MPIM Topology Seminar

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Duality
and
Dualizing Spheres

Mean Streets of Evanston.

14 June 2021

Duality and dualizing

Basic Example

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "\subseteq"] & \mathbb{R}^k \arrow[d, "\cap"] \\ M_+ = M \cup \{\infty\} \arrow[r] & S^k \end{tikzcd}$$

C^∞ manifold

Alexander Duality: $\tilde{H}_{k-q-1}(S^k - M_+) \cong \tilde{H}^q M_+ \cong H^q M$.

Atiyah: Let $D_k M_+ = \Sigma(S^k - M_+) \simeq T(\nu)$

Tubular Nbhd

Thom space of normal bundle.

$$\tilde{H}_{k-q} D_k M_+ \cong H^q M$$

+ Thom Iso $\Rightarrow$ Poincaré Duality

Both $T(\nu)$ and $D_k M_+$ are independent of k for large k.

Spanier-Whitehead Dual

$$T(-\tau_M) \simeq \{T(\nu)\}_k \simeq \{D_k M_+\} \simeq DM_+$$

Duality and groups action

What is M has action by a compact Lie group.

DM_+ depends $M \hookrightarrow V$ G-embedding into a G-rep

Example: $M = pt$, $G = \{e\}$ $M_+ = S^0 \subseteq S^k$ so ~~M_+~~ $S^k - M_+ \simeq S^{k-1}$

So $\Sigma(S^k - M_+) \simeq S^k$. $D_+M \underset{\text{as}}{\simeq} S^0$ spectra

If G is nontrivial $M = \{pt\}$

$M_+ \subseteq S^V$, $S^V - M_+ \simeq S(V)$ unit in the representation sphere

Theorem: $M = pt$, $DM_+ \simeq S^{-\mathfrak{g}_a}$

where $\mathfrak{g}_a$ = Adjoint representation of G

Example: $n\times n$ matrices of $\mathbb{Z}_p = M_n(\mathbb{Z}_p) \supseteq M_n(\mathbb{Z}_p)^\times = Gl_n(\mathbb{Z}_p)$

$\cap$

$M_n(\mathbb{Q}_p)$

$Gl_n(\mathbb{Z}_p) \supseteq \underbrace{1 + pM_n(\mathbb{Z}_p)}_{\Gamma_1} \supseteq \underbrace{1+p^2M_n(\mathbb{Z}_p)}_{\Gamma_2} \supseteq \dots \qquad \cap \Gamma_i = \{e\}$

$Gl_n(\mathbb{Z}_p) \cong \lim Gl_n(\mathbb{Z}_p)/\Gamma_i \cong Gl_n(\mathbb{Z}/p^n)$ is a profinite group

$\Gamma_k / \Gamma_{k+1} \cong (\mathbb{Z}/p)^{n^2} \qquad d = n^2 = \dim.$

$g_a \cong M_n(\mathbb{Z}_p)$ with the conjugation action. $\operatorname{rank}(g_a) \cong d = n^2$.

Example 2: $\mathcal{O}_n \supseteq \mathcal{O}_n^\times = S_n \supseteq \Gamma_i \supseteq \dots$ all the same properties g_a.

$\cap$

D_n

$\mathcal{O}_n = W(\mathbb{F}_{p^n})\langle S\rangle / (S^n - p) \qquad aS = \phi(a)S$

ϕ = Frobenius.

$\uparrow$ normed division algebra of $\mathbb{Q}_p$ of invariant $1/n$

Cohomology and Serre Duality

Let M be a G-module (continuous)

G = p-adic analytic group.

$$M \to H^0(G,M) = M^G$$

is right exact and has derived functors $H^n(G,M)$.

Serre: $\omega_G = \Lambda^d \mathfrak{g}_a$ = top exterior $\cong \mathbb{Z}_p$ ↻ G action. [Serre duality

Let $(-)^\# = \mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z}_{(p)})$ $= \mathbb{Z}/p^\infty$ = p-torsion in S^1

Hypotheses needed

$$H^q(G,M)^\# \cong H^{d-q}(G, \mathrm{Hom}(M, \mathbb{Z}/p^\infty \otimes \omega_G))$$

G acts by conjugation

M finite

Derived version over the $\mathbb{Z}_p$.

Turn this into homotopy theory analog of $S^{\mathfrak{g}_a}$.

The dualizing sphere

Formal answer: p-adic analytic group $G \supseteq \Gamma_i$. $\dim(G) = d$.

If F is finite BF = space of M is an F-module

$$H^q(BF, M) \cong H^q(F, M).$$

$F_1 \subseteq F_2$ $\quad BF_1 \xrightarrow{res} BF_2$ $\qquad H^q(BF_1, M) \xleftarrow{res} H^q(BF_2, M)$

$\Sigma^\infty_+ BF_1 \xleftarrow{tr} \Sigma^\infty_+ BF_2$ $\qquad H^q(BF_1, M) \xrightarrow{tr} H^q(BF_2, M)$.

Define $\quad B\Gamma_i \overset{\text{def}}{=} \operatorname{holim}_k B(\Gamma_i / \Gamma_{i+k})$ $\qquad$ G acts by conj

Then $\quad I_G = \operatorname{hocolim}_{tr} \left(\Sigma^\infty_+ B\Gamma_i \xrightarrow{tr} \Sigma^\infty_+ B\Gamma_{i+1} \to \cdots \right)$

Serre

Theorem $H_* I_G \cong \omega_G$ concentrated in degree d

$I_G \simeq S^d$ with some G-action

A linear version

A computable Version: $\mathfrak{g} = \mathfrak{g}_a \supseteq p^i \mathfrak{g}_a$ for $i \geq 0$.

$\mathfrak{g}_a \cong \mathbb{Z}_p^d$ $\qquad p^i \mathfrak{g}_a \cong (\mathbb{Z}_p)^d$

As before you form $B p^i \mathfrak{g} \simeq (S^1)^d_{p\text{-completed}}$

$$S^{\mathfrak{g}_a} = S^{\mathfrak{g}} = \operatorname{colim}_{tr} \Sigma^\infty_+ B p^i \mathfrak{g}.$$

Theorem: $H_* S^{\mathfrak{g}} \simeq \omega_{\cancel{\mathfrak{g}}G}$ concentrated in degree d.

Linearization ~~Hypothesis~~ Theorem: $I_G \simeq S^{\mathfrak{g}}$ G-equivariant equivalence

Clausen: Big question what category does $S^{\mathfrak{g}}$ live in?

LH for finite subgroups

<u>Theorem</u>: Let $F \subseteq G$ is a finite subgroup and suppose F has the property p-Sylow subgroup of $F/F \cap \text{center of } G$ is $\cong (\mathbb{Z}/p)^t$ for some t. Then $\boxed{I_G \cong S^{\mathfrak{g}}_F}$.

$\hookrightarrow$ Proof uses Lannes's Theory.

<u>Grodal</u>

<u>Castellana</u>: Trans of the AMS which is the key input.

For $S_n \subseteq Q_n$ it happens surprisingly often.

Geometric input

Theorem: Let $F \subseteq G$ be finite and suppose F-

F-invariant lattice $L = \mathbb{Z}^d \subseteq \mathfrak{g} = \mathfrak{g}_a$ for G.

$[\Rightarrow \mathbb{Q}_p \otimes L \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathfrak{g}]$.

Let $V = \mathbb{R} \otimes L$ - F representation. Then there is a

F-map

$$S^V \longrightarrow S^{\mathfrak{g}}$$

which is an equivalence after completion.

Example: if $n = 2 = p$, then copy $Q_8 \subseteq S_2$ which arises

for automorphisms of elliptic curves and Q_8 acts on $\mathbb{H}$

= quaternions.