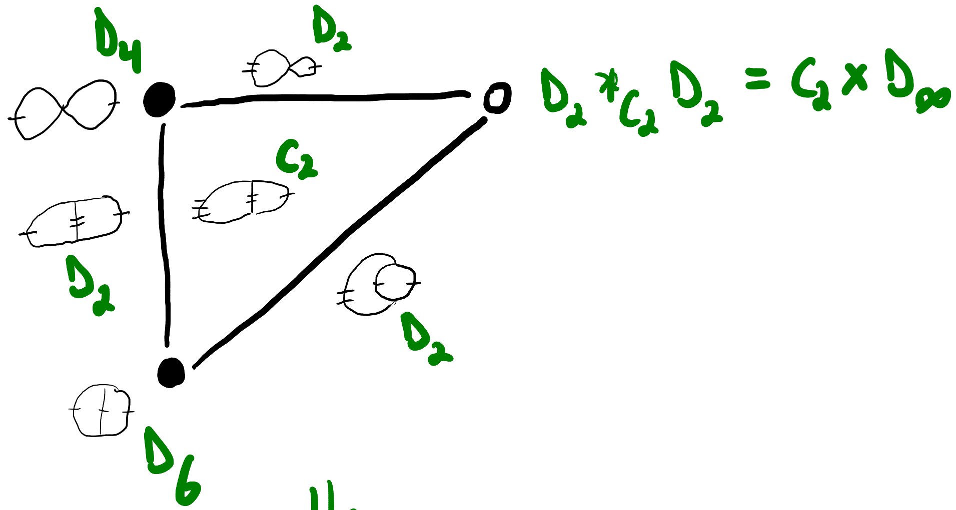
3b

$\boxed{\text{Classification of } [\phi] \in Out(F_2)}$

Outtake #2

i, Using $\tilde{M}^{spine}_{red}(F_2)$: $[\phi]$ elliptic $\Leftrightarrow$ fin. order; conj. into D_4 or D_6 (or both)

ii, Using $\tilde{M}_{red.}(F_2)$ $\Rightarrow$ quotient



$\Downarrow$

$[\phi]$ fixes pt. $\Leftrightarrow$ fin. order; conj. into D_4 or D_6

$[\phi]$ fixes ideal pt. $\Leftrightarrow$ reducible; conj into $C_2 \times D_\infty$

$[\phi]$ fixes no pt. nor ideal pt. $\Leftrightarrow$ ∞-order irred.; Anosov-like