Flat life on Surfaces

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What is a geometry?

X model space (manifold)

a group $G \curvearrowright X$ faithful
transitive

<u>def</u>: A geometry in the Klein' sense is a couple (G, X) as above

If $Y \subseteq X$, $H < G$ and H preserves Y
then (H, Y) is a SUB-GEOMETRY of (G, X)

Some Examples

$X = \mathbb{E}^2 \quad G = \mathrm{Iso}^+(\mathbb{E}^2)$ Euclidean Geometry

$X = \mathbb{E}^2 \quad G = \mathbb{R}^2 \simeq \mathbb{C}$ (translations)

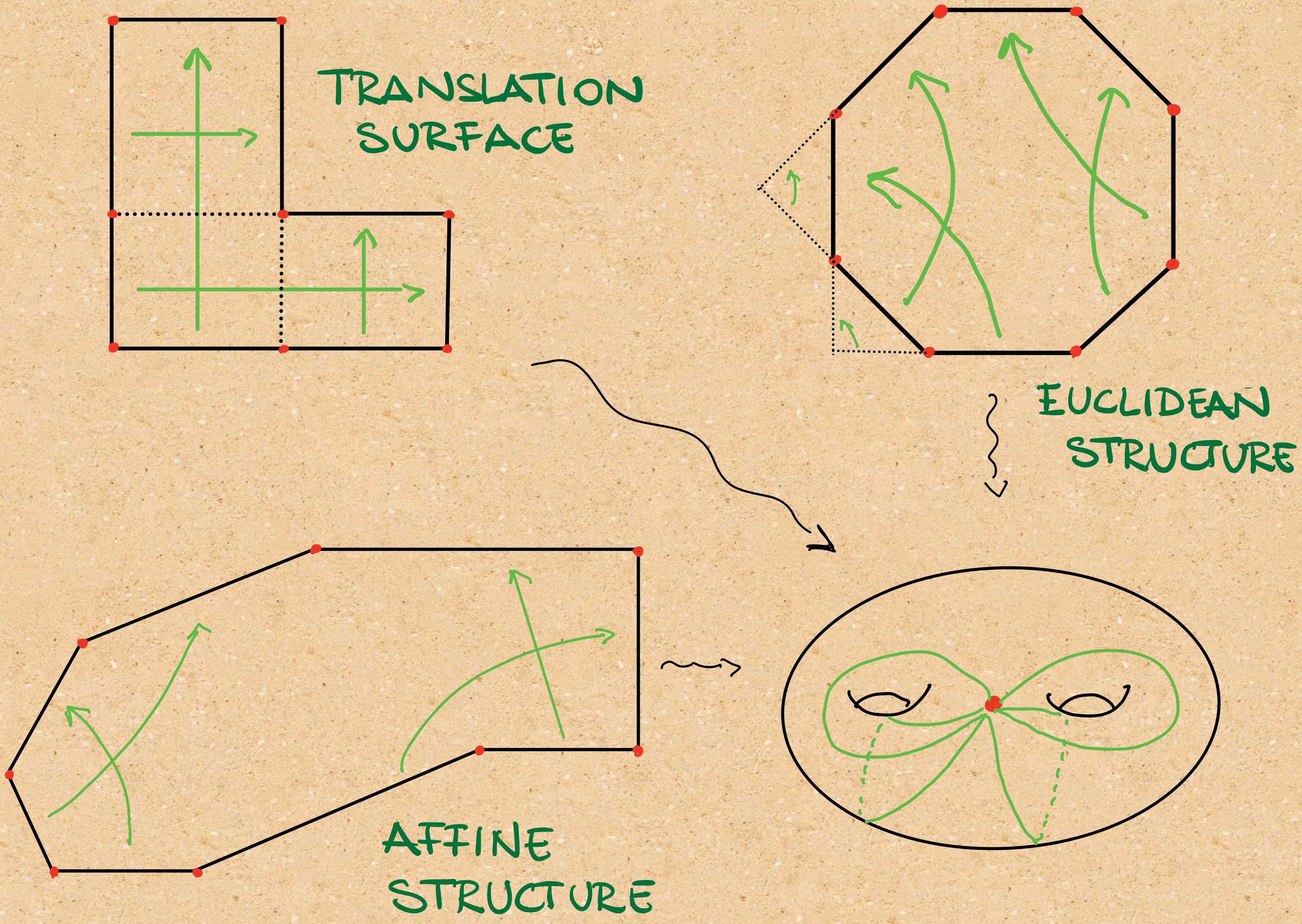
$X = \mathbb{C} \quad G = \mathrm{Aff}(\mathbb{C})$ Affine Geometry

$X = \mathbb{CP}^1 \quad G = \mathrm{PSL}(2, \mathbb{C})$ Complex Proj geometry

Geometric structures

$S_{g,n}$ is a connected, oriented surface of genus $g \geq 0$ and $n \geq 0$ punctures (if $n = 0$, S_g is closed)

def: a (G,X)-structure on $S_{g,n}$ is a maximal atlas of (possibly branched) charts taking values on X with change of coordinates in G


TRANSLATION SURFACE
EUCLIDEAN STRUCTURE
AFFINE STRUCTURE

Developing - Holonomy pair

a (G,X)-structure is always specified by a

representation $\rho : \pi_1(S_{g,n}) \longrightarrow G$

developing map $f : \widetilde{S_{g,n}} \longrightarrow X$

such that $f \circ \gamma = \rho(\gamma) \circ f \quad \forall\, \gamma \in \pi_1(S_{g,n})$

Problem: Given $\rho : \pi_1(S_{g,n}) \longrightarrow G$, does it appear as the holonomy of of some (G,X)-structure on $S_{g,n}$?

Motivation

2^{nd}-order ODEs on Riemann Surfaces

$$
\begin{tikzcd}
\left\{\begin{array}{l}2^{nd}\text{-order} \\ \text{ODEs on } S_{g,n}\end{array}\right\} \arrow[r, "\text{monodromy}"] \arrow[d, <->, dashed] & \left\{\begin{array}{l}\text{representations} \\ \pi_1(S_{g,n}) \to PSL(2,\mathbb{C})\end{array}\right\} \\
\left\{\begin{array}{l}\text{complex projective} \\ \text{structures on } S_{g,n}\end{array}\right\} \arrow[ur, bend right, "\text{holonomy}"'] &
\end{tikzcd}
$$

Two Issues

- Some representations are not realizable if we do not allow branched charts
- Whenever a representation $\rho : \pi_1(S_{g,n}) \longrightarrow PSL(2,\mathbb{C})$ is realized, the nature of the corresponding structure is not always specified!

Problem: Give a better description of the image and fibres of the holonomy map

$$\left\{ \begin{array}{c} \text{complex projective} \\ \text{structures on } S_{g,n} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{representations} \\ \pi_1(S_{g,n}) \to PSL(2,\mathbb{C}) \end{array} \right\}$$

Holonomy of Translation Surfaces

Theorem (Haupt, 1920)

A representation $\rho : \pi_1(S_g) \longrightarrow \mathbb{C}$ is the holonomy of a translation surface on S_g if and only if:

- $\mathrm{vol}(\rho) = \sum_{i=1}^{g} \Im\left(\overline{\rho(\alpha_i)}\, \rho(\beta_i)\right) > 0$

  $\{\alpha_i, \beta_i\}_{i=1,\dots,g}$ is a symplectic basis

- If $\mathrm{Im}(\rho) = \Lambda < \mathbb{C}$ lattice, then

  $$\mathrm{vol}(\rho) \geq 2\, \mathrm{Area}(\mathbb{C}/\Lambda)$$

How to extend Haupt's theorem to open surfaces?

a translation surface can be seen as a couple (R, ω) where R is a Riemann surface and ω is a holomorphic 1-differential

If R is a punctured Riemann Surface ω is allowed to have finite order poles at the punctures

One way is...

Theorem (Chenakkod - F. - Gupta)

Let $n \geq 1$, then any representation $\rho : \pi_1(S_{g,n}) \longrightarrow \mathbb{C}$ is the holonomy of a Translation surface with poles.

Another way is:

<u>Theorem</u> (F.-Gupta)

Let $n \geq 2$ and $2-2g-n<0$.
Then a non-trivial representation
$\rho: \pi_1(S_{g,n}) \longrightarrow \mathbb{C}$ is the
holonomy of a Translation
surface with all singularities
at the punctures.

For $n=1$ there are other
exceptions

What about the fibres?

Closed surfaces ($n=0$)

Calsamiglia - Deroin - Francaviglia

They provided conditions for the fibres to be connected.

$\Longrightarrow$ Two structures can be deformed one into another

$\Longrightarrow$ Ergodicity of the Isoperiodic foliations in the principal stratum

The same has been obtained independently by Hamenstädt

Problem: Study the fibres of the holonomy map for Translation surfaces with poles and translation surfaces without singularities.