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Noncommutative and Symplectic Geometry

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Homological Mirror Symmetry is supposed to involve an identification

$$\mathcal{A} = D^{\pi}Fuk(X) \simeq D^b Coh(\check{X}) = \mathcal{B}$$

a symplectic manifold — an algebraic variety

For us, X will be a Liouville domain/sector (Nate's talk) with $c_1(X) = 0$ and a Spin structure.

Underlying $\mathcal{A}, \mathcal{B}$ are certain A_∞/dg categories. (also denoted $\mathcal{A}, \mathcal{B}$)

Kontsevich: "HMS is an identity of noncommutative spaces".

"Defn": A noncommutative space is an A_∞/dg category $\mathcal{A}$ over a field $\mathbb{k}$. "Perf $\bar{A} = A$".

Basic Properties of noncommutative spaces.

- $\mathcal{A}$ is proper if $\dim H^\bullet(Hom(X,Y)) < \infty$ for $X, Y \in Ob\,\mathcal{A}$.

  $\mathcal{A} = D^b Coh\, X$: X proper $\Rightarrow \mathcal{A}$ proper. $\Leftarrow$ (Orlov)

- $\mathcal{A}$ has a category of modules. $Mod(\mathcal{A}) =$ "$QCoh\,\bar{A}$".

  (Mental shortcut: $\mathcal{A}$ has one object $\Rightarrow \mathcal{A}$ is an algebra)

  $M \in Mod(\mathcal{A})$
  $M(X) \in \mathbb{k}$-Vect
  for each $X \in Ob(\mathcal{A})$

  - $N \subset M$ sub-module $\Rightarrow$ quotient module M/N.
  - M (f.s.) free if $M \simeq \mathcal{Y}(X_1)[n_1] \oplus \mathcal{Y}(X_2)[n_2] \cdots$. $X \in Ob(\mathcal{A}) \Rightarrow Hom(-,X) = \mathcal{Y}(X)$
  - M a twisted complex if it has a filtration M^i st M^{i+1}/M^i is free. $Cone(A^{\oplus n} \to A^{\oplus m})$.

- M is a perfect module if it is homotopy equivalent to

M/M' is ...

- M is a <u>perfect module</u> if it is homotopy equivalent to $\operatorname{Im} p$, $p^2 = p$, for $p \in H^0(\operatorname{Hom}(N,N))$, N a twisted complex.

$$\rightsquigarrow \operatorname{Perf}(A) = \text{"}\operatorname{Perf} \bar{A}\text{"}.$$

<u>Product</u>: Can form A-A-bimod $= A \otimes A^{op}$ mod.
$= \text{"}QCoh(\bar{A} \times \bar{A})\text{"}.$

$F \in A$-A-bimod
$M \in A$-mod $\Rightarrow F(A) = F \underset{A}{\otimes} M \in A$-mod.

$\Leftarrow$ Often, functors are representables $\operatorname{Mod}(A) \xrightarrow{F} \operatorname{Mod}(A)$

$\underline{A} \in A$-A bimod $\Longleftrightarrow \mathcal{O}_\Delta \in QCoh(\bar{A} \times \bar{A})$
diagonal bimodule

Defn A is <u>smooth</u> if diagonal bimodule is perfect.

Idealted one of a finite nu. tw. of yoneda modules.

Prop: $A = \operatorname{Perf} X \rightarrow A$ smooth iff X smooth
separated <u>scheme</u> of finite type $/\Bbbk$.

$\cong$
A

Prop: $A = D^b Coh X \Rightarrow A$ smooth! ϟ

<u>Generation</u>: We say that $X_1, \ldots, X_n \in Ob(A)$ split-generate if Yoneda module of $X \in Ob(A)$ is gis to summand of a module filtered by sums/shifts of yoneda modules on X_i.

Prop: $M \in \operatorname{Mod}(A)$, A smooth, $\dim H^*(M(X)) < \infty$ proper modules.
$\Rightarrow M \in \operatorname{Perf}(A)$.

Idea: "Resolution of diagonal". Suppose $\dim_{\Bbbk} M(X) < \infty$. finite dimensional complex of $\Bbbk$ vector spaces.

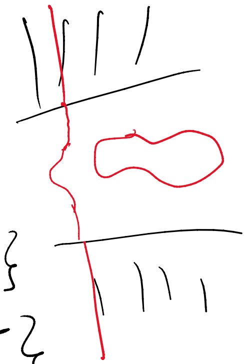
Then
$$M = A \underset{A}{\otimes} M = \begin{bmatrix} A \otimes A \\ \Uparrow \\ A \otimes A \end{bmatrix} \underset{A}{\otimes} M = \begin{bmatrix} A \otimes \Bbbk^{n} \\ \downarrow \\ A \otimes \Bbbk^{m} \end{bmatrix} \leftarrow \text{perfect complex}$$

resolve A

Symplectic geometry:

X Liouville domain/sector.

In this case there are several variants of $Fuk(X)$; we will consider $Fuk_{cpct}(X) = \{$closed exact Lagrangians (equipped w/ grading data)$\}$



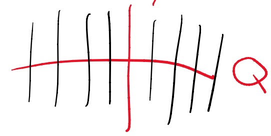
$WF(X) = \{$exact Lagns conical at ∞ (equipped w/ grading data)$\}$.

$Fuk_{cpct}(X) \longrightarrow$ proper — Floer homology between compact Lagrangians is finite dimensional.

$WF(X) \longrightarrow$ smooth (always!) $\{$Like $D^b Coh\, X$!$\}$.

Model case:

$X = T^*Q$. — Q is compact manifold (Spin, simply connected). T_q^*Q



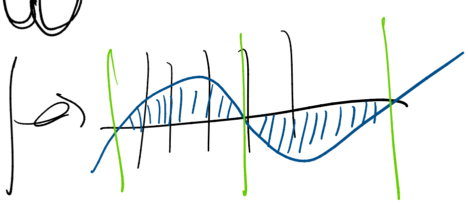
$T_q^*Q,\ Q \in WF(T^*Q)$.

$L \in WF(T^*Q) \xrightarrow{AS} \tilde{L} \in Mod\ Hom(T_q^*Q, T_q^*Q)$

$= Mod\ C_*(\Omega_q Q)$

Thm (Abouzaid): AS is an equivalence

$WF(T^*Q) \xrightarrow{\sim} Perf\ C_*(\Omega_q Q)$.



Basic Invariants of nc spaces.

$HH_*(A)$, $HH^*(A)$ — Hochschild (co)homology of A.

$\bigoplus_{k \geq 0} \underbrace{Hom(X_0, X_1) \otimes Hom(X_1, \ldots) \ldots}_{\mu_i} \otimes Hom(X_k, X_0)$. bar complex.

$\{ X_0 - X_k,\ X_1 \}$ differential $= b\ z$

apply every possible μ_i

$\left\{ x_1, \ldots, x_2, \ldots, x_k \right\}$ apply every possible μ_k $\qquad \mu_i$

Suppose A is a unital associative algebra.
With $\bar{A} = A/1 \cdot \mathbb{k}$.

(WF is not unital but its cohomology category is unital)

$H^*(\mathrm{Hom}(L,L))$
$\ni$
1.

Equivalent complex $= \bigoplus_k A \otimes \bar{A}^{k-1}$.

$a_0\, da_1 \cdots da_{k-1}$.

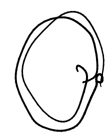
$HH_* =$ "noncommutative differential forms on A".

Define operator B as cyclically symmetric version of exterior derivative

$$B(a_0 \otimes \cdots \otimes a_k) = 1 \otimes a_0 \otimes \cdots \otimes a_k \pm 1 \otimes a_1 \otimes \cdots \otimes a_0 \pm \ldots$$

Ex: $b^2 = B^2 = bB + bB = 0$. $\qquad C_{-*}(S^1) = \mathbb{k}[b]/b^2 = 0$.

Connes: B defines an S^1-action on $HH_*(A)$.



$$\Rightarrow HH_*^{S^1}(A) = HC^-(A), \quad HP(A) = HC^-(A)[u^{-1}]$$

$\qquad\qquad$ (at the level of complexes)

$HH_*(A) \otimes \mathbb{k}[[u]], \; d^{S^1} = b + uB$.

Symplectic Geometry:

$$HH_{n-*}(WF(X)) \xrightarrow{OC} SH^*(X) \xrightarrow{CO} HH^*(WF(X)).$$

symplectic cohomology

$\ni$
$[1]$

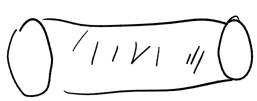
unital commutative algebra.

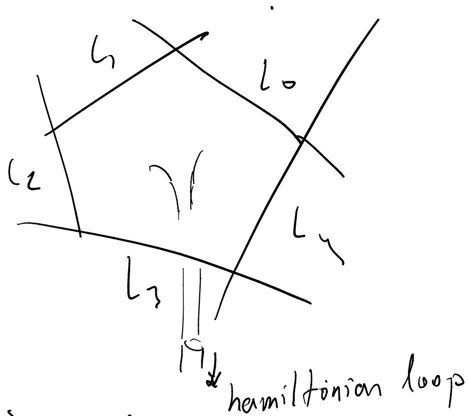
$SH^*(X) =$

[Fix H "quadratic @ ∞"
(think of $\frac{1}{2}|p|^2$ on T^*Q)]

Generators = time 1 fixed points of flow of X_H $\qquad \partial_s u + J(\partial_t u - X_H)$

d: counts cylinders.





Generation criterion: If $1 \in \mathrm{Im}(OC)$ then

corresponds to

Generation criterion: If $1 \in \mathrm{Im}(OC)$ then

X a Liouville manifold (no stops)

- OC, CO are isos
- $WF(X)$ is **smooth**!

unit corresponds to the fundamental class of X

Example: $X = T^*Q$.

$$\begin{tikzcd} HH_*(C_*(\Omega X)) \arrow[r, "OC"] \arrow[dr, "\text{Jones}"'] \arrow[rr, bend right=60, "\sim"'] & SH^*(X) \arrow[r, "OC"] \arrow[d, equal, "AS"] & HH^*(C_*(\Omega X)) \\ & H_*(\mathcal{L}X) \arrow[ur] & \end{tikzcd}$$

Manifestation of Calabi-Yau property of Fukaya category!

Cor: If we find $L_0, \dots, L_k$ st $OC|_{HH_*(WF\langle L_0,\dots,L_k\rangle)}$ hits 1 then $L_0, \dots, L_k$ generate WF!

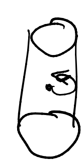
Ganatra-Pardon-Shende ! : Local-to-global principle for generation criterion.

$$\begin{tikzcd} \operatorname{hocolim} HH_*(WF(L_i)) \arrow[r, "\sim"] \arrow[d, "OC"', "\text{hit unit}"] & HH_*(WF(L_i)) \arrow[d, "OC"', "\text{have to hit unit.}"] \\ \operatorname{hocolim} SH^*(L_i) \arrow[r] & SH^*(L) \\ H^*(L_i, \partial L_i) \arrow[u, phantom, "\cup"] \arrow[r] & H^*(L) \ni 1. \arrow[u, phantom, "\Downarrow"] \end{tikzcd}$$

Algebraic Geometry:

HKR: $HH_k(\mathrm{Perf}(X)) \simeq \bigoplus_p H^{p-k}(X, \Omega^p_X)$.

OC: $HH_* \Downarrow SH^*$ $\quad B = d \Downarrow$



Φ Noncommutative space S,

HH nc-calculus 6_2 products

$\swarrow$ X hamiltonian dynamics $\qquad$ differential forms $X^\vee$ $\searrow$

... from

X hamiltonian dynamics

enumerative invariants $\Leftarrow$ 6, products $\Rightarrow$ Hodge theory

differential forms $X^\vee$

Classical mirror symmetry: Curve counts on X agree with Hodge theory on $X^\vee$.

If S comes from X or from $X^\vee$ then classical invariants of $X, X^\vee$ are captured by noncommutative invariants of S.

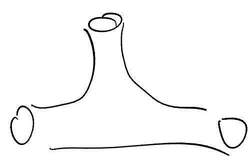
$$\begin{array}{ccc} HH^*(A) - E_2\text{-algebra.} \\ \uparrow \mathcal{CO} \qquad \qquad \uparrow S. \\ SH^*(X) - E_2\text{-algebra.} \end{array}$$

Analog of generation criterion in algebraic geometry would be:

If 1 is "

$$HH_*(\mathrm{Perf}\, X) \xrightarrow{HKR} H^*(\Omega^\circ\ )$$

Then X is smooth.

If $X =$ 

$$X^\vee = \{xy = 0\} \subset \mathbb{C}^2.$$

$\uparrow$ not smooth,

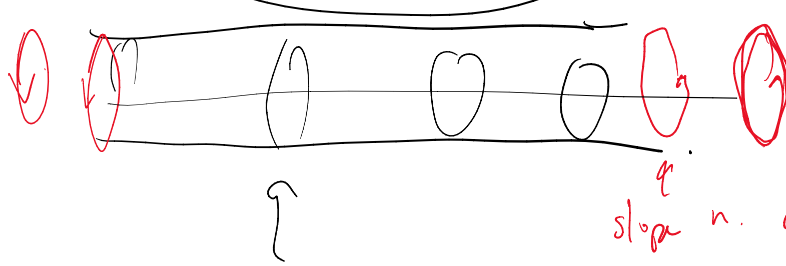
$$\mathcal{WF}(X) \simeq D^b\mathrm{Coh}(X^\vee).$$

A nice example:

$H = \frac{1}{2}|p|^2$

$$X = T^*S^1.$$

$H.$



slope n. $\longleftrightarrow$ x^n, $d(x^n)$

$$SH^*(X) = \bigoplus H_*(S^1).$$

$$\simeq \Omega^*(\mathbb{k}[x, x^{-1}]).$$

Symplectic manifold + SYZ fibration

... structure.

Symplectic manifold + SYZ fibration

$\Rightarrow$ candidate monoidal structure.

To have a (graded) A_∞ cat have to require that $2c_1 = 0$