A brief visit to flag country: reminiscences and recent results

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Yu. Manin, Grassmannians and flags in supergeometry. In: Some Problems of Modern Analysis, Moscow State University, 1984, pp 83-101.

See also Yu. Manin, Gauge Fields and Complex Geometry, Nauka, 1984

Recall the definition of a complex supermanifold $N = (N_{red}, \mathcal{O}_N)$. Let $T = T_0 \oplus T_1$ be finite-dimensional $\mathbb{Z}_2$-graded complex vector space

$$d = d_0|d_1, \quad d_0 \leqslant \dim T_0, \quad d_1 \leqslant \dim T_1.$$

$Gr(d,T)$: N-points of $Gr(d,T)$ = locally free subsheaves of rank d in $T \otimes \mathcal{O}_{Gr(d,T)}$, where N is an arbitrary complex supermanifold. Yu. I. showed

$$Gr(d,T)_{red} = Gr(d_0,T_0) \times Gr(d_1,T_1), \qquad \mathrm{gr}\, \mathcal{O}_{Gr(d,T)} \simeq \bigwedge\nolimits^{\bullet}(S_{+l} \otimes S_{-r} \oplus S_{-l} \otimes S_{+r}),$$

where

$$S_+ = S_a, \quad S_- = ((X \otimes \mathcal{O}_{Gr(a,X)})/S_a)^*$$

are the two non-ample tautological bundles on a usual grassmannian $Gr(a,X)$.

Already in 1981(?) the question arose:

$$\text{is } \mathcal{O}_{Gr(d,T)} \text{ isomorphic to } \bigwedge^{\bullet}(S_{+l} \otimes S_{-r} \oplus S_{-l} \otimes S_{+r}), \text{ i.e., does } Gr(d,T) \text{ split?}$$

The answer was given by Igor Skornyakov who showed that $Gr(1|1, \mathbb{C}^{2|2})$ does not split. This implies then that $Gr(d_0|d_1, T)$ splits if and only if $d_0 d_1(\dim T_0 - d_0)(\dim T_1 - d_1) = 0$.

Moreover, $Gr(d, T)$ is projective (embeds in $G(1|0, T')$ for some T') if and only if $Gr(d, T)$ splits. Igor and I characterized all projective flag supermanifolds: for instance all flag supermanifolds of maximal length are projective but do not neccessarily split.

Without details, I will mention that a Bott-Borel-Weil theory for flag supermanifolds started being developed already in the 1980's: see the recent book I.P., C. Hoyt, Classical Lie algebras at infinity, Springer Monographs in Mathematics, 2022.

However, this theory is incomplete: it concerns only "generic" line bundles of flag supermanifolds. For instance, the cohomology of all line bundles on all supergrassmannians seems not to have been systematically computed.

Recently, the cohomology of the sheaf $\mathcal{O}_{Gr(d,T)}$ was computed in S. Sam, A. Snowden, Cohomology of flag supermanifolds and resolution of determinantal ideals, arXiv 2108.00504

3. A theorem and a conjecture

The following is proved in I. P., A. Tikhomirov, Linear Ind-Grassmannians, Pure and Appl. Math Quarterly 10 (2014), 289-323.

Theorem: Let $\varphi\colon Gr(a,X) \hookrightarrow Gr(b,Y)$ be an embedding satisfying $\varphi^*\mathcal{O}_{Gr(b,Y)}(1) \simeq \mathcal{O}_{Gr(a,X)}(1)$. Then φ factors through a projective subspace of $Gr(b,Y)$ or is a standard extension as defined below.

An embedding $\varphi\colon Gr(a,X) \hookrightarrow Gr(b,Y)$ is a strict standard extension if it has the form $X_a \mapsto X_a \oplus Y_{\mathrm{fixed}}$ for some isomorphism $Y = X \oplus \overline{X}$ and some subspace $Y_{\mathrm{fixed}} \subset \overline{X}$.

A standard extension is the composition of a standard extension with a duality map

$$Gr(b,Y) = Gr(\mathrm{dim}Y - b, Y^*) \text{ or } Gr(a,X) = Gr(\mathrm{dim}X - a, X^*).$$

In view of the non-projectivity of a generic supergrassmannian, embeddings of supergrassmannians into other supergrassmannians should play a more interesting role than in commutative geometry. Note that the notion of standard extension generalizes to the super case.

Conjecture: Let $\varphi\colon Gr(d,T) \hookrightarrow Gr(f,Q)$ be an embedding of supergrassmannians inducing an isomorphism on Picard groups. Assume that $Gr(d,T)$ is not projective. Then φ is a standard extension.

Note: see also the recent paper by E. Shemyakova and T. Voronov, arXiv 1906.12011, in which a "super Plücker" embedding is defined.

4. Flag ind-varieties

Let $G = GL(\infty) = GL(E, V)$, where V is a countable-dimensional complex vector space and E is a fixed basis of V. By definition, $GL(E, V)$ is the group of automorphisms of V, each of which leaves all but finitely many elements of E fixed.

$$GL(E, V) = \varinjlim GL(n, V_n), \; V_n = \mathrm{span}(e_1, \dots, e_n).$$

Flag ind-varieties are ind-varieties G/P where P are parabolic subgroups of G such that $P \cap GL(n, V_n) = P_n$. Then

$$G/P \simeq \varinjlim \left(GL(n, V_n)/P_n\right).$$

Ivan Dimitrov and I gave a flag realization of flag ind-varieties: I. Dimitrov, I.P. Ind-varieties of generalized flags as homogeneous spaces for classical ind-groups, IMRN 2004.

The key notion is that of a generalized flag in V: this is a chain $\mathcal{F}$ of subspaces of V satisfying the following two conditions

- each element of F has an immediate successor or an immediate predecessor
- $\forall v \in V$, $v \neq 0$, we have $v \in F'' \backslash F'$ for some $F' \in \mathcal{F}$ and its immediate successor F''

A generalized flag $\mathcal{F}$ is E-compatible if each subspace in $\mathcal{F}$ is spanned by vectors of E.

The basis E determines an ind-torus $H \subset G$ (Cartan subgroup). Parabolic subgroups $P \subset G$ containing H are in $1-1$ correspondence with E-compatible generalized flags in V:

$$\mathcal{F} \leftrightarrow \mathrm{Stab}_G \mathcal{F} = P_{\mathcal{F}}$$

Moreover, $G/P_{\mathcal{F}} = Fl(\mathcal{F}, E, V)$, where $Fl(\mathcal{F}, E, V) =$
{all generalized flags $\overline{\mathcal{F}}$ isomorphic to $\mathcal{F}$ as ordered sets, such that $\mathcal{F}$ and $\overline{\mathcal{F}}$ are commensurable and $\overline{\mathcal{F}}$ is E'-compatible for a basis E' (depending on $\overline{\mathcal{F}}$) differing from E in finitely many vectors}

Analogies with the finite-dimensional super case:

- not all Borel subgroups of G are conjugate
- existence of non-ind-projective manifolds of generalized flags

In fact, $Fl(\mathcal{F}, E, V)$ is ind-projective $\Leftrightarrow$ $\mathcal{F}$ is ordered by $\mathbb{Z}_{\geqslant 0}$, $\mathbb{Z}_{\leqslant 0}$, $\mathbb{Z}$, or a finite set.

Differences with the super case:

- if $\mathcal{F} = \{0 \subset W \subset V\}$ then $Gr(W, E, V) := Fl(\mathcal{F}, E, V)$ is ind-projective.
- if $\mathcal{F}$ is maximal, then $Fl(\mathcal{F}, E, V)$ is almost never ind-projective, while flag supermanifolds of maximal length are always projective.

7. Automorphism groups

1. Finite-dimensional flag varieties:

$$\operatorname{Aut} Fl(k_1, \ldots, k_r, X) \simeq \begin{cases} PGL(X) \\ PGL(X) \rtimes \mathbb{Z}_2 \text{ if } k_i = \dim X - k_{r+1-i} \text{ for all } i \end{cases}$$

Very little dependence on $k_1, \ldots, k_r$!

Should be worked out also in the super case!

2. Ind-varieties of generalized flags:

Two easy cases. Consider $\mathbb{P}(V) := Fl(\mathcal{F}, E, V)$ for $\mathcal{F} = (0 \subset W \subset V)$, $\dim W = 1$, and $Fl(\mathcal{F}', E, V)$ for $\mathcal{F}' = (0 \subset W' \subset V)$, $\operatorname{codim}_V W' = 1$. Then

$$\operatorname{Aut} \mathbb{P}(V) = PGL(V), \;\; \operatorname{Aut} Fl(\mathcal{F}', E, V) = PGL(V_*),$$

where $V_* = \operatorname{span}\{E^*\}$ and E^* is the system dual to E.

Observation: the connected component of unity in $\operatorname{Aut} Fl(\mathcal{F}, E, V)$ is not a subgroup of $PGL(V)$.

Next case: $\operatorname{Aut} Fl(\mathcal{F}, E, V)$ for $\mathcal{F} = (0 \subset W \subset V)$, $\operatorname{codim}_V W = \dim W = \infty$.

$Gr(W, E, V) := Fl(\mathcal{F}, E, V)$ is the Sato grassmannian. To describe $\operatorname{Aut} Gr(W, E, V)$ we need a definition.

Let $X \otimes Y \to \mathbb{C}$ be a non-degenerate form. Then $Y \subset X^*$.

$M(X, Y) = \{\text{invertible } \varphi \colon X \to X \mid \varphi^*(Y) = Y\}$ is the Mackey group of the form $X \otimes Y \to \mathbb{C}$.

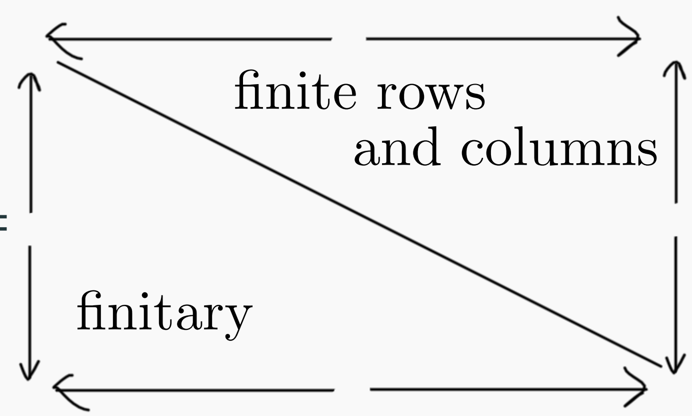
$\operatorname{Aut} Gr(W, E, V) \simeq P\left(M(U^* \oplus U, U^* \oplus U)^0\right) \rtimes \mathbb{Z}_2$ where U is a countable-dimensional vector space.

The elements of $M(U^* \oplus U, U^* \oplus U)^0$ have the following matrix form

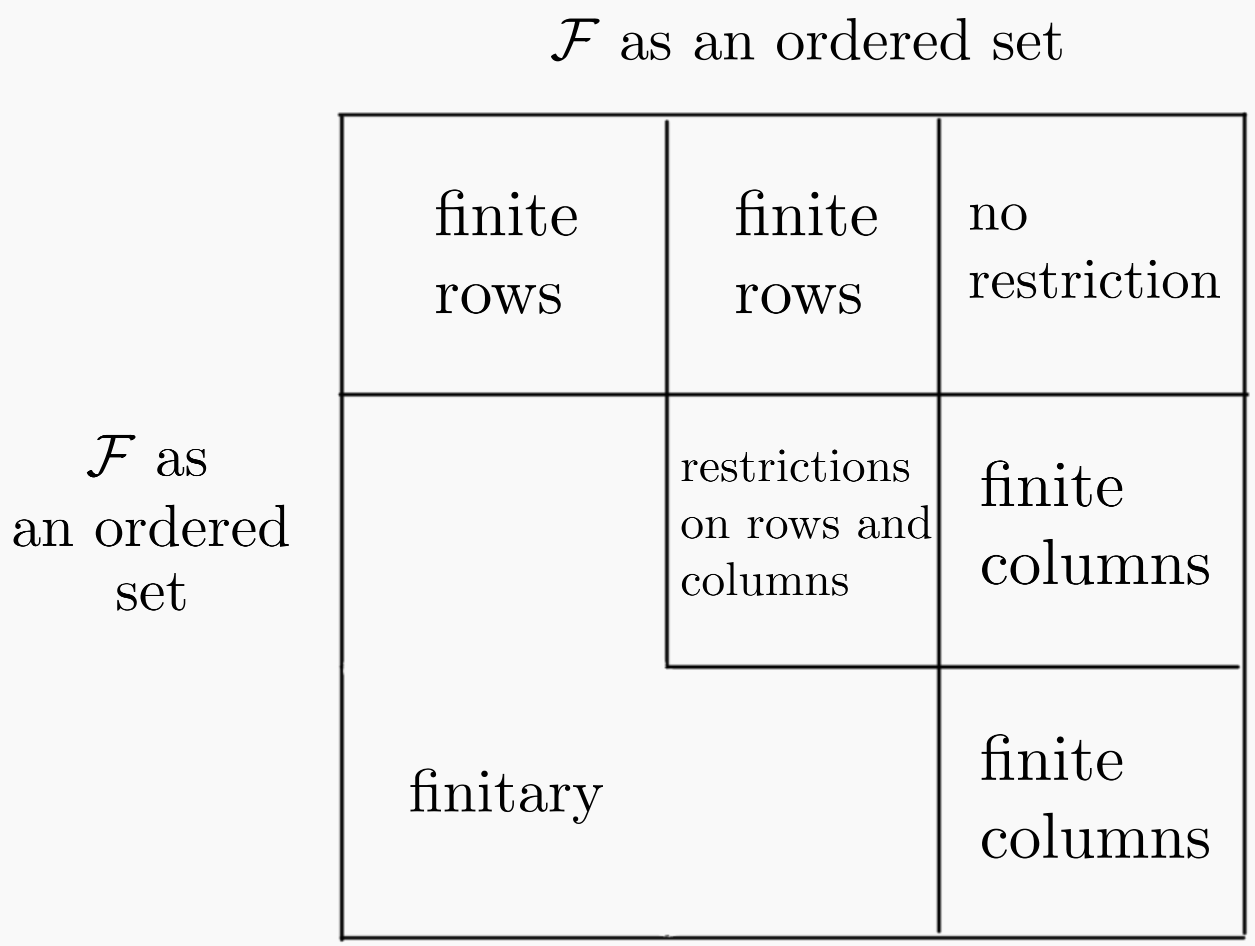
$$A = \left(\begin{array}{c|c} \text{finite rows} & \text{no restriction} \\ \hline S \text{ finitary} & \text{finite columns} \end{array} \right) \quad A^{-1} = \left(\begin{array}{c|c} \text{finite rows} & \text{no restriction} \\ \hline S' \text{ finitary} & \text{finite columns} \end{array} \right)$$

$$rkS = rkS'$$

Another example: $\mathcal{F} = \{0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset V\}$, maximal generalized flag

$\operatorname{Aut} Fl(\mathcal{F}, E, V) = \Bigg\{ A =$  and A^{-1} has the same form $\Bigg\}$

In general:



The details see in M. Ignatyev, I. P., Automorphism groups of ind-varieties of generalized flags, arXiv 2106.00989.

Conjecture. $\mathrm{Aut}\, Fl(\mathcal{F}, E, V)^{\mathrm{con}} \simeq \mathrm{Aut}\, Fl(\mathcal{F}', E', V)^{\mathrm{con}}$ if one of the following holds:

- for some $\varphi\colon V \to V$ with $\varphi(E) = E'$, $\mathcal{F}'$ and $\varphi(\mathcal{F})$ differ by adding or deleting finitely many spaces each of which yields a new finite-dimensional quotient.
- for some $\psi\colon V \to V_*$ with $\psi(E) = E'^*$, $\mathcal{F}'^{\perp}$ and $\psi(\mathcal{F})$ differ by adding or deleting finitely many spaces each of which yields a new finite-dimensional quotient.