# THREE LECTURES ON MOTIVIC COHOMOLOGY

This is a short course giving an introduction to the parallel theories of motivic cohomology furnished by Bloch's higher Chow groups and Voevodsky's motivic cohomology. We will introduce a number of motivic categories and describe how they give a framework for motivic cohomology. We will say a bit about two applications of the theory: constructions of the "motivic" spectral sequence from motivic cohomology to algebraic $K$-theory, and the proof of the Bloch-Kato conjectures relating mod $n$-motivic cohomology with mod $n$ étale cohomology. Finally, we will describe theories of motivic cohomology over a general base-scheme.

### Lecture 1: An introduction to higher Chow groups and triangulated categories of motives

In the first lecture, we will introduce Bloch's cycle complex, and Bloch's higher Chow groups. After detailing the construction, we will briefly describe the basic properties of the higher Chow groups as a Bloch-Ogus twisted duality theory: functoriality, homotopy invariance, projective bundle formula, and the crucial localization sequence. We conclude with an introduction to the relation of the higher Chow group with algebraic $K$-theory via the Chern character and the Bloch-Lichtenbaum spectral sequence.

In the second part of this lecture, we will introduces some categories of motives. We start with Grothendieck's category of Chow motives for smooth projective varieties. Next, we discuss Voevodsky's triangulated category of geometric motives and the resulting theory of motivic cohomology.

### Lecture 2: Motivic cohomology and triangulated categories of motives

We discuss Voevodsky's sheaf-theoretic version of motives, the triangulated category of effective motives. We describe Voevodsky's embedding theorem and the categorical description of Suslin homology. We discuss the comparison theorem identifying motivic cohomology with Bloch's higher Chow groups, and the relation of mod $n$-motivic cohomology with étale cohomology.

We use the comparison theorem to embed the category of Chow motives in the category of geometric motives, and briefly describe realization functors associated to de Rham cohomology and Betti (co)homology.

### Lecture 3: Applications and perspectives

We introduce the construction by Morel and Voevodsky of $\mathbb{A}^1$-homotopy theory and briefly describe how Voevodsky's triangulated category of motives fits into this picture via the motivic Eilenberg-MacLane spectrum. We discuss two applications of $\mathbb{A}^1$-homotopy theory to motivic cohomology: the slice spectral sequence and the solution of the Bloch-Kato conjectures.

The second part of this lecture is devoted to extensions of the theory to more general base-schemes. This includes the Déglise-Cisinski category of Beilinson motives, and it's use by Spitzweck in constructing a motivic cohomology spectrum over an arbitrary base. We conclude with a description of Hoyois' construction of this motivic cohomology spectrum, which relies on the theory of framed correspondences.

## Contents



## 0. An overview of these lectures

We can start the story with Grothendieck's idea of motives of smooth projective varieties as giving a framework for the "universal Weil cohomology for smooth projective varieties over an algebraically closed field". This has never been carried out fully successfully, although the construction of motives for an adequate equivalence relation makes perfectly good sense and is very much in use today.

The next step was the introduction of what would now be known as "oriented cohomology with additive formal group law" but really goes under the name of Bloch-Ogus twisted duality theory. This consists of assigning a bi-graded ring $X \mapsto \oplus H^a(X, \Lambda(b))$ for $X$ smooth over $k$, together with Gysin isomorphisms $H^a_Z(X, \Lambda(b)) \cong H^{a-2c}(Z, \Lambda(b-c))$ for smooth closed codimension $Z \subset X$ (plus other stuff, like 1st Chern lf line bundles ...). The need for the second grading (twist) arose from concrete examples, like $\ell$-adic étale cohomology over a non-algebraically closed field. Beilinson pointed out that Gillet's theory of Chern classes made Adams graded algebraic $K$-theory the universal theory with $\mathbb{Q}$-coefficients, and the search was soon on for the universal integral theory.

Beilinson and Lichtenbaum produced axioms for sheaves of complexes $\Gamma(q)$ (Beilinson used Zariski sheaves, Lichtenbaum used étale sheaves) that would produce the universal theory by taking hypercohomology. The axioms included the known relation with algebraic $K$-theory (enchanced to an Atiyah-Hirzebruch type spectral sequence giving an integral relation), a connection with the classical Chow ring and with Milnor $K$-theory of fields, as well as the identification of the mod $n$ theory with a truncated version of étale cohomology.

Another axiom, the so-called Beilinson-Soul'e vanishing conjectures, may have inspired Beilinson to reframe this conjectural universal theory as arising as Ext-groups in an abelian category of "motivic sheaves" over each scheme $X$, with a six functor formalism on the derived category. In doing so, the universal Bloch-Ogus cohomology theory became rechristened as "motivic cohomology".

This vision has not been completely realized, but nearly so. The first construction of complexes that (partly) satisfied the Beilinson-Lichtenbaum axioms was constructed by Bloch, with his cycle complex and higher Chow groups. The categorical framework was attacked by many, but Voevodsky came up with the most successful version, at least of a triangulated category that has the "feel" of the derived category of the conjectural category of motivic sheaves over a field. Remarkably, the motivic cohomology that arises out the this categorical construction agrees with Bloch's higher Chow groups. Work of others (Cisinski-Déglise, Ayoub, Röndigs-Østvær, Spitzweck, Hoyois,...) has extended Voevodsky's triangulated category to a very satisfying theory (actually several different constructions that yield more or less the same theory) over arbitrary base-schemes.

In these three lectures, I will begin in the first lecture with a description of Bloch's cycle complex and the first of Voevodsky's categorical construction. This is a very straightforward matter, but has the disadvantage that the motivic cohomology resulting from it is nearly impossible to compute or even relate to other theories, such as the classical Chow groups.

In the second lecture, I will discuss Voevodsky's great innovation, which was to put this rather naive theory into a sheaf-theoretic context, and using a mixture of fairly classical construction with the cohomological methods made available by the use of sheaves, was able to realize the abstract motivic cohomology as arising as the hypercohomolgy of a sheaf of complexes, just as envisaged by Beilinson and Lichtenbaum. Moreover, a similar combination of geometry and sheaf theory allowed him (with Friedlander and Suslin) to connect the motivic cohomology with

Bloch's higher Chow groups, and his category with Grothendieck's category of Chow motives.

In the third lecture, I will look at the interplay of the categories of motives with a finer theory: motivic homotopy theory. This will clarify the relation of motivic cohomology with algebraic $K$-theory, as well as providing many of the tools used by Voevodsky and others to verify the Beilinson-Lichtenbaum conjectures describing the relation of mod $n$ motivic cohomology with mod $n$ étale cohomology. In addition, this gives a framework for expanding the theory over a field to one over an arbitrary base-scheme, using methods that are interesting in their own right.

A partial table of the historical development of motivic cohomology and its categorical framework:

| Contributor | Cohomology | Category |
|---|---|---|
| Weil | Weil cohomology | |
| Grothendieck | Universal Weil cohomology | Motives for smooth projective varieties |
| Bloch, Ogus | Twisted duality theories | |
| Beilinson, Lichtenbaum | Universal cohomology, motivic complexes | |
| Beilinson | Motivic cohomology | Abelian categories of motivic sheaves |
| Bloch | Cycle complexes, higher Chow groups | |
| Suslin | algebraic singular complex, algebraic homology | |
| Voevodsky | motivic cohomology on $\mathbf{Sm}_k$ | triangulated categories of motives over a field |
| Morel-Voevodsky | generalized motivic cohomology | motivic homotopy categories |
| Röndigs-Østvær | motivic cohomology on $\mathbf{Sm}_k$ | modules over $H\mathbb{Z}$ |
| Cisinski-Déglise | rational motivic cohomology over a base | Beilinson motives |
| Spitzweck | motivic cohomology spectrum in $\mathrm{SH}(B)$ | modules over $H\mathbb{Z}$ |
| Hoyois | motivic cohomology spectrum in $\mathrm{SH}^{frame}(B)$ | modules over $H\mathbb{Z}$ |

## 1. Lecture 1: An introduction to higher Chow groups, motivic cohomology and the triangulated category of motives

### 1.1. **Higher Chow groups.**

1.1.1. *Motivation.* The idea behind Bloch's construction of his higher Chow groups is to give a "resolution" of the classical Chow group of dimension $q$ cycles on a variety (reduced finite-type $k$-scheme) $X$ modulo rational equivalence $\mathrm{CH}_q(X) := Z_q(X)/R_q(X)$, with $Z_q(X)$ the group of dimension $q$ algebraic cycles

$$Z_q(X) := \oplus_{W\subset X,\text{integral, closed, dimension }q}\mathbb{Z}\cdot W$$

and $R_q(X) \subset Z_q(X)$ the subgroup of cycles rationally equivalent to zero. This latter group has many equivalent definitions, but for us, let's use

$$R_q(X) := \{i_0^*(W) - i_1^*(W) \mid W \in Z_{q+1}(\mathbb{A}^1 \times X)_{0,1}\}$$

Here $Z_{q+1}(\mathbb{A}^1\times X)_{0,1}$ is the subgroup of $Zq+1(\mathbb{A}^1\times X)$ freely generated by integral $W \subset \mathbb{A}^1 \times X$ not contained in $\{0,1\} \times X$ and $i_0, i_1 : X \to \mathbb{A}^1 \times X$ are the 0- and 1-sections. For such $W$, the pullback cycles $i_t^*(W)$, $t = 0, 1$ are well defined dimension $q$ cycles on $X$.

In other words, we have the coequalizer sequence

$$Z_{q+1}(\mathbb{A}^1 \times X)_{0,1} \underset{i_0^*}{\overset{i_1^*}{\rightrightarrows}} Z_q(X) \to \mathrm{CH}_q(X)$$

1.1.2. *Bloch's cycle complex and higher Chow groups.* Bloch's idea is to extend this sequence to the left, using cycles on algebraic versions of the topological $n$-simplices.

**Definition 1.1.** 1. For $n \geq 0$ an integer let

$$\Delta_k^n \subset \mathbb{A}_k^{n+1} := \operatorname{Spec} k[t_0, \ldots, t_n]$$

be the affine hyperplane defined by $\sum_i t_i = 1$. A *codimension c face* of $\Delta_k^n$ is a subscheme $F \subset \Delta_k^n$ defined by equations of the form $t_{i_1} = \ldots = t_{i_c} = 0$, with $0 \leq i_1 < \ldots < i_c \leq n$, $c \leq n$.
2. Let $X$ be a $k$-variety. For $n, q \geq 0$, let $z_q(X, n) \subset Z_{q+n}(\Delta_k^n \times X)$ be the subgroup freely generated by integral closed subschemes $W$ such that, for each codimension $c$ face $F$ of $\Delta_k^n$, each irreducible coomponent of $W \cap X \times F$ has dimension $q+n-c$.

The collection of varieties $\{\Delta_k^n\}_{n \geq 0}$ forms a smooth, cosimplicial scheme over $k$. Letting **Ord** be the category with objects the finite ordered sets $[n] := \{0, \ldots, n\}$ (with the standard order) and maps the order-preserving maps of sets, we have

$$\Delta_k : \mathbf{Ord} \to \mathbf{Sm}_k$$

by $\Delta_k([n]) = \Delta_k^n$ and for $g : [n] \to [m]$ an order-preserving map, the map

$$\Delta_k(g) : \Delta_k^n \to \Delta_k^m$$

is the affine-linear map

$$\Delta_k(g)(t_0, \ldots, t_n) = (\Delta_k(g)_0, \ldots, \Delta_k(g)_m); \quad \Delta_k(g)_j = \sum_{i \in g^{-1}(j)} t_i$$

Note that $\Delta_k(g)$ factors as a smooth map $p_(g) : \Delta_k^n \to F$ followed by the inclusion $i_F : F \hookrightarrow \Delta_k^m$ for some face $F$, so we have a well-defined pullback

$$g^* := (\mathrm{Id}_X \times \Delta_k(g))^* : z_q(X, m) \to z_q(X, n)$$

This gives us the simplicial abelian group $[n] \mapsto z_q(X, n)$, with corresponding homological complex $(z_q(X, *), d)$; as usual, $d_n : z_q(X, n) \to z_q(X, n-1)$ is the map $\sum_{i=0}^n (-1)^i \delta_i^n$, with $\delta_i^n : [n-1] \to [n]$ the unique injective order-preserving map with $i$ not in the image.

**Definition 1.2.** Let $X$ be a $k$-variety. The complex $(z_q(X, *), d)$ is Bloch's *dimension q cycle complex* and the homology is Bloch's *higher Chow group*

$$\mathrm{CH}_q(X, n) := H_n((z_q(X, *), d))$$

If $X$ is equi-dimensional over $k$ of dimension $d$, we may index by codimension, defining $(z^q(X, *), d) := (z_{d-q}(X, *), d)$, $\mathrm{CH}^q(X, n) = \mathrm{CH}_{d-q}(X, n)$; we extend the notation to $X$ locally equi-dimensional over $k$ (e.g., $X$ smooth over $k$) by taking the direct sum over the connected components of $X$.

1.1.3. *Basic properties.* Here is a list of the fundamental properties of the cycle complexes and higher Chow groups.

**Theorem 1.3.** *1. Let $z_q(X,0) \to \mathrm{CH}_q(X,0)$, $Z_q(X) \to \mathrm{CH}_q(X)$ be the canonical surjections. There is a unique isomorphism $\mathrm{CH}_q(X,0) \cong \mathrm{CH}_q(X)$ making*

$$\begin{array}{ccc} z_q(X,0) & = & Z_q(X) \\ \downarrow & & \downarrow \\ CH_q(X,0) & \xrightarrow{\sim} & \mathrm{CH}_q(X) \end{array}$$

*commute.*
*2. Let $f : Y \to X$ be a proper map of $k$-varieties. The push-forward maps $(\mathrm{Id}\times f)_* : Z_{n+q}(\Delta^n_k \times Y) \to Z_{n+q}(\Delta^n_k \times X)$ define a functorial pushfoward map of complexes*

$$f_* : (z_q(Y,*), d) \to (z_q(X,*), d)$$

*and on homology, $f_* : \mathrm{CH}_q(Y,n) \to \mathrm{CH}_q(X,n)$.*
*3. Let $f : Y \to X$ be a flat map of $k$-varieties, of relative dimension $d$. The flat pullback maps*

$$(\mathrm{Id} \times f)^* : Z_{q+n}(\Delta^n_k \times X) \to Z_{q+n+d}(\Delta^n_k \times Y)$$

*give rise to well-defined functorial maps*

$$f^* : z_q(X,*) \to z_{q+d}(Y,*)$$

*and on homology $f^* : \mathrm{CH}_q(X,n) \to \mathrm{CH}_{q+d}(Y,n)$.*
*3'. Let $f : Y \to X$ be a morphism with $X$ smooth and $Y$ locally equi-dimensional over $k$. The "partially defined" pullback maps*

$$(\mathrm{Id} \times f)^* : Z^q(\Delta^n_k \times X)_{\mathrm{Id}\times f} \to Z^q(\Delta^n_k \times Y)$$

*give rise to well-defined functorial maps*

$$f^* : z^q(X,*) \to z^q(Y,*)$$

*in the derived category $D^-(\mathbf{Ab})$ and on the homology $f^* : \mathrm{CH}^q(X,n) \to \mathrm{CH}^q(Y,n)$.*
*4. Let $p : E \to X$ be an affine-space bundle (torsor for a vector bundle over $X$) of relative dimension $d$. Then $p^* : \mathrm{CH}_q(X,n) \to \mathrm{CH}_{q+d}(E,n)$ is an isomorphism.*
*5. For $X, X'$ $k$-varieties, we have well-defined external products*

$$\boxtimes_{X,X'} : z_q(X,*) \otimes_{\mathbb{Z}} z_{q'}(X',*) \to z_{q+q'}(X\times Y,*)$$

*in $D^-(\mathbf{Ab})$, inducing external products on homology $\boxtimes_{X,X'} : \mathrm{CH}_q(X,n)\otimes\mathrm{CH}_{q'}(X',n') \to \mathrm{CH}_{q+q'}(X\times X', n+n')$.*

*For $f : Y \to X$, $f' : Y' \to X'$ we have*

$$(f\times f')_* \circ \boxtimes_{Y,Y'} = \boxtimes_{X,X'} \circ (f_* \otimes f'_*)$$

*if $f, f'$ are proper.*

$$(f\times f')^* \circ \boxtimes_{X,X'} = \boxtimes_{Y,Y'} \circ (f^* \otimes f'^*)$$

*if $f$ and $f'$ are flat or if $X$, $X'$ are smooth and $Y, Y'$ are locally equi-dimensional.*
*5'. If $X$ is smooth over $k$, let $\delta_X : X \to X\times X$ be the diagonal. Then $\cup_X := \delta^* \circ \boxtimes_{X,X}$ makes $\mathrm{CH}^*(X,*) := \oplus_{q,n\geq 0}\mathrm{CH}^q(X,n)$ a bi-graded $\mathbb{Z}$-algebra, commutative*

*in $q$ and graded-commutative in $n$. Moreover, if $f : Y \to X$ is a morphism with $Y$ smooth, then we have the projection formula*

$$f_* \circ (f^* \cup \mathrm{Id}_Y) = \mathrm{Id}_X \cup f_*.$$

*6. (Projective bundle formula). Let $V \to X$ be a rank $n+1$ vector bundle over some smooth $X$, with associated projective space bundle $\mathbb{P}(V) := \mathrm{Proj}(\mathrm{Sym}^* V) \xrightarrow{q} X$. We have $\xi : c_1(\mathcal{O}(1)) \in \mathrm{CH}^1(\mathbb{P}(V))$, and via $q^*$, $\mathrm{CH}^*(\mathbb{P}(V), *)$ is a bi-graded $\mathrm{CH}^*(X, *)$-module. Then $\mathrm{CH}^*(\mathbb{P}(V), *)$ is a free $\mathrm{CH}^*(X, *)$-module with basis $1, \xi, \ldots, \xi^n$.*

*Finally, all these structures restrict to the classical classical ones for $\mathrm{CH}_*(-)$ and $\mathrm{CH}^*(-)$ via the isomorphism (1).*

A crucial property of the higher Chow groups is the long exact *localization sequence*. Let $X$ be a $k$-variety, $i : W \to X$ a closed subvariety with open complement $j : U \to X$. We have the classical right-exact localization sequence

$$\mathrm{CH}_q(W) \xrightarrow{i_*} \mathrm{CH}_q(X) \xrightarrow{j^*} \mathrm{CH}_q(U) \to 0$$

**Theorem 1.4** (Bloch, 1992)**.** *With $i : W \to X$, $j : U \to X$ as above, the sequence*

$$z_q(W, *) \xrightarrow{i_*} z_q(X, *) \xrightarrow{j^*} z_q(U, *)$$

*extends canonically to a distinguished triangle*

$$z_q(W, *) \xrightarrow{i_*} z_q(X, *) \xrightarrow{j^*} z_q(U, *) \to z_q(W, *)[1]$$

*in $D^-(\mathbf{Ab})$, giving rise to the long exact localization sequence*

$$\ldots \to \mathrm{CH}_q(W, n) \xrightarrow{i_*} \mathrm{CH}_q(X, n) \xrightarrow{j^*} \mathrm{CH}_q(U, n) \xrightarrow{\partial_n} \mathrm{CH}_q(W, n-1) \to \ldots$$
$$\to \mathrm{CH}_q(W, 0) \xrightarrow{i_*} \mathrm{CH}_q(X, 0) \xrightarrow{j^*} \mathrm{CH}_q(U, 0) \to 0$$

*extending the classical sequence via the isomorphism $\mathrm{CH}_q(-, 0) \cong \mathrm{CH}_q(-)$.*

Some remarks:

1. Proper pushforward and flat pullback are straightforward. The homotopy property for the projection $\mathbb{A}^1 \times X \to X$ is proven by identifying $(\mathbb{A}^1, i_0, i_1)$ with $(\Delta^1, \delta^1_1, \delta^0_1)$ and using an algebraic version of the standard subdivision of $\Delta^n \times \Delta^1$. One needs to be careful here as the "new" faces introduced in $\Delta^n \times \Delta^1$ to make the subdivision need to be taken into account when defining the suitable cycle groups on $\Delta^n \times \Delta^1 \times X$, and one needs an elementary moving lemma to make the proof of homotopy invariance of homology from topology work in this setting.

The contraviant functoriality for morphisms to a smooth variety relies on a version of the classical Chow moving lemma, in case $X$ is affine or projective. To get this to work in the general smooth case, one needs to use the localization property to prove a Mayer-Vietoris sequence, which reduces to the affine case.

2. By a standard argument, the localization theorem gives a long exact Mayer-Vietoris sequence: For $X = U \cup V$, $U, V$ open, we have $W := X \setminus U = V \setminus (U \cap V)$ and one pieces the two resulting localization triangles together to give a Mayer-Vietoris distinguished triangle

$$z_q(X, *) \xrightarrow{(j_U^*, j_V^*)} z_q(U, *) \oplus z_q(V, *) \xrightarrow{j^{V*}_{U\cap V} - j^{U*}_{U\cap V}} z_q(U \cap V, *) \xrightarrow{\partial} z_q(X, *)[1]$$

The Mayer-Vietoris sequence is then used to extend the basic homotopy invariance, for $X \times \mathbb{A}^1 \to X$, to the general version, and is similarly used to prove the projective bundle theorem, starting from the case of a product (which is proven using localization for the standard cell decomposition of $\mathbb{P}^n$, plus homotopy invariance).

3. The localization theorem uses essentially new ideas introduced by Bloch. The basic problem is as follows. Let $i : W \to X$, $j : U \to X$ be as in the statement and let $z_q(U,*)_j \subset z_q(U,*)$ be the image $j^*(z_q(X,*))$. This gives us the degreewise exact sequence of complexes

$$0 \to z_q(W,*) \xrightarrow{i_*} z_q(X,*) \xrightarrow{j^*} z_q(U,*)_j \to 0$$

so it suffices to show that the inclusion $z_q(U,*)_j \to z_q(U,*)$ is a quasi-isomorphism (it is easy to construct examples for which this is a proper inclusion). The problem is that a subvariety $W \in z_q(U,n)$ may have closure $\bar{W}$ in $Z_{q+n}(\Delta^n \times X)$ that no longer intersects all faces properly. To deal with this, Bloch shows that after a sequence of blow-ups of faces $U \times F$ in $\Delta^n \times U$ (which creates a new "polyhedral" version $U \times \tilde{\Delta}^n$ of $\Delta^n \times U$, the inverse image of $W$ has closure in $\tilde{\Delta}^n \times X$ that intersects all "faces" properly. Then by a clever subdivision argument, Bloch puts this blow-up of $W$ back in $z_q(U,n)$ and shows that it actually lands in $z_q(U,n)_j$. Finally, Bloch shows that this transformation defines a retraction $z_q(U,*) \to z_q(U,*)_j$ in the derived category, giving an inverse to the inclusion $z_q(U,*)_j \to z_q(U,*)$.

1.1.4. *Relations with algebraic $K$-theory.* For a variety $X$, the Grothendieck group $G_0(X)$ of coherent sheaves on $X$ is closely related to the Chow groups by taking the topological filtration: Let $\mathrm{Coh}_q(X) \subset \mathrm{Coh}(X)$ be the full subcategory consisting of coherent sheaves with support in dimension $\leq q$ and let $F_q^{\mathrm{top}}G_0(X) \subset G_0(X)$ be the image of $K_0(\mathrm{Coh}_q(X))$. It is well-known that sending a dimension $q$ subvariety $W$ to $[\mathcal{O}_W] \in F_q^{\mathrm{top}}G_0(X)$ descends to a well-defined homomorphism

$$\mathrm{cl}_q : \mathrm{CH}_q(X) \to \mathrm{gr}_q^{\mathrm{top}}G_0(X)$$

and this map is in fact an isomorphism modulo torsion.

Bloch and Lichtenbaum extended this construction to the higher algebraic $G$-theory, $G_*(X) = G_*(\mathrm{Coh}(X))$ on the one side, and the higher Chow group on the other, by considering a "topological filtration" of thge cosimplicial scheme $[n] \mapsto \Delta^n_k \times X$. One lets $F_q \,\mathrm{Coh}(X,n) \subset \mathrm{Coh}(\Delta^n_k \times X)$ be the full subcategory of coherent sheaves $\mathcal{F}$ with support satisfying

$$\mathrm{dimsupp}\,(\mathcal{F}) \cap X \times F \leq n+q-c$$

for each fact $F \subset \Delta^n$ of codimension $c$. Applying Quillen $K$-theory gives a tower of simplicial spectra

$$[n] \mapsto (\ldots \to K(F_p \,\mathrm{Coh}(X,n)) \to K(F_{p+1} \,\mathrm{Coh}(X,n)) \to \\ \ldots \to K(F_{\dim X}(\mathrm{Coh}(X,n))) = K(\mathrm{Coh}(\Delta^n_k \times X)))$$

and associated tower of total spectra

$$\ldots \to F_p \,\mathrm{Coh}(X,*) \to F_{p+1} \,\mathrm{Coh}(X,*) \to \ldots \to F_{\dim X}(\mathrm{Coh}(X,*)) = \mathrm{Coh}(\Delta^*_k \times X)$$

Moreover, the natural map $\mathrm{Coh}(X) \to \mathrm{Coh}(\Delta^*_k \times X)$ induces a weak equivalence

$$G(X) := K(\mathrm{Coh}(X)) \xrightarrow{\sim} G\,\mathrm{Coh}(\Delta^* \times X)$$

The tricky part is to show that

$$\pi_n(K(F_p \operatorname{Coh}(X,*))/K(F_{p-1} \operatorname{Coh}(X,*))) \cong \mathrm{CH}_p(X,n)$$

but having done this (Bloch-Lichtenbaum for $X = \operatorname{Spec} F$, Friedlander-Suslin in general with a somewhat different construction, Levine in general with this construction), one has the *Atiyah-Hirzebruch spectral sequence*

$$E^1_{p,q} = \mathrm{CH}_p(X, p+q) \Rightarrow G_{p+q}(X)$$

For $X$ smooth, we have $G_*(X) = K_*(X)$, and one usually reindexes to an $E_2$ spectral sequence looking like

$$E_2^{p,q} = \mathrm{CH}^{-q}(X, -p-q) \Rightarrow K_{-p-q}(X)$$

Gillet's theorem of Chern classes for higher $K$-theory works to give Chern class maps (for $X$ smooth)

$$c_{q,n} : K_n(X) \to \mathrm{CH}^q(X,n)$$

extending the classical Chern classes $c_q : K_0(X) \to \mathrm{CH}^q(X)$. Using these, one can show that the AH spectral sequence degenerates rationally.

1.1.5. *Milnor $K$-theory.* For a field $F$, the Milnor $K$-theory of $F$ is the $\mathbb{N}$-graded commutative ring

$$K^M_*(F) = (F^\times)^{\otimes *}/(\{a \otimes (1-a) \mid a \in F \setminus \{0,1\}\})$$

For $x = (x_1, \ldots, x_n) \in (F^\times)^n$, let $\Sigma(x) = \sum_i x_i$. Nestorenko-Suslin showed that the map sending $(F^\times)^n \setminus \{\Sigma(x) \mid x \in (F^\times)^n\}$ to $z^n(F,n)$ by

$$x = (x_1, \ldots, x_n) \mapsto (\frac{-1}{1-\Sigma(x)}, (\frac{x_1}{1-\Sigma(x)}, \ldots, (\frac{x_n}{1-\Sigma(x)})$$

descends to give an isomorphism $K^M_n(F) \to \mathrm{CH}^n(F,n)$. This fact was also proven by Totaro, using a "cubical" construction of the higher Chow groups. These maps for varying $n$ yield an isomorphism of graded rings $K^M_*(F) \cong \oplus_n \mathrm{CH}^n(F,n)$.

## 1.2. **Triangulated categories of motives.**

1.2.1. *Grothendieck-Chow motives.* Grothendieck constructed a series of categories of motives for smooth projective varieties, depending on a choice of a so-called adequate equivalence relation for algebraic cycles. His ultimate goal was to construct the universal Weil cohomology theory using purely geometric means. This is still an open question. However, using the finest adequate relation, namely, rational equivalence, one arrives at the category of Chow motives, which was later expanded by Voevodsky to form the basis for a truly successful theory of motivic cohomology.

To fit better with Voevodsky's construction, we define a homological version of Chow motives.

**Definition 1.5.** The category of *Chow correspondences* over a field $k$, $\mathrm{Cor}_{\mathrm{CH}}(k)$, has objects $[X]$ for each smooth projective variety $[X]$ over $k$ and morphism groups (for irreducible $X$)

$$\mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}(k)}([X],[Y]) := \mathrm{CH}_{\dim X}(X \times Y);$$

in general, write $X = \amalg_i X_i$ as a disjoint union of its irreducible components and define $\mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}(k)}([X],[Y]) = \prod_i \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}(k)}([X_i],[Y])$.

The composition law is that of *correspondences*: For $W_1 \in \mathrm{CH}_{\dim X}(X \times Y)$ and $W_2 \in \mathrm{CH}_{\dim Y}(Y \times Z)$ define

$$W_2 \circ W_1 := p_{X \times Z*}(p^*_{X \times Y}(W_1) \cdot p^*_{Y \times Z}(W_2))$$

The identity on $[X]$ is given by the diagonal cycle $\Delta_X \in \mathrm{CH}_{\dim X}(X \times X)$.

Note that we need $Y$ to be projective for $p_{X \times Z*}$ to be defined, and we need $X, Y$ and $Z$ to be smooth, and we need to pass from cycles to cycles mod rational equivalence, for the intersection product to be defined.

We have the functor $\mathbf{SmProj}_k \to \mathrm{Cor}_{\mathrm{CH}}(k)$ sending $X$ to $[X]$ and $f : X \to Y$ to the graph $\Gamma_f \in \mathrm{CH}_{\dim X}(X \times Y)$.

$\mathrm{Cor}_{\mathrm{CH}}(k)$ is an additive category with $\oplus$ induced by disjoint union in $\mathbf{SmProj}_k$. Similarly, product over $k$ makes $\mathrm{Cor}_{\mathrm{CH}}(k)$ a tensor category. The next step is to adjoin formal summands; this is a formal process where one has objects $([X], \alpha)$ with $\alpha : [X] \to [X]$ in $\mathrm{Cor}_{\mathrm{CH}}(k)([X], [X])$ an idempotent endomorphism. This gives the category of effective Chow motives $\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}$, with

$$Hom_{\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}}((X, \alpha), (Y, \beta)) := \beta \cdot \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}(k)}(X, Y) \cdot \alpha$$

with composition induced from $\mathrm{Cor}_{\mathrm{CH}}(k)$; this is embedded in $\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}$ by sending $X$ to $(X, \mathrm{Id})$. In $\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}$, we have the *Lefschetz motive* $\mathbb{L}$, this being the summand of $[\mathbb{P}^1]$ given by $0 \times \mathbb{P}^1 \in \mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1)$.

**Definition 1.6.** The category $\mathrm{Mot}_{\mathrm{CH}}(k)$ of Chow motives is defined by inverting the endofunctor $- \otimes \mathbb{L}$ on $\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}$:

$$\mathrm{Mot}_{\mathrm{CH}}(k) := \mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}[(- \otimes \mathbb{L})^{-1}]$$

Write $M\langle n\rangle$ for $M \otimes \mathbb{L}^{\otimes n}$. Note that we have the *cancellation property*: For $M, N \in \mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}$, the canonical map

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}}(M, N) \to \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}}(M\langle 1\rangle N\langle 1\rangle)$$

is an isomorphism. Since the objects of $\mathrm{Mot}_{\mathrm{CH}}(k)$ are all of the form $M\langle n\rangle$ for $M \in \mathrm{Mot}^{\mathrm{eff}}_{\mathrm{CH}}(k)$ and $n \in \mathbb{Z}$, and

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)}(M_1\langle n_1\rangle, M_2\langle n_2\rangle) = \mathrm{colim}_m \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}}(M_1\langle n_1 + m\rangle, M_2\langle n_2 + m\rangle)$$

the canonical functor $\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}} \to \mathrm{Mot}_{\mathrm{CH}}(k)$ is a fully faithful embedding.

*Remark* 1.7. The Chow groups $\mathrm{CH}_n(X)$ are represented in $\mathrm{Mot}_{\mathrm{CH}}(k)$ by

$$\mathrm{CH}_n(X) = \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)}(\mathbb{L}^{\otimes n}, [X])$$

and $\mathrm{CH}^n(X)$ is similarly represented by

$$\mathrm{CH}^n(X) = \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)}([X], \mathbb{L}^{\otimes n})$$

$\mathrm{Mot}_{\mathrm{CH}}(k)$ into a tensor category in which each non-zero object has a dual. For instance the dual of $[X]$ is $[X]\langle -\dim X\rangle$ and the dual of $([X], \alpha)$ is $([X]\langle -\dim X\rangle, \alpha^t \otimes \mathrm{Id})$, where $\alpha^t = \tau_*(\alpha)$, with $\tau : X \times X \to X \times X$ the symmetry.

1.2.2. *Voevodsky's geometric motives.* Somewhat in line with Bloch's idea of resolving the Chow groups, Voevodsky defines his triangulated category of geometric motives by replacing cycles modulo rational equivalence with cycles; he also expands the basic objects to all smooth $k$-varieties, not just the smooth projective ones. To do this, and still have a well-defined composition law using correspondences, he needed the following lemma

**Lemma 1.8.** *Let $X$, $Y$ and $Z$ be smooth irreducible $k$-varieties, and take $W_1 \in Z_{\dim X}(X \times Y)$ and $W_2 \in Z_{\dim Y}(Y \times Z)$. Suppose that for each irreducible component $C_1$ of the support of $W_1$, and for each each irreducible component $C_2$ of the support of $W_2$ the projections $C_1 \to X$ and $C_2 \to Y$ are finite and surjective. Then*

- *The intersection product $W := p^*_{XY}(W_1) \cdot p^*_{YZ}(W_2)$ on $X \times Y \times Z$ is a well-defined cycle of dimension* $\dim X$.
- *The support of $W$ is finite over $X \times Z$, so we have a well-defined cycle $p_{XZ*}(W) \in Z_{\dim X}(X \times Z)$*
- *For each irreducible component $C_3$ of $p_{XZ*}(W)$, the projection $C_3 \to X$ is finite and surjective*

With this lemma, we can follow Voevodsky in defining the category $\mathrm{Cor}(k)$ of *finite correspondences on* $\mathbf{Sm}_k$

**Definition 1.9.** Let $k$ be a field. The category $\mathrm{Cor}(k)$ has objects $[X]$ for $X \in \mathbf{Sm}_k$ and morphism group $\mathrm{Hom}_{\mathrm{Cor}(k)}([X],[Y])$ the subgroup of $Z_{\dim X}(X \times Y)$ freely generated by subvarieties $W \subset X \times Y$ such that the projection $W \to X$ is finite, and is surjective onto an irreducible component of $X$. The composition law is given by composition of correspondences: For $W_1 \in \mathrm{Hom}_{\mathrm{Cor}(k)}([X],[Y])$ and $W_2 \in \mathrm{Hom}_{\mathrm{Cor}(k)}([Y],[Z])$ we set

$$W_2 \circ W_1 := p_{XZ*}(p^*_{XY}(W_1) \cdot p^*_{YZ}(W_2))$$

The identity $\mathrm{Id}_{[X]}$ is given by the diagonal on $X \times X$.

Sending $X \in \mathbf{Sm}_k$ to $[X]$ and a morphism $f : X \to Y$ to its graph gives a faithful embedding $[-] : \mathbf{Sm}_k \to \mathrm{Cor}(k)$, by which we consider $\mathbf{Sm}_k$ as a subcategory of $\mathrm{Cor}(k)$.

We now use methods from triangulated categories to promote $\mathrm{Cor}(k)$ to a "motivic" category. $\mathrm{Cor}(k)$ is an additive, tensor category, with $\oplus$ given by disjoint union and $\otimes$ given by product over $k$.We consider the bounded homotopy category $K^b(\mathrm{Cor}(k))$ (i.e. bounded complexes with morphisms chain homotopy classes of maps) and perform a Verdier localization.

**Definition 1.10.** The triangulated category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ of effective geometric motives is the localization of $K^b(\mathrm{Cor}(k))$ with respect to the thick subcategory generated by the following two types of complexes

- (Homotopy invariance) For $X \in \mathbf{Sm}_k$, the complex $([X \times \mathbb{A}^1] \xrightarrow{p_X} [X])$
- (Mayer-Vietoris) For $X \in \mathbf{Sm}_k$, suppose we have open subschemes $j_U : U \to X$, $j_V : V \to X$ with $X = U \cup V$, giviing the inclusions $j^U_{U\cap V} : U \cap V \to U$, $j^V_{U\cap V} : U \cap V \to V$, and the complex

$$[U \cap V] \xrightarrow{(j^U_{U\cap V}, -j^V_{U\cap V})} [U] \oplus [V] \xrightarrow{j_U + j_V} [X]$$

We let $M^{\mathrm{eff}} : \mathbf{Sm}_k \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ be the functor sending $X$ to the image of $[X]$, concentrated in degree 0, in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$

$\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is a triangulated tensor category. We have the *reduced motive* $\tilde{M}^{\mathrm{eff}}(\mathbb{P}^1)$ of $\mathbb{P}^1$, namely, the complex

$$[\mathbb{P}^1] \xrightarrow{p} [\mathrm{Spec}\, k]$$

with $[\mathbb{P}^1]$ in degree zero and $p : \mathbb{P}^1 \to \mathrm{Spec}\, k$ the projection. Define $\mathbb{Z}(1)$ by

$$\mathbb{Z}(1) := \tilde{M}^{\mathrm{eff}}(\mathbb{P}^1)[-2].$$

We define the triangulated tensor category of geometric motives, $\mathrm{DM}_{\mathrm{gm}}(k)$ by

$$\mathrm{DM}_{\mathrm{gm}}(k) := \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)[(-\otimes \mathbb{Z}(1))]^{\natural},$$

where $(-)^{\natural}$ means adjoin summands corresponding to idempotent endomorphisms.

The objects $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$, $n \in \mathbb{Z}$ are called the pure Tate motives. Concretely, the objects of $\mathrm{DM}_{\mathrm{gm}}(k)$ are of the form $M(m) := M \otimes \mathbb{Z}(m)$ for some $m \in \mathbb{Z}$, and with morphism groups

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(m), N(n))\\ &\qquad := \mathrm{colim}_{r \geq \max(-m,-n)} \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M(r+m), N(r+n))\end{aligned}$$

$\mathbb{Z}(0) := M(\mathrm{Spec}\, k)$ is the unit for the tensor structure.

A fundamental theorem of Voevodsky (the *cancellation theorem*) reduces the study of $\mathrm{DM}_{\mathrm{gm}}(k)$ to $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$

**Theorem 1.11.** *For all $M, N \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$, the stabilization map*

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M, N) \to \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M(1), N(1))$$

*is an isomorphism. In particular, the canonical functor $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)$ is a fully faithful embedding.*

Let $M : \mathbf{Sm}_k \to \mathrm{DM}_{\mathrm{gm}}(k)$ be the composition of $M^{\mathrm{eff}}$ with the canonical functor. The reason for inverting $-\otimes \mathbb{Z}(1)$ is the same as for the case of Chow motives: if $k$ has characteristic zero, then $M(X)$ is a dualizable object in $\mathrm{DM}_{\mathrm{gm}}(k)$ for all $X \in \mathbf{Sm}_k$; if $k$ has characteristic $p > 0$, this also holds after inverting $p$. Just as for $\mathrm{Mot}_{\mathrm{CH}}$, if $X$ is smooth and projective of dimension $d$, then

$$M(X)^{\vee} = M(X)(-d)[-2d].$$

Note that the object corresponding to the Lefschetz motive is $\tilde{M}(\mathbb{P}^1) = \mathbb{Z}(1)[2]$.

1.2.3. *Motivic cohomology.* Via $\mathrm{DM}_{\mathrm{gm}}(k)$, we have the categorical construction of motivic cohomology.

**Definition 1.12.** For $X \in \mathbf{Sm}_k$, define

$$H^p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X), \mathbb{Z}(q)[p])$$

More generally, for an arbitrary $M \in \mathrm{DM}_{\mathrm{gm}}(k)$, we set

$$H^p(M, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M, \mathbb{Z}(q)[p])$$

Immediate consequences of this construction include:

1. Functoriality. Each morphism $f : M \to N$ in $\mathrm{DM}_{\mathrm{gm}}(k)$ induces $f^* : H^p(N, \mathbb{Z}(q)) \to H^p(M, \mathbb{Z}(q))$. In particular, each $f : Y \to X$ in $\mathbf{Sm}_k$ induces $f^* := M(f)^* : H^p(Y, \mathbb{Z}(q)) \to H^p(X, \mathbb{Z}(q))$.

2. Mayer-Vietoris. Let $X = U \cup V$ be an open cover of $X \in \mathbf{Sm}_k$. Then we have the long exact Mayer-Vietoris sequence

$$\dots \to H^{p-1}(U \cap V, \mathbb{Z}(q)) \to H^p(X, \mathbb{Z}(q)) \to H^p(U, \mathbb{Z}(q)) \oplus H^p(V, \mathbb{Z}(q)) \to H^p(U \cap V, \mathbb{Z}(q)) \to \dots$$

Homotopy invariance. Let $p : \mathbb{A}^1_k \to \operatorname{Spec} k$ be the projection. For each $M \in \mathrm{DM}_{\mathrm{gm}}(k)$, the map $\mathrm{Id}_M \otimes M(p_1) : M \otimes M(\mathbb{A}^1) \to M$ and the induced map $(\mathrm{Id}_M \otimes M(p_1))^* : H^p(M, \mathbb{Z}(q)) \to H^p(M \otimes M(\mathbb{A}^1, \mathbb{Z}(q))$ is an isomorphism. In particular, $p_X^* : H^p(X, \mathbb{Z}(q)) \to H^p(X \times \mathbb{A}^1, \mathbb{Z}(q))$ is an isomorphism for all $X \in \mathbf{Sm}_k$.

Together with the Mayer-Vietoris sequence, this gives the extended homotopy property: For $p : E \to X$ an affine space bundle, the map $p^* : H^p(X, \mathbb{Z}(q)) \to H^p(E, \mathbb{Z}(q))$ is an isomorphism.

Variants:

Mod $n$ motivic cohomology. For a postive integer $n$, define $\mathbb{Z}/n(q)$ as the complex

$$\mathbb{Z}(q) \xrightarrow{\times n} \mathbb{Z}(q)$$

concentrated in degrees -1, 0, define

$$H^p(X, \mathbb{Z}/n(q)) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X), \mathbb{Z}/n(q)[p])$$

and define $H^p(M, \mathbb{Z}/n(q))$ for $M \in \mathrm{DM}_{\mathrm{gm}}(k)$ similarly. We thus have the long exact coefficient sequence

$$\dots \to H^{p-1}(M, \mathbb{Z}/n(q)) \to H^p(M, \mathbb{Z}(q)) \xrightarrow{\times n} H^p(M, \mathbb{Z}(q)) \to H^p(M, \mathbb{Z}/n(q)) \to \dots$$

and the motivic Milnor sequence

$$0 \to H^p(M, \mathbb{Z}(q))/n \to H^p(M, \mathbb{Z}/n(q)) \to H^{p+1}(M, \mathbb{Z}(q))_{n-\mathrm{torsion}} \to 0$$

Motivic cohomology with support: Let $i : Z \to X$ be a closed subscheme and $j : U \to X$ the open complement, with $X \in \mathbf{Sm}_k$. Define the motive with support $M_Z(X)$ as the complex

$$M(U) \xrightarrow{M(j)} M(X)$$

in degrees -1, 0, and

$$H^p_Z(X, \mathbb{Z}(q)) := H^p(M_Z(X), \mathbb{Z}(q)); \ H^p_Z(X, \mathbb{Z}/n(q)) := H^p(M_Z(X), \mathbb{Z}/n(q)).$$

This gives us the distinguished triangle

$$M(U) \xrightarrow{M(j)} M(X) \to M_Z(X) \to M(U)[1];$$

applying $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(-, \mathbb{Z}(q)[*])$ gives the long exact motivic cohomology sequence

$$\dots \to H^p_Z(X, \mathbb{Z}(q)) \to H^p(X, \mathbb{Z}(q)) \to H^p(U, \mathbb{Z}(q)) \to H^{p+1}_Z(X, \mathbb{Z}(q)) \to \dots$$

One can also define motivic homology:

$$H_p(X, \mathbb{Z}) := \mathrm{Hom}(\mathbb{Z}(0), M(X)[p]); \ H_p(M, \mathbb{Z}) := \mathrm{Hom}(\mathbb{Z}(0), M[p]).$$

We will revisit this construction later on, in the context of *Suslin homology*.

## 2. Lecture 2: Triangulated categories of motivic sheaves

### 2.1. **The category of effective motivic sheaves.**

2.1.1. *Presheaves and sheaves with transfer.* Voevodsky realized that it is usually impossible to compute morphisms in a localization. To understand the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$, he constructed a sheaf-theoretic version, $\mathrm{DM}^{\mathrm{eff}}(k)$, that contains $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ as the full subcategory of compact objects. We follow his treatment, detailed in [**?**], but with some refinements introduced later on by Cisinski-Déglise [**?**]. Mainly this involves replacing Voevodsky's use of bounded above derived categories $D^-$ with the unbounded versions.

**Definition 2.1.** 1. A *presheaf with transfers* (PST) on $\mathbf{Sm}_k$ is an additive functor

$$P : \mathrm{Cor}(k)^{\mathrm{op}} \to \mathbf{Ab}$$

We let $\mathrm{PST}(k)$ denote the category of such additive functors.
2. A PST $P$ is a *Nisnevich sheaf with transfers* (NST) if the restriction of $P$ to $\mathbf{Sm}_k \subset \mathrm{Cor}(k)$ is a Ninevich sheaf.
3. For $X \in \mathbf{Sm}_k$, we let $\mathbb{Z}^{\mathrm{tr}}(X)$ denote the representable functor $\mathrm{Hom}_{\mathrm{Cor}(k)}(-, X)$ on $\mathrm{Cor}(k)$.

Note that $\mathbb{Z}^{\mathrm{tr}}(X)$ is an NST, and we consider the NSTs as a full subcategory $\mathrm{NST}(k)$ of $\mathrm{PST}(k)$. $\mathrm{PST}(k)$ is an abelian category, with exactness determined objectwise. Here are some basic facts about these categories

**Proposition 2.2.** *1.* $\mathrm{NST}(k)$ *is a full abelian subcategory of* $\mathrm{PST}(k)$*, and the inclusion* $\mathrm{NST}(k) \hookrightarrow \mathrm{PST}(k)$ *admits the exact left adjoint* $P \mapsto P_{\mathrm{Nis}}$ *of Nisnevich sheafification.*
*2. Each* $P \in \mathrm{PST}(k)$ *admits the canonical surjection*

$$\oplus_{X \in \mathbf{Sm}_k, \alpha \in P(X)} \mathbb{Z}^{tr}(X) \to P$$

*via the Yoneda isomorphism* $\mathrm{Hom}_{\mathrm{PST}(k)}(\mathbb{Z}^{tr}(X), P) = P(X)$*. Applying this to the kernel of the above map and iterating gives the canonical left resolution*

$$\mathcal{L}_\bullet(P) \to P \to 0$$

*with each* $\mathcal{L}_n(P)$ *a direct sum of representable PSTs.*
*3. Define* $\mathbb{Z}^{tr}(X) \otimes^{tr} \mathbb{Z}^{tr}(Y) := \mathbb{Z}^{tr}(X \times Y)$*, and extend to arbitrary PSTs by*

$$P \otimes^{tr} Q := H_0(\mathcal{L}_\bullet(P) \otimes^{tr} \mathcal{L}_\bullet(Q))$$

*This makes* $\mathrm{PST}(k)$ *into an abelian tensor category. Extend this to* $\mathrm{NST}(k)$ *by sheafification:*

$$N \otimes^{tr}_{\mathrm{Nis}} M := (N \otimes^{tr} M)_{\mathrm{Nis}}$$

*This makes* $\mathrm{NST}(k)$ *into an abelian tensor category.*
*4. Let* $N$ *be an NST. Then for each* $n$*, the cohomology presheaf* $X \mapsto H^n(X_{\mathrm{Nis}}, N)$ *has a canonical structure of a PST.*

The property (4) is not at all obvious.

A crucial property enjoyed by a PST is that of *homotopy invariance* and for NSTs that of *strict homotopy invariance*

**Definition 2.3.** 1. A PST $P$ is homotopy invariant if the map $p_X^* : P(X) \to P(X \times \mathbb{A}^1)$ is an isomorphism for all $X \in \mathbf{Sm}_k$.
2. An NST $N$ is strictly homotopy invariant if for each $n$ the PST $H^n((-)_{\mathrm{Nis}}, N)$ is homotopy invariant, i.e., for each $X \in \mathbf{Sm}_k$, the map $p_X^* : H^n(X_{\mathrm{Nis}}, N) \to$

$H^n((X\times\mathbb{A}^1)_{\mathrm{Nis}}, N)$ is an isomorphism. Let $\mathrm{HI}(k)\subset\mathrm{NST}(k)$ be the full subcategory of strictly homotopy invariant NSTs.

Parallel to the definition of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$, one defines $\mathrm{DM}^{\mathrm{eff}}(k)$ as a localization. Let $\mathcal{T}$ be a triangulated category admitting arbitrary (set-indexed) direct sums. Recall that a *localizing* subcategory of $\mathcal{T}$ is a full subcategory, closed under arbitrary (set-indexed) direct sums; a localizing subcategory is automatically a thick subcategory in the sense of Verdier, in other words, closed under direct summands.

**Definition 2.4.** $\mathrm{DM}^{\mathrm{eff}}(k)$ is defined to be the localization of $D(\mathrm{NST}(k))$ by the localizing subcategory generated by objects of the form $\mathrm{Cone}(K_*\otimes\mathbb{Z}^{\mathrm{tr}}(\mathbb{A}^1_k))\xrightarrow{\mathrm{Id}_{K^*}\otimes\mathbb{Z}^{\mathrm{tr}}(p)} K_*)$, where $p:\mathbb{A}^1_k\to\mathrm{Spec}\,k$ is the projection. Let $Q: D(\mathrm{NST}(k))\to\mathrm{DM}^{\mathrm{eff}}(k)$ be the quotient functor.

The category $D_{\mathbb{A}^1}(\mathrm{NST}(k))$ is defined as the full subcategory of the derived category $D(\mathrm{NST}(k))$ with objects the complexes $K^*$ whose cohomology sheaves $h^n(K^*)_{\mathrm{Nis}}$ are strictly homotopy invariant.

2.1.2. *The Suslin complex.* The category $\mathrm{PST}(k)$ has an internal Hom with

$$\mathcal{H}om(P,Q)(X) := \mathrm{Hom}_{\mathrm{PST}(k)}(P\otimes^{\mathrm{tr}}\mathbb{Z}^{\mathrm{tr}}(X), Q)$$

This defines internal Hom functors on $\mathrm{NST}(k)$, $C(\mathrm{PST}(k))$, $C(\mathrm{NST}(k))$ and on $D(\mathrm{NST}(k))$.

Given a smooth cosimplicial scheme $[n]\mapsto D^n$ and $K^*\in C^-(\mathrm{PST}(k))$ this gives us the simplicial object $[n]\mapsto\mathcal{H}om(\mathbb{Z}^{\mathrm{tr}}(D^n), K^*)$, and the associated mapping complex $K^*(D^*)\in C^-(\mathrm{PST}(k))$, with

$$K^*(D^*)^{-n}(X) := \oplus_m K^m(X\times D^{m+n}),$$

and with differential the usual alternating sum of face maps. For $K^*\in C(\mathrm{PST}(k))$, write $K^*$ as a colimit of subcomplexes (using the canonical truncation $\tau_{\le n}$): $K^* = \mathrm{colim}_{n\to+\infty}\tau_{\le n}K^*$ and define $K^*(D^*) = \mathrm{colim}_{n\to+\infty}(\tau_{\le n}K^*)(D^*)$.

**Definition 2.5.** For $K^*\in C(\mathrm{PST}))$, define

$$C^{Sus}(K^*) := K^*(\Delta^*_k)$$

Extend to $K^*\in C(\mathrm{PST}(k))$ by taking the colimit of the $C^{Sus}(\tau_{\le n}K^*)$, where $\tau_{\le n}K^*\to K^*$ is the canonical truncation.

Using the triangulation of $\Delta^n_k\times\Delta^1_k\cong\Delta^n_k\times\mathbb{A}^1_k$ again, one shows that for $K^*\in C(\mathrm{PST}(k))$, the cohomology presheaves $h^i(C^{Sus}(K^*))$ are homotopy invariant.

**Definition 2.6.** For $K^*\in C(\mathrm{PST}(k))$, the *nth Suslin homology* $H^{\mathrm{Sus}}_n(K^*)$ is defined by

$$H^{\mathrm{Sus}}_n(K^*,\mathbb{Z}) := H_n(C^{Sus}(K^*)(\mathrm{Spec}\,k)).$$

For $X\in\mathbf{Sm}_k$, we write $H^{\mathrm{Sus}}_n(X,\mathbb{Z})$ for $H^{\mathrm{Sus}}_n(\mathbb{Z}^{\mathrm{tr}}(X),\mathbb{Z})$.

Here are the central results about homotopy invariant PSTs and the Suslin complex.

**Theorem 2.7.** *1. If $P$ is a homotopy invariant PST, then $P_{\mathrm{Nis}}$ is strictly homotopy invariant and the canonical map $P_{\mathrm{Zar}}\to P_{\mathrm{Nis}}$ is an isomorphism.*
*2.* $\mathrm{HI}(k)\subset\mathrm{NST}(k)$ *is an abelian subcategory, closed under extensions in* $\mathrm{NST}(k)$.
*3. Let $K^*\in C(\mathrm{PST}(k))$ be a complex in* $\mathrm{PST}(k)$. *Suppose that the Nisnevich*

*sheafification $K^*_{\mathrm{Nis}}$ is acyclic. Then the Zariski sheafification of the Suslin complex $C^{Sus}(K^*)_{\mathrm{Zar}}$ is also acyclic.*
*4. For $K^* \in C(\mathrm{PST}(k))$, the canonical map $K \to C^{Sus}(K^*)$ induces an isomorphism $Q(K) \to Q(C^{Sus}(K^*))$ in $\mathrm{DM}^{\mathrm{eff}}(k)$.*

We won't say anything about the proofs, except that some of the geometric input is a version of Chow's moving lemma, and the transfer structure is crucial; the analogous properties are not valid for arbitrary presheaves or Nisnevich sheaves.

2.2. **Motivic complexes.** The Suslin complex construction gives us presheaves of complexes $\mathbb{Z}(q)^*$ on $\mathbf{Sm}_k$ that will turn out to be the strictly functorial versions of Bloch's cycle complexes that satisfy (most of) the Beilinson-Lichtenbaum axioms.

**Definition 2.8.** For $q \geq 0$ be an integer, $\mathbb{Z}(q)^*$ is the presheaf of complexes on $\mathbf{Sm}_k$ defined by

$$\mathbb{Z}(q)^*(X) := C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(q))^*(X);$$

Explicitly, $\mathbb{Z}^{\mathrm{tr}}(q) := \mathbb{Z}^{\mathrm{tr}}(1)^{\otimes^{\mathrm{tr}}_{\mathrm{Nis}} q}$ and

$$\mathbb{Z}(q)^n(X) = \oplus_m \mathbb{Z}^{\mathrm{tr}}(q)^m(\Delta^{m-n} \times X)$$

*Remark* 2.9. Recall that

$$\mathbb{Z}^{\mathrm{tr}}(q)[2q] = (\mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1) \to \mathbb{Z}^{\mathrm{tr}}(\operatorname{Spec} k))^{\otimes q} \xleftarrow[\sim]{\text{q-iso}} [\ker(\mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1) \to \mathbb{Z}^{\mathrm{tr}}(\operatorname{Spec} k))]^{\otimes q},$$

with $\mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1)$ in degree 0, so $\mathbb{Z}^{\mathrm{tr}}(q)$ is quasi-isomorphic to a complex supported in degree $2q$, and thus $\mathbb{Z}(q)^*$ is quasi-isomorphic to a complex supported in degrees $\leq 2q$. Actually, we shall see that $\mathcal{H}^n(\mathbb{Z}(q)^*) = 0$ for $n > q$.

2.2.1. *The localization theorem and the embedding theorem.* The Suslin complex construction gives an effective way of understanding the localization $Q : D(\mathrm{NST}(k)) \to \mathrm{DM}^{\mathrm{eff}}(k)$. We note that $C^{Sus}(K^*)_{\mathrm{Nis}}$ is in $D_{\mathbb{A}^1}(\mathrm{NST}(k))$ for $K^*$ in $C(\mathrm{PST}(k))$.

**Theorem 2.10.** *Sending $K^* \in C(\mathrm{PST}(k))$ to $C^{Sus}(K^*)_{\mathrm{Nis}} \in D_{\mathbb{A}^1}(\mathrm{NST}(k))$ defines an exact functor*

$$RC^{\mathrm{Sus}} : D(\mathrm{NST}(k)) \to D_{\mathbb{A}^1}(\mathrm{NST}(k))$$

*that is left adjoint to the inclusion $\mathrm{DM}^{\mathrm{eff}}(k) \hookrightarrow D(\mathrm{NST}(k))$. Moreover, $RC^{\mathrm{Sus}}$ factors through the localization $Q : D(\mathrm{NST}(k)) \to D_{\mathbb{A}^1}(\mathrm{NST}(k))$ and defines an equivalence of $D_{\mathbb{A}^1}(\mathrm{NST}(k))$ with $\mathrm{DM}^{\mathrm{eff}}(k)$, and*

$$RC^{\mathrm{Sus}}(\mathbb{Z}^{tr}(X)) = C^{\mathrm{Sus}}(\mathbb{Z}^{tr}(X))_{\mathrm{Nis}}$$

*for all $X \in \mathbf{Sm}_k$.*

We henceforth identify $\mathrm{DM}^{\mathrm{eff}}(k)$ with the subcategory $D_{\mathbb{A}^1}(\mathrm{NST}(k))$ of $D(\mathrm{NST}(k))$ via $RC^{\mathrm{Sus}}$.

Via the localization functor $Q = RC^{\mathrm{Sus}} : D(\mathrm{NST}(k)) \to \mathrm{DM}^{\mathrm{eff}}(k)$, the tensor structure $\otimes^{\mathrm{tr}}_{\mathrm{Nis}}$ on $D(\mathrm{NST}(k))$ induces a tensor structure on $\mathrm{DM}^{\mathrm{eff}}(k)$, making $\mathrm{DM}^{\mathrm{eff}}(k)$ a tensor triangulated category, with internal Hom.

We let $\mathbb{Z}(n) = C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n))_{\mathrm{Nis}}$, $\mathbb{Z}(X) := C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}}$ and note that $\mathbb{Z}(0)$ is the unit for the tensor structure on $\mathrm{DM}^{\mathrm{eff}}(k)$. For an object $M$ of $\mathrm{DM}^{\mathrm{eff}}(k)$, we write $M(n)$ for $M \otimes \mathbb{Z}(n)$.

**Corollary 2.11.** *1. For $X \in \mathbf{Sm}_k$ and $n \in \mathbb{Z}$, we have a canonical isomorphism*

$$H_n^{\mathrm{Sus}}(X,\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(0)[n], \mathbb{Z}(X))$$

*Moreover $H_n^{\mathrm{Sus}}(X,\mathbb{Z}) = 0$ for $n < 0$.*
*2. Suppose $X \in \mathbf{Sm}_k$ is a union of open subschemes $U, V$. Then we have a long exact Mayer-Vietoris sequence*

$$\ldots \to H_n^{\mathrm{Sus}}(U\cap V,\mathbb{Z}) \to H_n^{\mathrm{Sus}}(U,\mathbb{Z}) \oplus H_n^{\mathrm{Sus}}(V,\mathbb{Z}) \to H_n^{\mathrm{Sus}}(X,\mathbb{Z}) \xrightarrow{\partial} H_{n-1}^{\mathrm{Sus}}(U\cap V,\mathbb{Z}) \to \ldots \to H_0^{\mathrm{Sus}}(X,\mathbb{Z}) \to 0$$

*Proof.* (1) uses the adjoint property:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(0)[n], \mathbb{Z}(X)) &= \mathrm{Hom}_{D_{\mathbb{A}^1}(\mathrm{NST}(k))}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(\mathrm{Spec}\, k))[n], C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X)))\\
&= \mathrm{Hom}_{D(\mathrm{NST}(k))}(\mathbb{Z}^{\mathrm{tr}}(\mathrm{Spec}\, k)[n], C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}})\\
&= H_n(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))(\mathrm{Spec}\, k))\\
&= H_n^{\mathrm{Sus}}(X,\mathbb{Z})
\end{aligned}$$

Since $C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))(\mathrm{Spec}\, k)$ is concentrated in homological degrees $\geq 0$, we have $H_n^{\mathrm{Sus}}(X,\mathbb{Z}) = 0$ for $n < 0$.

For (2), we have the right exact sequence in $\mathrm{PST}(k)$

$$0 \to \mathbb{Z}^{\mathrm{tr}}(U\cap V) \to \mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(V) \xrightarrow{j_{U*}+j_{V*}} \mathbb{Z}^{\mathrm{tr}}(X)$$

but the last map is not in general a surjective map of presheaves, for example, if $X$ is irreducible and $U$ and $V$ are proper open subsets, then the diagonal in $\mathbb{Z}^{\mathrm{tr}}(X)(X) \subset Z_{\dim X}(X \times X)$ is not in the image. However, $\mathbb{Z}^{\mathrm{tr}}(Y)$ is a Nisnevich sheaf for all $Y \in \mathbf{Sm}_k$ and we claim that the map $\mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(V) \to \mathbb{Z}^{\mathrm{tr}}(X)$ is a surjective map in $\mathrm{NST}(k)$.

Indeed, the points in the Nisnevich topology are hensel rings $\mathcal{O}^h_{Y,y}$ for $y \in Y \in \mathbf{Sm}_k$. Given $Z \in \mathbb{Z}^{\mathrm{tr}}(X)(\mathcal{O}^h_{Y,y})$, we have the restriction $Z_y \in \mathbb{Z}^{\mathrm{tr}}(X)(k(y))$, and

$$|Z_y| = \{z_1, \ldots, z_s\}$$

with the $z_i$ closed points in $X_{k(y)}$. We can arrange the $z_i$ so that $z_1, \ldots, z_r$ is in $U_{k(y)}$ and $z_{r+1}, \ldots, z_s$ is in $V_{k(y)}$. Since $\mathcal{O}^h_{Y,y}$ is hensel, we can write the support of $Z$ as a disjoint union

$$|Z| = \amalg_{i=1}^s Z_i$$

with $Z_i \cap X_{k(y)} = z_i$ (as closed subset). Since $\mathcal{O}^h_{Y,y}$ is local, this says that $Z_i \subset U_{\mathcal{O}^h_{Y,y}}$ for $i = 1, \ldots, r$ and $Z_i \subset V_{\mathcal{O}^h_{Y,y}}$ for $i = r+1, \ldots, s$, thus $Z$ is in the image of $\mathbb{Z}^{\mathrm{tr}}(U)(\mathcal{O}^h_{Y,y}) \oplus \mathbb{Z}^{\mathrm{tr}}(V)(\mathcal{O}^h_{Y,y}) \to \mathbb{Z}^{\mathrm{tr}}(X)(\mathcal{O}^h_{Y,y})$.

Letting $\overline{\mathbb{Z}}^{\mathrm{tr}}(X)$ denote the presheaf image of $j_{U*} + j_{V*}$, we thus have the exact sequences in $\mathrm{PST}(k)$

$$0 \to \mathbb{Z}^{\mathrm{tr}}(U\cap V) \to \mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(V) \xrightarrow{j_{U*}+j_{V*}} \overline{\mathbb{Z}}^{\mathrm{tr}}(X) \to 0$$

$$0 \to \overline{\mathbb{Z}}^{\mathrm{tr}}(X) \to \mathbb{Z}^{\mathrm{tr}}(X) \to \mathbb{Z}^{\mathrm{tr}}(X)/\overline{\mathbb{Z}}^{\mathrm{tr}}(X) \to 0$$

and $(\mathbb{Z}^{\mathrm{tr}}(X)/\overline{\mathbb{Z}}^{\mathrm{tr}}(X))_{\mathrm{Nis}} = 0$.

By Theorem 2.7(3), $C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X)/\overline{\mathbb{Z}}^{\mathrm{tr}}(X))(\operatorname{Spec} k)$ is acyclic, and as $C^{\mathrm{Sus}}(-)$ transforms exact sequences in $\mathrm{PST}(k)$ to termwise exact sequences in $C^{-}(\mathrm{PST}(k))$, we see that

$$C^{\mathrm{Sus}}(\overline{\mathbb{Z}}^{\mathrm{tr}}(X))(k) \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))(k)$$

is a quasi-isomorphism and thus

$$C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U \cap V))(k) \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U))(k) \oplus C^{\mathrm{Sus}}((\mathbb{Z}^{\mathrm{tr}}(V))(k) \xrightarrow{j_{U*}+j_{V*}} C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))(k)$$

extends to a distinguished triangle in $D^{-}(\mathbf{Ab})$. This yields the desired Mayer-Vietoris sequence by taking homology. □

We have the evident map of triangulated categories $K^b(\mathrm{Cor}(k)) \to D(\mathrm{NST}(k))$ sending $[X]$ to $\mathbb{Z}^{\mathrm{tr}}(X)$, and the localization map $q : K^b(\mathrm{Cor}(k)) \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

**Theorem 2.12** (Embedding theorem)**.** *There is a unique exact functor $i : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \to \mathrm{DM}^{\mathrm{eff}}(k)$ sending $M^{\mathrm{eff}}(X)$ to $\mathbb{Z}(X) := C^{\mathrm{Sus}}(\mathbb{Z}^{tr}(X))_{\mathrm{Nis}}$ for each $X \in \mathbf{Sm}_k$, and making the diagram*

$$\begin{array}{ccc} K^b(\mathrm{Cor}(k)) & \longrightarrow & D(\mathrm{NST}(k)) \\ \Big\downarrow q & & \Big\downarrow RC^{\mathrm{Sus}} \\ \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) & \xrightarrow{\ i\ } & D_{\mathbb{A}^1}(\mathrm{NST}(k)) \end{array}$$

*commute. Moreover $i$ is a fully faithful embedding with dense image.*

*Proof.* We need to show that the map $X \mapsto C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}}$ sends the two complexes defining the localization $q$ to acyclic complexes in $D_{\mathbb{A}^1}(\mathrm{NST}(k))$. The cone of the map $C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X \times \mathbb{A}^1)) \xrightarrow{p_*} C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))$ is acyclic, by the homotopy invariance of the Suslin complex construction, hence the same holds for the Nisnevich sheafification. The argument used in the proof of Corollary 2.11(2) shows that the Mayer-Vietoris sequence

$$C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U \cap V))_{\mathrm{Nis}} \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U))_{\mathrm{Nis}} \oplus C^{\mathrm{Sus}}((\mathbb{Z}^{\mathrm{tr}}(V))_{\mathrm{Nis}} \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}}$$

gives a quasi-isomorphism of $C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}}$ with the cone of the map $C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U \cap V))_{\mathrm{Nis}} \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U))_{\mathrm{Nis}} \oplus C^{\mathrm{Sus}}((\mathbb{Z}^{\mathrm{tr}}(V))_{\mathrm{Nis}}$, so the total complex of the Mayer-Vietoris sequence is zero in $D(\mathrm{NST}(k))$. This shows the existence of the exact functor $i$.

To show that $i$ is fully faithful, one relies on results of Neeman. One considers a triangulated category $\mathcal{T}$ admitting arbitrary small direct sums and its full subcategory $\mathcal{T}_0$ of compact objects. If $\mathcal{L}_0$ is a thick subcategory of $\mathcal{T}_0$, generating a localizing subcategory $\mathcal{L}$ of $\mathcal{T}$, then Neeman shows that the induced exact functor

$$\mathcal{T}_0/\mathcal{L}_0 \to \mathcal{T}/\mathcal{L}$$

is fully faithful with dense image. Taking $\mathcal{T} = D(\mathrm{PST}(k))$, we need to consider the localizing subcategory $\mathcal{L}$ generated by complexes $\mathbb{Z}^{\mathrm{tr}}(X \times \mathbb{A}^1) \to \mathbb{Z}^{\mathrm{tr}}(X)$ together with all $P \in \mathrm{PST}(k)$ such that $P_{\mathrm{Nis}} = 0$, and then show that $\mathcal{L}$ is generated by the complexes $\mathbb{Z}^{\mathrm{tr}}(X \times \mathbb{A}^1) \to \mathbb{Z}^{\mathrm{tr}}(X)$ and the "Mayer-Vietoris" complexes

$$\mathbb{Z}^{\mathrm{tr}}(U \cap V) \to \mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(V) \xrightarrow{j_{U*}+j_{V*}} \bar{\mathbb{Z}}^{\mathrm{tr}}(X)$$

Letting $\mathcal{L}'$ be the localizing subcategory generated by these two types of complexes, we need to show that if $P_{\mathrm{Nis}} = 0$, then $P$ is in $\mathcal{L}'$.

For such a $P$, consider the functors

$$H^n(X) := \mathrm{Hom}_{D(\mathrm{PST}(k))/\mathcal{L}'}(\mathbb{Z}^{\mathrm{tr}}(X), P[n]).$$

It suffices to show that $H^n(X) = 0$ for all $X \in \mathbf{Sm}_k$ and all $n$ if $P_{\mathrm{Nis}} = 0$.

Since $\mathcal{L}'$ contains all Mayer-Vietoris complexes, the family $\{H^n(X)\}$ admits long exact Mayer-Vietoris sequences for Zariski open covers, so we need only show that the Zariski sheafifications $H^n_{\mathrm{Zar}}$ of the presheaves $X \mapsto H^n(X)$ are all zero. But the $H^n(-)$ are PSTs, hence by ***, we have

$$H^n_{\mathrm{Zar}} = H^n_{\mathrm{Nis}}$$

An element $\alpha \in H^n(X)$, i.e., a morphism $\alpha : \mathbb{Z}^{\mathrm{tr}}(X) \to P[n]$ in $D(\mathrm{PST}(k))/\mathcal{L}'$, is represented by a diagram in $D(\mathrm{PST}(k))$

$$\begin{array}{ccc} \mathbb{Z}^{\mathrm{tr}}(X) & & P[n] \\ & \searrow \quad \swarrow_{g} & \\ & K & \end{array}$$

such that the cone of $g$ is in $\mathcal{L}'$. The proof of homotopy invariance for $C^{\mathrm{Sus}}$ shows that the canonical map $K \to C^{\mathrm{Sus}}(K)$ is an isomorphism in $D(\mathrm{PST}(k))/\mathcal{L}'$. Moreover, $P_{\mathrm{Nis}} = 0$ implies that $C^{\mathrm{Sus}}(P)_{\mathrm{Nis}} = 0$ in $D(\mathrm{NST}(k))$. Since the functor $C^{\mathrm{Sus}}(-)$ inverts the elements of $\mathcal{L}'$, this implies that $C^{\mathrm{Sus}}(K)_{\mathrm{Nis}} = 0$ in $D(\mathrm{NST}(k))$ as well. But then there is a Nisnevich cover $U \to X$ such that the composition

$$\mathbb{Z}^{\mathrm{tr}}(U) \to \mathbb{Z}^{\mathrm{tr}}(X) \to K \to C^{\mathrm{Sus}}(K)$$

is zero, hence the pullback of $\alpha$ to $H^n(U)$ is zero, and thus $H^n_{\mathrm{Nis}} = 0$. □

**Corollary 2.13.** *For $X \in \mathbf{Sm}_k$, $p, q \in \mathbb{Z}$, $q \geq 0$, we have a canonical isomorphism*

$$H^p(X, \mathbb{Z}(q)) \cong \mathbb{H}^p(X_{\mathrm{Nis}}, \mathbb{Z}(q)^*)$$

*natural in $X$.*

*Proof.*

$$\begin{aligned} H^p(X, \mathbb{Z}(q)) &:= \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M^{\mathrm{eff}}(X), \mathbb{Z}(q)[p]) \\ &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(X), \mathbb{Z}(q)[p]) \\ &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}}, C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(q))_{\mathrm{Nis}}[p]) \\ &\cong \mathrm{Hom}_{D(\mathrm{NST}(k))}(\mathbb{Z}^{\mathrm{tr}}(X), \mathbb{Z}(q)^*[p]) \\ &\cong \mathrm{Hom}_{D(\mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}_k))}(\mathbb{Z}(X)_{\mathrm{Nis}}, \mathbb{Z}(q)^*[p]) \\ &\cong \mathrm{Hom}_{D(\mathrm{Sh}^{\mathrm{Nis}}(X))}(\mathbb{Z}_X, \mathbb{Z}(q)^*[p]) \\ &= \mathbb{H}^p(X_{\mathrm{Nis}}, \mathbb{Z}(q)^*) \end{aligned}$$

□

2.3. **Motivic cohomology and the higher Chow groups.** We have seen an interpretation of Suslin homology as maps in $\mathrm{DM}^{\mathrm{eff}}(k)$. However, Suslin homology is not closely related to Bloch's higher Chow groups; they have quite different functoriality. There is a natural map $H^{\mathrm{Sus}}_0(X, \mathbb{Z}) \to \mathrm{CH}_0(X)$, which is always surjective, and even an isomorphism if $X$ is proper. To connected motivic cohomology with the higher Chow groups, we need to introduce some new NSTs; these play an important role in duality.

2.3.1. *Equidimensional cycles and quasi-finite cycles.*

**Definition 2.14.** Let $X$ be a finite type $k$-scheme. For an integer $r \geq 0$ and a $Y \in \mathbf{Sm}_k$, define $z_{\mathrm{equi},r}(X)(Y)$ to be the subgroup of $z_{\dim Y+r}(Y \times X)$ freely generated by integral closed subschemes $W \subset Y \times X$ such that the projection $p_Y : W \to Y$ is equi-dimensional of relative dimension $r$. Precisely, this means that for each point $y \in Y$ and each irreducible component $Z$ of $W \cap y \times X$, we have $\dim_{k(y)} Z = r$.

We let $z_{\mathrm{qfin}}(X)(Y) := z_{\mathrm{equi},0}(X)(Y)$.

*Remarks* 2.15. 1. $z_{\mathrm{qfin}}(X)(Y)$ is the subgroup of $z_{\dim Y}(Y \times X)$ freely generated by integral closed subschemes $W \subset Y \times X$ that are quasi-finite over $Y$.
2. $Y \mapsto z_{\mathrm{equi},r}(X)(Y)$ extends canonically to an NST: For $Z \in \mathrm{Cor}_k(Y', Y)$ we have

$$Z^* : z_{\mathrm{equi},r}(X)(Y) \to z_{\mathrm{equi},r}(X)(Y')$$

defined by the usual formula for correspondences

$$Z^*(W) := p_{Y' \times X*}(p^*_{Y \times X}(W) \cdot p_{Y' \times Y}(Z))$$

This makes sense even for $X$ not smooth, by taking local closed embeddings of $X$ in a smooth $k$-scheme $M$, taking the intersection product on $Y' \times Y \times M$ and noting that the resulting cycle is supported in $Y' \times Y \times X$.

**Definition 2.16.** For $X \in \mathbf{Sm}_k$, define $\mathbb{Z}(X)^c$ to be the image in $\mathrm{DM}^{\mathrm{eff}}(k)$ of the NST $z_{\mathrm{qfin}}(X)$, i.e., $\mathbb{Z}(X)^c := C^{\mathrm{Sus}}(z_{\mathrm{qfin}}(X))_{\mathrm{Nis}}$.

We have the localization distinguished triangle:

**Theorem 2.17.** *Let $i : W \to X$ a closed immersion in $\mathbf{Sch}_k$, with open complement $j : U \to X$, and let $r \geq 0$ be an integer, giving the right exact sequence in* $\mathrm{NST}(k)$

$$0 \to z_{\mathrm{equi},r}(W) \xrightarrow{i_*} z_{\mathrm{equi},r}(X) \xrightarrow{j^*} z_{\mathrm{equi},r}(U)$$

*Then the induced sequence*

$$C^{\mathrm{Sus}}(z_e qui(W))_{\mathrm{Nis}} \xrightarrow{i_*} C^{\mathrm{Sus}}(z_e qui(X))_{\mathrm{Nis}} \xrightarrow{j^*} C^{\mathrm{Sus}}(z_e qui(U))_{\mathrm{Nis}}$$

*extends canonically to a distinguished triangle in* $\mathrm{DM}^{\mathrm{eff}}(k)$ *(after inverting* $\mathrm{char} k$ *if this is positive).*

For the proof, we need the extension of Theorem 2.7(3) to the setting of the *cdh topology.* This is the Grothendieck topology on $\mathbf{Sch}_k$ generated by the Nisnevich topology and the coverings given by "abstract blow-up squares": a cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & Y \\ \downarrow & & \downarrow f \\ X' & \xrightarrow{i} & X \end{array}$$

with $i$ and $i'$ closed immersions, $f$ proper and inducing an isomorphism $f_0 : Y \setminus Y' \to X \setminus X'$. For such a square, the map $X' \amalg Y \to X$ is a cdh cover.

We have the fundamental result:

**Theorem 2.18.** *Take $K \in C(\mathrm{PST})$ and suppose that the cdh-sheafification $K_{\mathrm{cdh}}$ is acyclic Then $C^{\mathrm{Sus}}(K)_{\mathrm{Zar}}$ is acyclic (after inverting* $\mathrm{char} k$ *if this is $> 0$).*

To apply this to the localization theorem, note that for $Y \in \mathbf{Sm}_k$ and $z \in z_{\mathrm{equi},r}(U)(Y) \subset Z_{r+\dim Y}(U \times Y)$. there is a blow-up $Y' \to Y$ of $Y$ (with $Y'$ smooth in characteristic zero) such that the proper transform $z'$ of $z$ to $Z_{r+\dim Y}(U \times Y')$ has closure in $X \times Y'$ that is in $z_{\mathrm{equi},r}(X)(Y')$. In other words $\mathrm{coker}(j^*)_{\mathrm{cdh}} = 0$, which shows that $j^*(C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X)))_{\mathrm{Nis}}) \hookrightarrow C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(U))_{\mathrm{Nis}}$ is a quasi-isomorphism, as desired. In positive characteristic, one needs to use alterations.

We consider the Suslin complex $C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X))$. In homological degree $n$, $C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X))_n(k)$ is the subgroup of $Z_{n+r}(\Delta^n_k \times X)$ freely generated by integral closed subschemes $W \subset \Delta^n_k \times X$ that are equi-dimensional of dimension $r$ over $\Delta^n_k$. In particular each such $W$ intersects $F \times X$ properly for each face $F$ of $\Delta^n_k$, so we have a natural inclusion of (homological) complexes

$$C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(X))(k) \hookrightarrow z_r(X,*)$$

2.3.2. *Moving lemmas of Suslin and Friedlander-Lawson.*

**Theorem 2.19.** *In the following, we invert* $\mathrm{char} k$ *if* $\mathrm{char} k > 0$.
*1. Suppose $X$ is a finite-type $k$-scheme. Then the inclusion $C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(X))(k) \hookrightarrow z_r(X,*)$ is a quasi-isomorphism.*
*2. $\mathbb{Z}(X)^c \in \mathrm{DM}^{\mathrm{eff}}(k)$ is in the image of an object $M^c(X)$ of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$.*
*3. There are canonical isomorphisms*

$$C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X)) \cong \underline{\mathcal{H}om}(\mathbb{Z}(r)[2r], \mathbb{Z}(X)^c)$$

*in* $\mathrm{DM}^{\mathrm{eff}}(k)$.
*4. For $X \in \mathbf{Sm}_k$ of pure dimension $d$ over $k$, we have*

$$M(X)^\vee = M^c(X)(-d)[-2d]$$

*in* $\mathrm{DM}_{\mathrm{gm}}(k)$.

Some comments on (1). The proof is in two parts. For $X$ affine, Suslin constructs an explict homotopy, i.e., a geometric moving lemma, that "moves" cycles in $z_r(X,n)$ to cycles in $z_{\mathrm{equi},r}(X)(\Delta^n)$. For this, he takes a closed embedding $X \hookrightarrow \mathbb{A}^N_k$ for some $N$ and reduces to the case $X = \mathbb{A}^N$. Let $\partial\Delta^n \subset \Delta^n$ be the union of the codimension one faces, a divsor defined by $h_n := t_0 \cdot t_1 \cdots t_n = 0$. Suslin constructs inductively in $n$ maps $\Phi_n : \Delta^n \times \mathbb{A}^N \to \Delta^n \times \mathbb{A}^N$ over $\mathbb{A}^N$ with the property that $\Phi_n \circ (\delta^n_i \times \mathrm{Id}) = (\delta^n_i \times \mathrm{Id}) \circ \Phi_{n-1}$ for all $i = 0, \ldots, n$, $n \geq 1$, starting with $\Phi_0 = \mathrm{Id}_{\mathbb{A}^N}$ This implies that the maps $\Phi_{n-1}$ on the codimension one faces of $\Delta^n$ fit together to give a map $\partial\Phi_n : \partial\Delta^n \times \mathbb{A}^N \to \partial\Delta^n \times \mathbb{A}^N$ over $\mathbb{A}^N$, $\partial\Phi_n(t,y) = (\partial\phi_n(t,y), y)$. Since $\partial\Delta^n \times \mathbb{A}^N$ is a closed subscheme of the affine space $\Delta^n \times \mathbb{A}^N \cong \mathbb{A}^{n+N}$, we can extend $\partial\phi_n$ to a map $\widetilde{\partial\phi}_n : \Delta^n \times \mathbb{A}^N \to \Delta^n \times \mathbb{A}^N$. Let $p_n$ be a map $p_n : \Delta^n \to \mathbb{A}^N$ and define $\Phi_n$ by

$$\Phi_n(t,y) := (\widetilde{\partial\phi}_n(t,y) + h_n(t) \cdot p(t), y)$$

Suslin shows that for a given finite set of cycles $\{Z_1, \ldots, Z_s\} \subset z_r(X,n)$, by choosing the $p_n = (p_{n,1}(t), \ldots, p_{n,N}(t))$ with the $p_{n,}$ general polynomials of sufficiently high degree (inductively in $n$), the cycles $\Phi_n(t,y)^*(Z_i)$ are all in $z_{\mathrm{equi},r}(X)(\Delta^n)$. Using this, he shows that, for each finitely generated subcomplex $z_r(X,*)_{\mathcal{W}} \subset z_r(X,*)$, there is a map of complexes

$$\Phi^*_{\mathcal{W}} : z_r(X,*)_{\mathcal{W}} \to C_*\mathrm{Sus}(z_{\mathrm{equi},r}(X))(k)$$

such that the composition of $\Phi^*_{\mathcal{W}}$ with the inclusion $C_*\mathrm{Sus}(z_{\mathrm{equi},r}(X))(k) \hookrightarrow z_r(X,*)$ is homotopic to the inclusion $z_r(X,*)_{\mathcal{W}} \hookrightarrow z_r(X,*)$. This shows that $C_*\mathrm{Sus}(z_{\mathrm{equi},r}(X))(k) \hookrightarrow z_r(X,*)$ is a quasi-isomorphism, at least for $X$ affine.

Now take $X$ to be a finite type $k$-scheme and we prove (1) by noetherian induction, the case dimension zero being trivially true. Let $Y \subset X$ be a proper closed subscheme such that $X \setminus Y$ is affine. Applying Theorem 2.17 and evaluating at $\mathrm{Spec}\, k$, this gives us the commutative diagram of distinguished triangles in $D(\mathbf{Ab})$

$$\begin{array}{ccccccc}
C_*^{\mathrm{Sus}}(z_{\mathrm{equi},r}(Y))(k) & \xrightarrow{i_*} & C_*^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X))(k) & \xrightarrow{j^*} & C_*^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X\setminus Y))(k) & \to & C_*^{\mathrm{Sus}}(z_{\mathrm{equi},r}(Y))(k)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
z_r(Y,*) & \xrightarrow{i_*} & z_r(X,*) & \xrightarrow{j^*} & z_r(X\setminus Y,*) & \longrightarrow & z_r(Y,*)[1]
\end{array}$$

the bottom row being a distinguished triangle by Bloch's moving lemma. Induction shows the 1st and 4th vertical arrows are quasi-isomorphisms and the affine case shows that the 3rd vertical arrow is a quasi-isomorphism as well.

For (3), this relies on Voevodsky's extension of the Friedlander-Lawson moving lemma. For $X \in \mathbf{Sch}_k$, $Y, U \in \mathbf{Sm}_k$, with $U$ of dimension $u$, we have the map

$$z_{\mathrm{equi},s}(X)(U \times Y) \to z_{\mathrm{equi},s+u}(X \times U)(Y)$$

by noting that if $W \subset X \times U \times Y$ is equidimensional of relative dimension $s$ over $Y \times U$, then $W$ is equidimensional of relative dimension $s+u$ over $Y$. The induced map

$$C^{\mathrm{Sus}}(z_{\mathrm{equi},s}(X))(U \times Y) \to C^{\mathrm{Sus}}(z_{\mathrm{equi},s+u}(X \times U))(Y)$$

thus gives the map of complexes of presheaves

$$\underline{\mathcal{H}om}(\mathbb{Z}^{\mathrm{tr}}(U), C^{\mathrm{Sus}}(z_{\mathrm{equi},s}(X))) \to C^{\mathrm{Sus}}(z_{\mathrm{equi},s+u}(X \times U)) \tag{2.1}$$

**Theorem 2.20.** *The map* (2.1) *defines a quasi-isomorphism in* $C(\mathrm{NST}(k))$.

We won't give a proof of this, except to note that the original result of Friedland-Lawson on "moving families of algebraic cycles of bounded degree" [**?**, Theorem 3.7] says as a special case that, given smooth (irreducible) varieties $X, Y \in \mathbf{Sm}_k$ with $Y$ projective, and $W \in Z_n(X \times Y)$ a dimension $n$ cycle with $n \geq \dim Y$, then $W$ is rationally equivalent to some $W' \in z_{equi,n-\dim Y}(X \times Y) \subset Z_n(X \times Y)$. This fact is the starting point of the proof of the above result.

Taking $U = (\mathbb{P}^1)^r$, and understanding $\mathbb{Z}^{\mathrm{tr}}(r)[2r]$ as a summand of $\mathbb{Z}^{\mathrm{tr}}((\mathbb{P}^1)^r)$ gives the isomorphism in $DM^{\mathrm{eff}}(k)$

$$\underline{\mathcal{H}om}(\mathbb{Z}(r)[2r], C^{\mathrm{Sus}}(z_{\mathrm{equi},s}(X))) \cong C^{\mathrm{Sus}}(z_{\mathrm{equi},r+s}(X)))$$

proving (3).

The proof of (4) in case $X$ projective is just by noting that $\mathbb{Z}(X)^c = \mathbb{Z}(X)$ for projective $X$, and use duality in $\mathrm{DM}_{\mathrm{gm}}(k)$. The general result (assuming resolution of singularities) follows by taking a smooth projective compactification of $X$ with normal crossing divisor as complement, using the localization and Gysin triangles (see below) and induction on dimension.

2.3.3. *Gysin triangle, the projective bundle formula and Chern classes.* Let $i : W \to X$ be a closed immersion in $\mathbf{Sch}_k$ with open complement $j : U \to X$. Taking $r = 0$ in Theorem 2.17 gives the distinguished localization triangle in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$

$$M^c(W) \xrightarrow{i_*} M^c(X) \xrightarrow{j^*} M^c(U) \xrightarrow{\partial} M^c(W)[1]$$

If $X$ and $W$ are smooth, and $W$ has codimension $c$ in $X$, we have the Gysin distinguished triangle

$$M(U) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(W)(c)[2c] \to M(U)[1]$$

The map $i^*$ gives us the map

$$i_* : H^p(W, \mathbb{Z}(q)) \to H^{p+2c}(X, \mathbb{Z}(q + c))$$

by applying $i^*$ to $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(-, \mathbb{Z}(q + c)[p + 2c])$.

We also have first Chern classes for line bundles, the projective bundle formula and the resulting theory of Chern classes for vector bundles: For $L \to Y$ a line bundle, we have $c_1(L) \in H^2(Y, \mathbb{Z}(1)) = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M(Y), \mathbb{Z}(1)[2])$ defined by

$$c_1(L) := s^* s_*(1_X)$$

where $s : X \to L$ is the 0-section and $1_X \in H^0(X, \mathbb{Z}(0))$ is the class of the map $M(X) \to M(\mathrm{Spec}\, k) = Z(0)$ induced by the structure map $p : X \to \mathrm{Spec}\, k$.

For $V \to X$ a rank $n+1$ vector bundle over $X \in \mathbf{Sm}_k$, with associated projective space bundle $q : \mathbb{P}(V) \to X$ and tautological quotient line bundle $\mathcal{O}(1)$, we have

$$M(\mathbb{P}(V)) \cong \oplus_{i=0}^n M(X)(i)[2i]$$

where one maps $M(\mathbb{P}(V))$ to $M(X)(i)[2i]$ by

$$M(\mathbb{P}(V)) \xrightarrow{\times c_1(\mathcal{O}(1))^i} M(\mathbb{P}(V))(i)[2i] \xrightarrow{q_*} M(X)(i)[2i]$$

This induces the usual isomorphism

$$H^p(\mathbb{P}(V), \mathbb{Z}(q)) \cong \oplus_{i=0}^n H^{p-2i}(X, \mathbb{Z}(q - i))$$

This extends the pushforward $i_*$ for closed immersions to pushforward for projective morphisms $f : Y \to X$ by factoring $f$ (of relative dimension $d$) as $Y \xrightarrow{i} \mathbb{P}^n \times X \xrightarrow{p} X$ with $i$ a closed immersion and $p$ the projection. Then $f_* : H^p(Y, \mathbb{Z}(q)) \to H^{p-2d}(X, \mathbb{Z}(q - d))$ is the composition of

$$i_* : H^p(Y, \mathbb{Z}(q)) \to H^{p+2n-2d}(\mathbb{P}^n \times X, \mathbb{Z}(q+n-d)) = \oplus_{i=0}^n H^{p+2n-2d-2i}(\mathbb{P}^n \times X, \mathbb{Z}(q+n-d-i))$$

with projection on the factor $i = n$,

$$\oplus_{i=0}^n H^{p+2n-2d-2i}(\mathbb{P}^n \times X, \mathbb{Z}(q + n - d - i)) \to H^{p-2d}(X, \mathbb{Z}(q - d)).$$

Finally, one has the blow-up formula: for $Z \subset X$ codimension $c$, with $X, Z$ smooth, then

$$M(\mathrm{Bl}_Z X) \cong M(X) \oplus_{i=1}^{c-1} M(Z)(i)[2i]$$

This follows by comparing the Gysin sequence for $Z \subset X$ and $E \subset \mathrm{Bl}_Z X$, where $E$ is the exceptional divisor, and using the projective bundle formula for $M(E)$.

2.3.4. *Motivic cohomology and higher Chow groups.*

**Corollary 2.21.** *For $X \in \mathbf{Sm}_k$, we have a natural isomorphism $H^p(X, \mathbb{Z}(q)) \cong \mathrm{CH}^q(X, 2q-p)$.*

*Proof.* Suppose $X$ is integral of dimension $d$ over $k$ and that $q \le d$. Then

$$\begin{aligned} H^p(X,\mathbb{Z}(q)) &:= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X), \mathbb{Z}(q)[p]) \\ &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(\mathbb{Z}(0), M(X)^\vee \otimes \mathbb{Z}(q)[p]) \\ &\cong \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(\mathbb{Z}(0), M^c(X) \otimes \mathbb{Z}(q-d)[p-2d]) \\ &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(\mathbb{Z}(d-q)[2d-2q], M^c(X)[p-2q]) \\ &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(0), \underline{\mathcal{H}om}(\mathbb{Z}(d-q)[2d-2q], M^c(X))[p-2q]) \\ &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(0), C^{\mathrm{Sus}}(z_{\mathrm{equi},d-q}(X))[p-2q]) \\ &= \mathrm{Hom}_{D(\mathrm{NST}(k))}(\mathbb{Z}^{\mathrm{tr}}(k), C^{\mathrm{Sus}}(z_{\mathrm{equi},d-q}(X))[p-2q]) \\ &= H_{2q-p}(C^{\mathrm{Sus}}(z_{\mathrm{equi},d-q}(X))(k)) \\ &= H_{2q-p}(z^q(X,*)) = \mathrm{CH}^q(X, 2q-p). \end{aligned}$$

If $q > d$, we replace $X$ with $X \times \mathbb{A}^{q-d}$ and use homotopy invariance for the higher Chow groups. □

**Corollary 2.22.** *1. For $X \in \mathbf{Sm}_k$ and $q \ge 0$ an integer, $H^{2q+i}(X, \mathbb{Z}(q)) = 0$ for $i > 0$.*
*2. For $X \in \mathbf{Sm}_k$, $H^p(X, \mathbb{Z}(0)) = 0$ for $p \ne 0$ and $H^0(X, \mathbb{Z}(0)) = H^0(X, \mathbb{Z}_{\mathrm{Zar}})$*
*3. For $X \in \mathbf{Sm}_k$,*

$$H^p(X,\mathbb{Z}(1)) = \begin{cases} \mathrm{Pic}(X) & \text{for } p = 2 \\ \Gamma(X, \mathcal{O}_X^\times) & \text{for } p = 1 \\ 0 & \text{else.} \end{cases}$$

*4. For $X \in \mathbf{Sm}_k$ and $q > 0$ an integer, $H^p(X, \mathbb{Z}(-q)) = 0$ for all $p \in \mathbb{Z}$*
*5. For $F$ a field, $H^{n+i}(F, \mathbb{Z}(n)) = 0$ for $i > 0$*

*Proof.* (1) $H^{2q+i}(X, \mathbb{Z}(q)) = \mathrm{CH}^q(X, -i) = 0$ for $i > 0$.
(2) $H^p(X, \mathbb{Z}(0)) = \mathrm{CH}^0(X, -p)$. For $X$ integral, the complex $z^0(X,*)$ is $\mathbb{Z}$ in every degree with differentials alternating between 0 and the identity, so $\mathrm{CH}^0(X,-p) = 0$ except for $p = 0$ and $\mathrm{CH}^0(X, 0) = \mathbb{Z}$.
(3) $H^p(X, \mathbb{Z}(1)) = \mathrm{CH}^1(X, 2-p)$. Bloch shows that

$$\mathrm{CH}^1(X,n) = \begin{cases} \mathrm{CH}^1(X) = \mathrm{Pic}(X) & \text{for } n = 0 \\ \Gamma(X, \mathcal{O}_X^\times) & \text{for } n = 1 \\ 0 & \text{else.} \end{cases}$$

(4) We may assume $X$ is integral.

$$H^p(X,\mathbb{Z}(-q)) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X), \mathbb{Z}(-q)[p]) = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M^{\mathrm{eff}}(X)(q)[2q], \mathbb{Z}(0)[p+2q]).$$

Note that $M^{\mathrm{eff}}(X)(q)[2q]$ is a summand of $M^{\mathrm{eff}}(X \times \mathbb{P}^q) = \oplus_{i=0}^q M(X)(i)[2i]$, and thus $H^p(X, \mathbb{Z}(-q))$ is the corresponding summand of $H^{p+2q}(X \times \mathbb{P}^q, \mathbb{Z}(0))$. This latter group is zero for $p+2q \neq 0$, and the projection $M^{\text{eff}}(X\times\mathbb{P}^q) \to M^{\text{eff}}(X)(0)[0]$ induces the isomorphism

$$p_X^* : \mathbb{Z} = H^0(X, \mathbb{Z}(0)) \to H^0(X \times \mathbb{P}^q, \mathbb{Z}(0)) = \mathbb{Z}$$

Thus $\mathrm{Hom}_{\mathrm{DM}^{\text{eff}}_{\text{gm}}(k)}(M^{\text{eff}}(X)(q)[2q], \mathbb{Z}(0)) = 0$ if $q > 0$.
(5) Changing $k$ to $F$, we have

$$H^{n+i}(F, \mathbb{Z}(n)) = \mathrm{CH}^n(\mathrm{Spec}\, F, n-i)$$

Note that $z^n(\mathrm{Spec}\, F, n-i)$ is a subgroup of $Z^n(\Delta_F^{n-i})$. But since $\Delta_F^{n-i}$ has dimension $n-i$ over $F$, there are no subvarieties of codimension $n$ on $\Delta_F^{n-i}$ if $i > 0$, so $z^n(\mathrm{Spec}\, F, n-i) = 0$ and hence $\mathrm{CH}^n(\mathrm{Spec}\, F, n-i) = 0$ for $i > 0$. □

2.3.5. *Chow motives and Voevodsky motives.*

**Corollary 2.23.** *For $X, Y$ smooth and projective over $k$, we have a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{DM}^{\text{eff}}_{\text{gm}}(k)}(M(X), M(Y)[i]) \cong\cong H^{2\mathrm{dim}Y+i}(X\times Y, \mathbb{Z}(\mathrm{dim}Y)) \cong \mathrm{CH}_{\mathrm{dim}X}(X\times Y, -i)$$

*In particular, for $i > 0$, $\mathrm{Hom}_{\mathrm{DM}^{\text{eff}}_{\text{gm}}(k)}(M(X), M(Y)[i]) = 0$. For $i = 0$,*

$$\mathrm{Hom}_{\mathrm{DM}^{\text{eff}}_{\text{gm}}(k)}(M(X), M(Y)) \cong \mathrm{CH}_{\mathrm{dim}X}(X \times Y, 0) = \mathrm{CH}_{\mathrm{dim}X}(X \times Y)$$

*and composition in $\mathrm{DM}^{\text{eff}}_{\text{gm}}(k)$ transforms to composition of correspondences.*

*Proof.* Since $Y$ is projective, we have $z_{\text{qfin}}(Y) = \mathbb{Z}^t r(Y)$, so $M^c(Y) = M(Y)$ and thus $M(Y)^\vee = M(Y)(-d)[-2d]$, where $d$ is the dimension of $Y$ over $k$. Thus

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}^{\text{eff}}_{\text{gm}}(k)}(M(X), M(Y)[i]) &= \mathrm{Hom}_{\mathrm{DM}_{\text{gm}}(k)}(M(Y)^\vee \otimes M(X), \mathbb{Z}(0)[i]) \\ &= \mathrm{Hom}_{\mathrm{DM}_{\text{gm}}(k)}(M(X \times Y), \mathbb{Z}(d)[2d+i]) \\ &= H^{2d+i}(X \times Y, \mathbb{Z}(d)) \\ &= \mathrm{CH}^d(X \times Y, -i)\end{aligned}$$

To show that composition in $\mathrm{DM}^{\text{eff}}_{\text{gm}}(k)$ corresponds to composition of correspondences (in cycles mod rational equivalence) one uses the Friedlander-Lawson moving lemma to show that $\mathrm{CH}_{\mathrm{dim}X}(X\times Y)$ is generated by cycles each component of which is finite over $X$. For such cycles, the composition as correspondences is exactly the composition in $\mathrm{Cor}(k)$, hence in $\mathrm{DM}^{\text{eff}}_{\text{gm}}(k)$. □

**Corollary 2.24.** *Sending a smooth projective $X$ to $M(X) \in \mathrm{DM}_{\text{gm}}(k)$ extends to a fully faithful embedding*

$$\mathrm{Mot}_{\mathrm{CH}}(k) \to \mathrm{DM}_{\text{gm}}(k)$$

2.4. **Realizations.** Realizations form an important tool for the study of motives and its use in arithmetic. Here we sketch the construction of the de Rham and Betti realizations, and say a word about étale reallizations.

For the de Rham realization, the main point is to show that the sheaf $\Omega^p_{-/k}$ on $\mathbf{Sm}_k$ extends to a Nisnevich sheaf with transfers. For simplicity, we work in characteristic zero. The main point is the following result. For a normal $k$-scheme $Y$, we let $\Omega^{p**}_{Y/k}$ denote the double dual of $\Omega^p_{Y/k}$; of course if $Y$ is smooth over $k$, $\Omega^{p**}_{Y/k} = \Omega^p_{Y/k}$.

**Lemma 2.25.** *Let $f : W \to X$ be a finite Galois cover of normal schemes, with Galois group $G$. Then the map $f^* : \Omega^{p**}_{X/k} \to f_*\Omega^{p**}_{W/k}$ identifies $\Omega^{p**}_{X/k}$ with the $G$-invariants $(f_*\Omega^{p**}_{W/k})^G$.*

For a general finite map of $k$-schemes $f : W \to X$, with $X$ smooth, this gives rise to a transfer map of sheaves

$$\mathrm{Tr}_{W/X} : f_*\Omega^p_{W/k} \to \Omega^p_{X/k}$$

as follows: We assume $W$ and $X$ are irreducible. Let $g : W^* \to W$ be the (normal) Galois closure of the normalization $W^N$ of $W$, with induce map $h : W^* \to X$, let $d$ denote the degree of $W^* \to W^N$ and let $G$ be the Galois group of $W^*$ over $X$. Define $\mathrm{Tr}_{W/X}$ as the composition

$$f_*\Omega^p_{W/k} \xrightarrow{g^*} h_*\Omega^{p**}_{W^*/k} \xrightarrow{(1/d)\mathrm{Tr}_G} (h_*\Omega^{p**}_{W^*/k})^G \cong \Omega^p_{X/k}$$

Here $\mathrm{Tr}_G$ is the map $\eta \mapsto \sum_{g\in G} g^*\eta$.

For $W \subset X \times Y$ an integral closed subscheme, finite over $X$, we thus have

$$W_* : \Omega^p_{Y/k}(Y) \to \Omega^p_{X/k}(X)$$

sending $\eta \in \Omega^p_{Y/k}(Y)$ to $\mathrm{Tr}_{W/X}(p_Y^*\eta)$. One then shows that this makes the de Rham complex $X \mapsto (\Omega^*_{X/k}, d)$ into a complex in $\mathrm{PST}(k)$, and the well-known properties of de Rham cohomology make this into an object of $D_{\mathbb{A}^1}(\mathrm{NST}(k))$, i.e., an object $\Omega^*/k$ of $\mathrm{DM}^{\mathrm{eff}}(k)$, representing de Rham cohomology via

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(M^{\mathrm{eff}}(X), \Omega^*/k[n]) \cong \mathbb{H}^n(X_{\mathrm{Nis}}, \Omega^*_{X/k}) =: H^n_{dR}(X/k).$$

Similarly, sending $p_X : X \to \mathrm{Spec}\, k$ $\mathbf{Sm}_k$ to the derived pushforward $\mathbb{R}p_{X*}\Omega^*_{X/k}$ extends to a functor

$$\Re_{dR} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)^{\mathrm{op}} \to D(k - \mathbf{Vec})$$

and then extends further to

$$\Re_{dR} : \mathrm{DM}_{\mathrm{gm}}(k)^{\mathrm{op}} \to D(k - \mathbf{Vec})$$

noting that $\Re_{dR}(M^{\mathrm{eff}}(X) \otimes \mathbb{Z}(1)) \cong \Re_{dR}(M^{\mathrm{eff}}(X))$ (there is a shift in the Hodge filtration, but that's another story).

The Betti realization is done similarly, using a Čech complex for the constant sheaf $\mathbb{Z}$ (or whatever). The point is that for $p : W \to Y$ finite with $Y$ smooth (all over $\mathbb{C}$), one can take a Leray open cover $\mathcal{U}$ of $Y(\mathbb{C})$ that pulls back to a Leray open cover of $W$. Moreover, for each $U_I = \cap_{i\in I}U_i$ the locus over which $p^{-1}(U_I) \to U_I$ is not étale has codimension $\geq 1$ in $U_I$, so the complement is connected, and thus any section of the constant sheaf on the "étale" locus extends uniquely to a section over $U_I$.

Finally, there are versions of étale realizations. One can repeat the basic idea for the Betti realization using the étale topology. Alternatively, one can repeat the construction of $\mathrm{DM}^{\mathrm{eff}}(k)$, replacing the Nisnevich topology with the étale topology. One can also have a theory in sheaves of $\mathbb{Z}/n$-modules. For $k$ of finite $n$-cohomological dimension, with $n$ prime to the characteristic of $k$ if this is positive, the resulting category $\mathrm{DM}_{\text{ét}}(k; \mathbb{Z}/n)$ is equivalent to the derived category of $n$-torsion étale sheaves on the small étale site of $\mathrm{Spec}\, k$, while the étale theory with

$\mathbb{Q}$-coefficients is equivalent to $\mathrm{DM}(k;\mathbb{Q})$. In particular, the change of topology functor gives the mod $n$ étale realization map

$$H^p(X,\mathbb{Z}/n(q)) \to H^p(X,\mathbb{Z}/n(q)_{\text{ét}}) \cong H^p_{\text{ét}}(X,\mu_n^{\otimes q})$$

Huber [**?**] has given a quite general construction of realization functors, which extend the constructions sketched above. In particular, she constructs exact tensor realization functors corresponding to rational mixed Hodge structures and continuous $\mathbb{Q}_\ell$-étale cohomology from $\mathrm{DM}_{\mathrm{gm}}(k)$ (for the MHS, one needs to be given an embedding $k \hookrightarrow \mathbb{C}$):

$$\Re_{MHS} : \mathrm{DM}_{\mathrm{gm}}(k) \to D_{MHS}(\mathbb{Q})$$
$$\Re_{\text{ét},\ell} : \mathrm{DM}_{\mathrm{gm}}(k) \to D_{\text{ét},ctn}(\mathbb{Q}_\ell, k)$$

Here $D_{MHS}(\mathbb{Q})$ is Beilinson's triangulated tensor category of $\mathbb{Q}$-mixed Hodge complexes and $D_{\text{ét},ctn}(\mathbb{Q}_\ell, k)$ is Ekedahl's triangulated tensor category of constructible complexes of $\mathbb{Q}_\ell$-étale sheaves on $\mathrm{Spec}\,(k)$. Neither of these is the derived category of an abelian category.

For $X \in \mathbf{Sm}_k$, these induce natural maps

$$\Re^{p,q}_{MHS}(X) : H^p(X,\mathbb{Q}(q)) \to H^p_{MHS}(X_\mathbb{C},\mathbb{Q}(q))$$

with $H^p_{MHS}(X,\mathbb{Q}(q))$ the $\mathbb{Q}$-mixed Hodge structure on the cohomology of the $\mathbb{C}$-scheme $X_\mathbb{C}$ and

$$\Re^{p,q}_{\text{ét},\ell}(X) : H^p(X,\mathbb{Q}(q)) \to H^p_{\text{ctn},\text{ét}}(X,\mathbb{Q}_\ell(q))$$

with $H^p_{\text{ctn},\text{ét}}(X,\mathbb{Q}_\ell(q))$ continuous $\ell$-adic étale cohomology, compatible with the various structures described above, e.g., products, projective pushforward, Gysin sequences, Chern classes, etc.

## 3. Lecture 3: Motivic cohomology and motivic stable homotopy

3.1. **The Beilinson-Lichtenbaum conjectures.** The Beilinson-Lichtenbaum conjectures are what started off the whole search for a motivic cohomology. All but one of them (the so-called Beilinson-Soulé vanishing conjectures) have up to now been verified. Most of these are covered by the existence of suitable triangulated categories of motives over a general base-scheme, which is an extension of the constructions we have described above, and which will be discussed below. The most difficult part of these conjectures (aside from the vanishing conjectures) concerns the comparison map from motivic to étale cohomology

$$H^p(X,\mathbb{Z}/n(q)) \to H^p_{\text{ét}}(X,\mu_n^{\otimes q})$$

for the case $X = \mathrm{Spec}\,F$, $F$ a field, and $n$ prime to the characteristic of $F$, and asserts that this map is an isomorphism for $p \le q$. This part of the overall set of conjectures of Beilinson and Lichtenbaum is often referred to as "the" Beilinson-Lichtenbaum conjectures.

Note that the isomorphism $H^q(F,\mathbb{Z}(q)) \cong K^M_q(F)$ (described in §1.1.5) and the fact that $H^{q+1}(F,\mathbb{Z}(q)) = 0$ shows that $H^q(F,\mathbb{Z}/n(q)) = K^M_q(F)/n$, so the case $p = q$ of the Beilinson-Lichtenbaum conjecture asserts that the map (the Galois symbol)

$$K^M_q(F)/n \to H^q_{\text{ét}}(X,\mu_n^{\otimes q}) \tag{3.1}$$

is an isomorphism for all $F$ and for all $n$ prime to $\mathrm{char}F$. This map is defined concretely by first noting that Kummer theory gives the isomorphism $F^\times/(F^\times)^n \cong$

$H^1_{\text{ét}}(F,\mu_n)$. Tate showed long ago that the map $F^\times \otimes F^\times \to H^2_{\text{ét}}(F,\mu_n^{\otimes 2})$ sends elements $a\otimes(1-a)$ to zero, so using products again gives the map (3.1).

The assertion that (3.1) is an isomorphism (for all $F$ and all $n$ prime to char$F$) is now known as the Bloch-Kato conjecture, although it is just part of the Beilinson-Lichtenbaum conjectures (which were stated about 12 years earlier). In fact, the Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture:

**Theorem 3.1** (Suslin-Voevodsky, Geisser-Levine). *Let $F_0$ be the prime field and let $\ell$ be a prime $\neq$ char$F_0$. Suppose the Galois symbol $K^M_n(F)\to H^n_{\acute{e}t}(F,\mu_\ell^{\otimes n})$ is surjective for all fields $F\supset F_0$. Then the change of topology map*

$$H^p(F,\mathbb{Z}/n(q))\to H^p_{\acute{e}t}(F,\mathbb{Z}/n(q))$$

*is an isomorphism for all fields $F\subset F_0$, and all $p,q$ with $0\le q\le n$ and $p\le q$.*

For $q=2$, the Bloch-Kato conjecture is the theorem of Merkurjev-Suslin theorem (proven some 14 years before Bloch and Kato formulated their "conjecture" and about two years before the Beilinson-Lichtenbaum conjectures arrived). An essential part of their argument considers the Severi-Brauer variety $SB(a,b,\zeta)$ for $a,b\in F^\times$ and $\zeta$ a primitive $\ell$th root of 1. The main point is to show that the kernel of the map $K^M_2(F)/\ell\to K^M_2(F(SB(a,b,\zeta)))\ell$ is exactly the $\mathbb{Z}/\ell$-span of the symbol $\{a,b\}$; this being before the advent of motivic cohomology, they use Quillen $K$-theory instead, via Matsumoto's theorem: $K^M_2(F)=K_2(F)$, and Gillet's Riemann-Roch theorem for higher $K$-theory that allows them to compare the relatively easy to understand $K_2(SB(a,b,\zeta))$ with $K_2(F(SB(a,b,\zeta))$.

A few years later, Merkurjev-Suslin and independently Levine used a "relativization" method to extend this to give an isomorphism

$$H^1(F,\mathbb{Z}/n(2))\cong H^1_{\text{ét}}(F,\mu_n^{\otimes 2})$$

(although their result was phrased in terms of the so-called "indecomposable $K_3$", as motivic cohomology was not yet around). This of course was before the general Bloch-Kato $\Rightarrow$ Beilinson-Lichtenbaum result mentiond above was proven and in a sense was its precursor.

For $q=3$, Rost extended the Merkurjev-Suslin method to prove the Bloch-Kato conjecture in weight 3. After a long development, Voevodsky put together his work on DM together with his construction of motivic Steenrod operations plus results of Rost and others to prove the Beilinson-Lichtenbaum conjectures in general, first for $n$ a power of 2, and then the general case.

The case $\ell=2$ was handled first (by Voevodsky, relying on some results of Rost). The proof is again based (at least in part) on the method used by Merkurjev-Suslin for $q=2$, but is more complicated. The Severi-Brauer varieties that play a central role as splitting varieties for a symbol $\{a,b\}$ mod $\ell$ in the proof of the Merkurjev-Suslin are replaced by the Pfister neighbor quadrics $Q_{\underline{a}}$ associated to an element $\underline{a}=(a_1,\ldots,a_n)\in(k^\times)^n$. Voevodsky also streamlines the argument by reducing to showing that the étale version of motivic cohomology, *Lichtenbaum motivic cohomology* $H^p_L(-,\mathbb{Z}(q))$ vanishes on fields in bi-degree $(n+1,n)$. The Merkurjev-Suslin arguments involving the Severi-Brauer became showing that, for $\underline{a}=(a_1,\ldots,a_n)\in(k^\times)^n$, one has:

(1) The symbol $\{a_1,\ldots,a_n\}\in K^M_n(k)$ goes to zero in $K^M_n(k(Q_{\underline{a}})/\ell$.
(2) The map $H^{n+1}_L(k,\mathbb{Z}(n))\to H^{n+1}_L(k(Q_{\underline{a}}),\mathbb{Z}(n))$ is injective.

Having these facts at hand, the argument is exactly as for the Merkurjev-Suslin theorem: Starting with $k$, one takes the maximal prime to 2 extension $k'$ of $k$, and then take the compositum of $k'$ with all fields $k(Q_{\underline{a}})$ for $\underline{a} = (a_1, \ldots, a_n)$ with $\{a_1, \ldots, a_n\} \neq 0$ in $K^M_n(k)/\ell$, forming the field $k_1$. Then repeat, forming the field $k_\infty$ with the property that $K^M_n(k_\infty)/\ell = 0$, and with $H^{n+1}_L(k, \mathbb{Z}(n)) \to H^{n+1}_L(k_\infty, \mathbb{Z}(n))$ injective. In induction in $n$ is used to show that $H^{n+1}_L(F, \mathbb{Z}(n)) = 0$ if $K^M_n(F)/\ell = 0$, so $H^{n+1}_L(k, \mathbb{Z}(n)) = 0$.

The injectivity (2) is quite hard to prove, and relies on an intricate application of the *motivic Steenrod operations*, as well as some deep results of Rost on the motives of the quadrics $Q_{\underline{a}}$ and a certain triviality of the motivic cohomology $H^{2n-1}(Q, \mathbb{Z}(n))$ for $Q$ a smooth $n$-dimensional quadric.

In a bit more detail, one main technical point is to show that $H^{n+1,n}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) = 0$, where $\mathcal{X}_{\underline{a}}$ is the Čech simplicial scheme $[n] \mapsto associatedtoQ^n_{\underline{a}}$ associated to $Q_{\underline{a}}$. Letting $\tilde{\mathcal{X}}_{\underline{a}}$ be the "reduced" version of $\mathcal{X}_{\underline{a}}$ (cofiber of the map $\mathcal{X}_{\underline{a}}, \to \operatorname{Spec} k$), this is the same as the vanishing of $H^{n+2,n}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}_{(2)})$. A sequence of Milnor operations maps this group (injectively!) to $H^{2^n,2^{n-1}}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}_{(2)}) = H^{2^n-1,2^{n-1}}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)})$. This latter group is a subgroup of $H^{2^n-1,2^{n-1}}(Q_{\underline{a}}, \mathbb{Z}_{(2)})$ and Rost's injectivity theorem show that pushforward by the structure map defines injection $H^{2^n-1,2^{n-1}}(Q_{\underline{a}}, \mathbb{Z}) \hookrightarrow H^1(k, \mathbb{Z}(1)) = k^\times$.

But now this says the base-change to $\bar{k}$ defines an injection

$$H^{2^n-1,2^{n-1}}(Q_{\underline{a}}, \mathbb{Z}) \hookrightarrow H^{2^n-1,2^{n-1}}(Q_{\underline{a}} \times_k \bar{k}, \mathbb{Z})$$

Running the construction in reverse, this says that

$$H^{n+1,n}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) \to H^{n+1,n}(\mathcal{X}_{\underline{a}} \times_k \bar{k}, \mathbb{Z}_{(2)})$$

is injective. But since $Q_{\underline{a}}(\bar{k}) \neq \emptyset$, $\mathcal{X}_{\underline{a}} \times_k \bar{k} \sim \operatorname{Spec} \bar{k}$ and

$$H^{n+1,n}(\mathcal{X}_{\underline{a}} \times_k \bar{k}, \mathbb{Z}_{(2)}) \cong H^{n+1,n}(\bar{k}, \mathbb{Z}_{(2)}) \cong \mathrm{CH}^n(\bar{k}, n-1) \otimes \mathbb{Z}_{(2)} = 0.$$

This is the essential point, the proof that $H^{n+1,n}_L(k, \mathbb{Z}_{(2)}) \to H^{n+1,n}_L(k(Q_{\underline{a}}), \mathbb{Z}_{(2)})$ is injective, follows from this and the fundamental distinguished triangle relating the 'Rost motive" $M_{\underline{a}}$) of $Q_{\underline{a}}$), $M(\mathcal{X}_{\underline{a}})$, and $M(\mathcal{X}_{\underline{a}})((2^{n?1}?1)[2^n?1]$:

$$M(\mathcal{X}_{\underline{a}})((2^{n?1}?1)[2^n-2] \to M_{\underline{a}}) \to M(\mathcal{X}_{\underline{a}}) \to M(\mathcal{X}_{\underline{a}})((2^{n?1}?1)[2^n?1].$$

The case of odd prime $\ell$ is treated in many way the same as for $\ell = 2$, but numerous technical problems arise, for instance, there is no nice collection of smooth projective varieties that play the role of the Severi-Brauer varieties for weight two, and the Pfister neighbor quadrics in higher weight for $\ell = 2$.

The construction of the Steenrod operations, an integral part of the proof, requires the introduction of the motivic stable homotopy category and understanding its relation with the triangulated category of motives; in the case $q = 2$, the use of the Steenrod operations was in part replaced by using $K$-theory and the Riemann-Roch theorem.

### 3.2. **The motivic stable homotopy category.**

3.2.1. *The unstable and stable motivic homotopy categories.* Subsequent to Voevodsky's construction of the category $\mathrm{DM}^{\mathrm{eff}}(k)$, Morel and Voevodsky developed a parallel $\mathbb{A}^1$ *homotopy theory.* In essence, this is much simpler, although this requires more input from the theory of model categories to carry out the construction; we will suppress this crucial technical side in our overview.

First of all, correspondences no longer play a role, and parallel to classical homotopy theory, one relies on presheaves of simplicial sets rather than presheaves of complexes of abelian. One can work over an arbitrary noetherian base-scheme $S$. Formally, one has the presheaf category $\mathrm{Psh}^{\mathbf{s}\mathrm{Sets}}(\mathbf{Sm}_S)$ where $\mathbf{Sm}_S$ is the category of smooth separated $S$-schemes of finite type, and $\mathbf{s}$Sets is the category of simplicial sets; this is called the category of *spaces over* $S$, $\mathbf{Spc}(S)$. Working in the Nisnevich topology again, one inverts morphisms $P \to Q$ that are weak equivalences (i.e. induce bijections on $\pi_0$ and all homotopy groups $\pi_n(-, x)$ for $n \geq 1$ and all choice of base-point) of simplicial sets on all Nisnevich stalks $P_x \to Q_x$, where a Nisnevich points is some $x \in X \in \mathbf{Sm}_k$, and $P_x$ is the colimit of $P(U)$ over all $(U, u) \to (X, x)$ Nisnevich neighborhoods of $x$. Formally, one can write $P_x = P(\mathcal{O}^h_{X,x})$ where $\mathcal{O}^h_{X,x}$ is the henselization of the local ring $\mathcal{O}_{X,x}$,

$$\mathcal{O}^h_{X,x} = \mathrm{colim}_{(U,u)\to(X,x)}\mathcal{O}_{U,u}$$

In addition, one inverts all morphisms $p^* : P \to P^{\mathbb{A}^1}$, where for a presheaf $P : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{s}\mathrm{Sets}$, $P^{\mathbb{A}^1}$ is the presheaf $X \mapsto P(X \times \mathbb{A}^1)$, and $p^*$ is the collection of maps $p_X^* : P(X) \to P(X \times \mathbb{A}^1)$. This gives us the $\mathbb{A}^1$ *unstable homotopy category* $\mathcal{H}(S)$ with canonical morphism $\mathbf{Sm}_S \to \mathcal{H}(S)$.

A striking feature of this construction is that it allows the mixing of algebraic geometry and classsical homotopy theory: algebraic geometry enters via the Yoneda functor $\mathbf{Sm}_S \to \mathbf{Spc}(S)$, sending $X$ presheaf with value at $Y \in \mathbf{Sm}_S$ the constanct simplicial set on $X(Y)$, while classical homotopy theory enters via the constant presheaf functor $c : \mathbf{s}\mathrm{Sets} \to \mathbf{Spc}(S)$. Moreover, the presheaf category $\mathbf{Spc}(S)$ inherits all the usual constructions of topology, especially limits, colimits, products (resp. smash products) and internal Homs, by operating objectwise on the presheaves (resp. pointed presheaves). For instance, we have the usual suspension and loops functors, $\Sigma_{S^1}$, $\Omega_{S^1}$, via $\mathcal{X} \mapsto S^1 \wedge \mathcal{X}$, $\mathcal{X} \mapsto \mathcal{H}om_{\mathbf{Spc}_\bullet(S)}(S^1, \mathcal{X})$. We have the Nisnevich sheaves of "connected components" $\pi_0^{\mathrm{Nis}}(\mathcal{X})$ and $\pi_0^{\mathbb{A}^1}(X)$, the first being the sheaf associated to the presheaf $U \mapsto \pi_0(\mathcal{X}(U))$ and the second the sheaf associated to the presheaf $U \mapsto [U, \mathcal{X}]_{\mathcal{H}(S)}$. Higher homotopy sheaves are defined similarly, with identities for $\mathcal{X} \in \mathbf{Spc}_\bullet(S)$

$$\pi_n^{\mathbb{A}^1}(\mathcal{X}) = \pi_0^{\mathbb{A}^1}(\Omega^n_{S^1}\mathcal{X});\ \pi_n^{\mathrm{Nis}}(\mathcal{X}) = \pi_0^{\mathrm{Nis}}(\Omega^n_{S^1}\mathcal{X}).$$

One uses the theory of model categories to show in the first place that such a localization exists, and to yield cofibrant and fibrant models in $\mathbf{Spc}(S)$, with the property that homotopy classes of map $P \to Q$, for $P$ cofibrant and $Q$ fibrant, compute the morphisms $[P, Q]_{\mathcal{H}(k)}$. The particular choice of model structure has the representable presheaves $X$ for $X \in \mathbf{Sm}_S$ being cofibrant, but the fibrant models are much more difficult to understand. Replacing simplicial sets with pointed simplicial sets $\mathbf{s}\mathrm{Sets}_\bullet$ yields the category of pointed spaces over $S$, $\mathbf{Spc}_\bullet(S)$, and the pointed $\mathbb{A}^1$ homotopy category $\mathcal{H}_\bullet(S)$.

Another notable feature of $\mathcal{H}_\bullet(S)$ is the two-parameter family of "spheres". Let $\mathbb{G}_m = (\mathbb{A}^1 \setminus \{0\}, \{1\})$ be the "Tate circle" and for $a \geq b \geq 0$, let $S^{a,b} := S^{a-b} \wedge \mathbb{G}_m^{\wedge b}$.

We have corresponding suspension functors $\mathcal{X} \mapsto \Sigma^{a,b}\mathcal{X} := \mathcal{X} \wedge S^{a,b}$ and loops functors $\mathcal{X} \mapsto \Omega^{a,b}\mathcal{X} := \mathcal{H}om(S^{a,b}, \mathcal{X})$. We also have the canonical isomorphism $(\mathbb{P}^1, \infty) \cong S^{2,1}$ in $\mathcal{H}_\bullet(S)$, given the natural isomorphism $\Sigma^n_{\mathbb{P}^1} \cong \Sigma^{2n,n}$. This gives us as well the bigraded family of homotopy sheaves

$$\pi^{\mathbb{A}^1}_{a,b}(\mathcal{X}) := \pi^{\mathbb{A}^1}_0(\Omega^{a,b}\mathcal{X})$$

and the similarly defined Nisnevich version $\pi^{\mathrm{Nis}}_{a,b}(\mathcal{X})$.

The stable theory is modeled on the classical case of suspension spectra of spaces, except that we replace $S^1$-suspension with $\mathbb{P}^1$-suspension, yielding the category of $\mathbb{P}^1$-spectra over $S$, $\mathrm{Sp}_{\mathbb{P}^1}(S)$. This is the category of sequences $E_* := (E_0, E_1, \ldots)$, $E_n \in \mathbf{Spc}_\bullet(S)$, together with bonding maps $\epsilon_n : \Sigma_{\mathbb{P}^1} E_n \to E_{n+1}$, where $\Sigma_{\mathbb{P}^1} E_n := E_n \wedge (\mathbb{P}^1, \infty)$. A morphism $E_* \to F_*$ in $\mathrm{Sp}_{\mathbb{P}^1}(S)$ is a collection of maps $f_n : E_n \to F_n$ in $\mathbf{Spc}_\bullet(S)$ that commute with the respective bonding maps. A map $f : E_* \to F_*$ is an $\mathbb{A}^1$ *stable weak equivalence* if the $f_n$ induce isomorphisms of $\mathbb{A}^1$ homotopy sheaves

$$\mathrm{colim}_n f_{n*} : \mathrm{colim}_n \pi^{\mathbb{A}^1}_{a+2n,b+n}(E_n) \to \mathrm{colim}_n \pi^{\mathbb{A}^1}_{a+2n,b+n}(F_n)$$

for all $a, b \in \mathbb{Z}$ Here the inductive system $\{\pi^{\mathbb{A}^1}_{a+2n,b+n}(E_n)\}$ has transition maps

$$\pi^{\mathbb{A}^1}_{a+2n,b+n}(E_n) \cong \pi^{\mathbb{A}^1}_{a+2n+2,b+n+1}(\Sigma_{\mathbb{P}^1} E_n) \xrightarrow{\epsilon_{n*}} \pi^{\mathbb{A}^1}_{a+2n+2,b+n+1}(E_{n+1})$$

and is defined for all $n$ sufficiently large (depending on $a, b$): $n \geq \max(0, -b, b - a)$. The $\mathbb{A}^1$ stable homotopy category is then defined by inverting stable weak equivalences in $\mathrm{Sp}_{\mathbb{P}^1}(S)$. This gives a triangulated tensor category, with translation functor induced by $\Sigma_{S^1}$ and with the suspension functors $\Sigma^{a,b}$ defined and invertible for all $a, b \in \mathbb{Z}$. As in the classical case one has the adjoint pair of infinite $\mathbb{P}^1$-suspension/infinite $\mathbb{P}^1$-loops functors

$$\Sigma^\infty_{\mathbb{P}^1} : \mathcal{H}_\bullet(S) \rightleftarrows \mathrm{SH}(S) : \Omega^\infty_{\mathbb{P}^1}$$

3.2.2. *The category* $\mathrm{DM}(k)$. One might ask, what about inverting $\mathbb{Z}(1)$ in $\mathrm{DM}^{\mathrm{eff}}(k)$? Here the naive Gabriel-Zisman localization $\mathrm{DM}^{\mathrm{eff}}(k)[(-\otimes\mathbb{Z}(1))^{-1}]$ is not really what one wants, as this category lacks arbitrary homotopy limits and colimits. A better construction is to be guided by homotopy theory, in forming the category of $\mathbb{Z}(1)[2]$-spectra, as above. Replacing $\mathbf{Spc}_\bullet(S)$ with $C(\mathrm{NST}(k))$, $\mathcal{H}_\bullet(S)$ with $\mathrm{DM}^{\mathrm{eff}}(k)$, and $\Sigma_{\mathbb{P}^1}$ with $- \otimes^{\mathrm{tr}} \mathbb{Z}^{\mathrm{tr}}(1)[2]$, we arrive at the triangulated tensor category $\mathrm{DM}(k)$, the localization of $\mathbb{Z}(1)[2]$-spectra in $C(\mathrm{NST}(k))$ with respect to stable $\mathbb{A}^1$-weak equivalence. The functor $- \otimes \mathbb{Z}(1)[2]$ on $\mathrm{DM}(k)$ is invertible and we have the adjoint pair of exact tensor functors

$$\Sigma^\infty_{\mathbb{Z}(1)[2]} : \mathrm{DM}^{\mathrm{eff}}(k) \rightleftarrows \mathrm{DM}(k) : \Omega^\infty_{\mathbb{Z}(1)[2]}$$

The situation here is a bit different as in $\mathbb{A}^1$-homotopy theory, in that the translation functor $M \mapsto M[1]$ is already invertible on triangulated category $\mathrm{DM}^{\mathrm{eff}}(k)$, while $\mathcal{H}_\bullet(S)$ does not have a triangulated structure and $\Sigma_{S^1}$ is not invertible.

The embedding theorem extends to show that

$$\mathrm{DM}_{\mathrm{gm}}(k) \to \mathrm{DM}(k)$$

is an exact, fully faithful embedding with dense image, identifying $\mathrm{DM}_{\mathrm{gm}}(k)$ with the subcategory of compact objects in $\mathrm{DM}(k)$.

*Remarks* 3.2. 1. $\mathrm{DM}(k)$ is a triangulated category admitting small coproducts. Using the theory of *symmetric spectra* gives $\mathrm{DM}(k)$ the structure of tensor triangulated category.

2. A general theory of stablization due to Hovey has a different definition of stable weak equivalence, but in the case of $\mathrm{DM}^{\mathrm{eff}}(k)$, Jardine (and also Voevodsky) shows that this agrees with the "naive" notion of stable weak equivalence described above.

3.2.3. $\mathrm{DM}(k)$ *and the category of* $H_{\mathrm{mot}}\mathbb{Z}$*-modules.* The objects of $\mathrm{SH}(S)$ represent bi-graded cohomology theories on $\mathbf{Sm}_S$ in the following way: Given $E \in \mathrm{SH}(S)$ and $U \in \mathbf{Sm}_S$, define

$$E^{a,b}(U) := [\Sigma^\infty_{\mathbb{P}^1} U_+, \Sigma^{a,b} E]_{\mathrm{SH}(S)}$$

Properties built into $\mathrm{SH}(S)$ give: contravariant functoriality, Mayer-Vietoris properties and homotopy invariance to this bi-graded family of abelian groups. If $E$ admits the structure of a commutative monoid in $\mathrm{SH}(S)$, then $E^{**}(U)$ has a bi-graded ring structure, with a certain form of graded commutativity.

Conversely, given a bi-graded "cohomology theory" $U \mapsto H^{**}(U) := \oplus_{a,b} H^{a,b}(U)$ on $\mathbf{Sm}_S$, one says that $H^{**}$ is represented by some $E \in \mathrm{SH}(S)$ if there is a natural isomorphism of functors $H^{**} \cong E^{**}$.

Motivic cohomology on $\mathbf{Sm}_k$ is in fact represented by a certain object $H_{\mathrm{mot}}\mathbb{Z} \in \mathrm{SH}(k)$, constructed as the sequence

$$H_{\mathrm{mot}}\mathbb{Z} := (\mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(0)), \mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(1)[2]), \ldots, \mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n)[2n]), \ldots)$$

Here EM is the Eilenberg-MacLane functor from $C(\mathbf{Ab})$ to usual suspension spectra, and the bonding maps are defined by applying EM to the maps of complexes

$$C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n)[2n]) \otimes^{\mathrm{tr}} \mathbb{Z}(1)[2] \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n+1)[2n+2])$$

induced by the natural map $\mathcal{H}om(X, A) \otimes^{\mathrm{tr}} B \to \mathcal{H}om(X, A \otimes^{\mathrm{tr}} B)$. One also uses the natural map (graph) of the representable functor $Y \mapsto \mathrm{Hom}_{\mathbf{Sm}_k}(Y, X)$ to $\mathbb{Z}^{\mathrm{tr}}(X)$ to define the map

$$\mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n)[2n])) \wedge (\mathbb{P}^1, \infty) \to \mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n)[2n]) \otimes^{\mathrm{tr}} \mathbb{Z}(1)[2]);$$

putting these together gives the bonding map

$$\Sigma_{\mathbb{P}^1} \mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n)[2n])) \to \mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n+1)[2n+2]))$$

One can give $H_{\mathrm{mot}}\mathbb{Z}$ the structure of an $E_\infty$ object in $\mathrm{Sp}_{\mathbb{P}^1}(k)$, which gives us the homotopy category of $H_{\mathrm{mot}}\mathbb{Z}$-modules, $H_{\mathrm{mot}}\mathbb{Z} - \mathbf{Mod}$, and the free-forget adjunction

$$H_{\mathrm{mot}}\mathbb{Z} \wedge - : \mathrm{SH}(k) \rightleftarrows H_{\mathrm{mot}}\mathbb{Z} - \mathbf{Mod} : \mathrm{EM}_{\mathrm{mot}}$$

In fact, we have the following fundamental theorem

**Theorem 3.3** (Röndigs-Østvær)**.** *Suppose that $k$ has characteristic zero. Then there is a natural isomorphism of tensor triangulated categories $H_{\mathrm{mot}}\mathbb{Z} - \mathbf{Mod} \cong \mathrm{DM}(k)$.*

This has been extended to characteristic $p > 0$, after inverting $p$, by Hoyois-Kelly-Østvær.

This connection of $\mathrm{DM}(k)$ with $\mathrm{SH}(k)$ has opened the way to a more "homotopical" approach to motives; we give a few examples.

3.2.4. *The motivic Steenrod algebra.* We have already mentioned the motivic Steenrod algebra and its role in the proof of the Beilinson-Lichtenbaum/Bloch-Kato conjectures. In classical homotopy theory, the mod $\ell$ Steenrod algebra is simply the (graded) endomorphism ring of the spectrum $H\mathbb{Z}/\ell$ representing mod $\ell$ singular cohomology in the stable homotopy category SH, that is

$$\oplus_n [H\mathbb{Z}/\ell, \Sigma^n H\mathbb{Z}/\ell]_{\mathrm{SH}}$$

The dual Steenrod algebra is the mod $\ell$ homology of $H\mathbb{Z}/\ell$, that is

$$\oplus_n \pi_n^s(H\mathbb{Z}/\ell \wedge H\mathbb{Z}/\ell)$$

where $\pi_n^s$ is stable homotopy: $\pi_n^s(E) := [\Sigma^n\mathbb{S}, E]_{\mathrm{SH}}$ and $\mathbb{S}$ is the sphere spectrum, $\mathbb{S} := \Sigma^\infty S^0$.

Making the obvious changes yields the motivic version: The motivic sphere spectrum over $S$ is $\mathbb{S}_S : \Sigma^\infty_{\mathbb{P}^1} S^0_S$, where $S^0_S = S_+$, i.e., the base-scheme $S$ with a disjoint copy of $S$ added as a base-point. $E$-cohomology of a spectrum $F \in \mathrm{SH}(S)$ is $E^{a,b}(F) := [F, \Sigma^{a,b}E]_{\mathrm{SH}(S)}$, $E$-homology is $E_{a,b}(F) := [\Sigma^{a,b}\mathbb{S}_S, E \wedge F]_{\mathrm{SH}(S)}$, giving us the mod $\ell$ motivic Steenrod algebra $H_{\mathrm{mot}}\mathbb{Z}/\ell^{**}(H_{\mathrm{mot}}\mathbb{Z}/\ell)$ and the dual Steenrod algebra $H_{\mathrm{mot}}\mathbb{Z}/\ell_{**}(H_{\mathrm{mot}}\mathbb{Z}/\ell)$.

Voevodsky was able to construct explicit elements in $H_{\mathrm{mot}}\mathbb{Z}/\ell^{**}(H_{\mathrm{mot}}\mathbb{Z}/\ell)$ in case $S = \operatorname{Spec} k$, $k$ a field of characteristic zero, and show that these give a presentation of $H_{\mathrm{mot}}\mathbb{Z}/\ell^{**}(H_{\mathrm{mot}}\mathbb{Z}/\ell)$ that is remarkably similar to the classical case. The main difference is that the classical Steenrod algebra is an algebra over $H\mathbb{Z}/\ell^*(pt) = \mathbb{Z}/\ell$ (concentrated in degree 0), whereas the motivic version is an algebra over $H_{\mathrm{mot}}\mathbb{Z}/\ell^*(k) = \oplus_{a,b} H^a(\operatorname{Spec} k, \mathbb{Z}/\ell(b))$. The classical version acts trivially on the cohomology of a point, while (in general) the motivic version acts non-trivially on the motivic cohomology of $k$, which makes the algebra structure more complicated. However, Voevodsky's generators correspond directly to the standard classical generators, and fulfill essentially the same relations (the Adem relations). The most notable difference occurs at $\ell = 2$. Here $-1 \in k^\times$ shows up in two different places, one as the element $\tau \in H^0(k, \mathbb{Z}/2(1)) = \mu_2(k) = \{\pm 1\}$ and a second time as $\rho \in H^1(k, \mathbb{Z}/2(1)) = k^\times/k^{\times 2}$ as the class of -1 modulo squares. $\tau$ behaves differently from $-1 \in H^0(pt, \mathbb{Z}/2)$ because, if $k$ does not contain $\sqrt{-1}$, the map $H^0(k, \mathbb{Z}/4(1)) \to H^0(k, \mathbb{Z}/2(1))$ is the trivial map, so the Bockstein of $\tau$ is non-zero. Similarly $H^1(pt, \mathbb{Z}/2) = 0$ but if $k$ does not contain $\sqrt{-1}$, then $\rho \neq 0$, so we have this additional "-1" to consider (in fact, the Bockstein of $\tau$ is $\rho$).

This was all extended to the positive characteristic case, at least for $\ell \neq \mathrm{char} k$, by Hoyois-Kelly-Østvær. The motivic Steenrod algebra shows up in many other foundational computations, for instance, in the theorem of Hopkins-Morel-Hoyois, describing the relationship of $H_{\mathrm{mot}}\mathbb{Z}$ with Voevodsky's algebraic cobordism spectrum MGL.

Frankland and Spitzweck have shown that the characteristic zero version of the mod $p$ motivic Steenrod algebra is a summand of the actual motivic Steenrod algebra over a field of characteristic $p$. The lack of a complete understanding of the mod $p$ motivic Steenrod algebra in characteristic $p$ is a significant hinderance to our understanding of motivic homotopy theory in positive or mixed characteristic.

3.2.5. *Voevodsky's slice tower.* Besides motivic cohomology, algebraic $K$-theory is also represented in $\mathrm{SH}(k)$. One of the main results of Morel-Voevodsky about the unstable category $\mathcal{H}(k)$ is that the infinite Grassmannian $\mathrm{Gr}(\infty, \infty) := \mathrm{colim}_{m,n}\mathrm{Gr}(m, n+$

$m$) represents Quillen's algebraic $K$-theory for $X \in \mathbf{Sm}_k$:

$$K_n(X) \cong [\Sigma^n_{S^1} X_+, \mathrm{Gr}(\infty,\infty)\times\mathbb{Z}]_{\mathcal{H}(k)}$$

Voevodsky promotes this to representability in $\mathrm{SH}(k)$ by the $\mathbb{P}^1$-spectrum KGL

$$\mathrm{KGL} := (\mathrm{Gr}(\infty,\infty)\times\mathbb{Z}, \mathrm{Gr}(\infty,\infty)\times\mathbb{Z}, \ldots)$$

with bonding map given by the maps

$$\mathrm{Gr}(m,\infty)\wedge(\mathbb{P}^1,\infty)\to\mathrm{Gr}(m,\infty)$$

representing the virtual bundle $p_1^*E_m\otimes p_2^*\mathcal{O}(1)-p_1^*E_m-p_2^*\mathcal{O}(1)+\mathcal{O}$ on $\mathrm{Gr}(m,\infty\times\mathbb{P}^1$. His idea is to define a version of the classical Postnikov tower, again replacing usual connectivity with $\mathbb{P}^1$-connectivity.

More precisely, let $\mathrm{SH}^{\mathrm{eff}}(k)$ be the localizing subcategory of $\mathrm{SH}(k)$ generated by suspension spectra $\Sigma^\infty_{\mathbb{P}^1}X_+$, $X\in\mathbf{Sm}_k$, and for $n\in\mathbb{Z}$, let $\Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k)$ denote the translate of $\mathrm{SH}^{\mathrm{eff}}(k)$ by the $n$-fold suspension functor. This gives the filtration of $\mathrm{SH}(k)$ by localizing subcategories

$$\ldots\subset\Sigma^{n+1}_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k)\subset\Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k)\subset\ldots\subset\mathrm{SH}(k)$$

The inclusion $i_n : \Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k)\to\mathrm{SH}(k)$ admits the right adjoint $r_n:\mathrm{SH}(k)\to\Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k)$, defining the truncation functor $f_n := i_n r_n : \mathrm{SH}(k)\to\mathrm{SH}(k)$; using the above tower gives the natural transformations $f_{n+1}\to f_n\to\mathrm{Id}_{\mathrm{SH}(k)}$, giving the tower of endofunctors on $\mathrm{SH}(k)$

$$\ldots\to f_{n+1}\to f_n\to\ldots\to\mathrm{Id}_{\mathrm{SH}(k)}$$

Taking the layers in this tower gives the distinguished triangles

$$f_{n+1}\to f_n\to s_n\to f_{n+1}[1]$$

Voevodsky calls $s_n$ the *nth slice.* Applying this to an $E\in\mathrm{SH}(k)$ gives the tower in $\mathrm{SH}(k)$

$$\ldots\to f_{n+1}E\to f_nE\to\ldots\to E$$

and the distinguished triangles

$$f_{n+1}E\to f_nE\to s_nE\to f_{n+1}E[1]$$

The tower is called the *slice tower* for $E$, and gives rise to the *slice spectral sequence*

$$E_2^{p,q}(n)(\mathcal{X}) := (s_{-q}E)^{p+q,n}(\mathcal{X})\Rightarrow E^{p+q,n}(\mathcal{X})$$

In general, this tower does not have good convergence properties, due in part to the fact that the filtration $\Sigma^*_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k)$ of $\mathrm{SH}(k)$ is neither exhaustive nor separated, so in using the slice spectral sequence, one needs to address convergence.

The classical Postnikov tower in SH can be constructed in the same way, replacing $\Sigma^n_{\mathbb{P}^1}$ with $\Sigma^n_{S^1}$ and taking $\mathrm{SH}^{\mathrm{eff}}$ to be the localizing subcategory generated by $\Sigma^\infty T_+$, for $T$ an arbitrary simplicial set. This is also the subcategory of $-1$-connected spectra, i.e. spectra $E$ such that $\pi^s_mE=0$ for $m<0$, and $\Sigma^n_{S^1}\mathrm{SH}^{\mathrm{eff}}$ is the subcategory of $n-1$-connected spectra ($\pi^s_mE=0$ for $m<n$). The corresponding $n$th slice of $E$ is the Eilenberg-MacLane spectrum $\mathrm{EM}(\pi_nE,n)$, characterised by

$$\pi^s_m\mathrm{EM}(\pi_nE,n)=\begin{cases}\pi_nE & \text{for } m=n\\ 0 & \text{else.}\end{cases}$$

and the spectral sequence is

$$E_2^{p,q}(X) := H^p(X, \pi_{-q}E) \Rightarrow E^{p+q}(X)$$

This is the *Atiyah-Hirzebruch* spectral sequence, so one often calls Voevodsky's version the motivic Atiyah-Hirzebruch spectral sequence. In the classical case, the filtration is separated, so one has much better convergence properties.

The some basic results regarding the slice tower are

**Theorem 3.4.** *1.* $s_n H_{\mathrm{mot}}\mathbb{Z} = 0$ *for* $n \neq 0$ *and* $s_0 H_{\mathrm{mot}}\mathbb{Z} = H_{\mathrm{mot}}\mathbb{Z}$.
*2.* $s_n \mathrm{KGL} = \Sigma^n_{\mathbb{P}^1} H_{\mathrm{mot}}\mathbb{Z}$

The first result con be promoted to

**Corollary 3.5.** *For each* $E \in \mathrm{SH}(k)$*,* $s_n E$ *has a canonical structure of an* $H_{\mathrm{mot}}\mathbb{Z}$*-module*

Thus, we have the *homotopy motive* $\pi_{\mathrm{mot},n}E \in \mathrm{DM}^{\mathrm{eff}}(k)$, with

$$s_n E = \Sigma^n_{\mathbb{P}^1} \mathrm{EM}_{\mathrm{mot}}(\pi_{\mathrm{mot},n}E) = \mathrm{EM}_{\mathrm{mot}}(\pi_{\mathrm{mot},n}E \otimes \mathbb{Z}(n)[2n])$$

and we can rewrite the slice spectral sequence as

$$E_2^{p,q}(n)(X) := H^{p-q}(X, \pi_{\mathrm{mot},-q}E(n)) \Rightarrow E^{p+q,n}(\mathcal{X})$$

For $E = \mathrm{KGL}$ and $n = 0$, this gives

$$E_2^{p,q}(n)(X) = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow \mathrm{KGL}^{p+q,0}(X) = K_{2q-p}(X).$$

Via the isomorphism $H^{p-q}(X, \mathbb{Z}(-q)) \cong \mathrm{CH}^{-q}(X, -p-q)$, this agrees with the $E_2$- reindexed Bloch-Lichtenbaum/Friedlander-Suslin spectral sequence described in §**??**.

3.2.6. *The algebraic Hopf map.* The classical stable Hopf map is the element of $\pi_1^s(\mathbb{S})$ induced by the generator $\eta_{\mathrm{top}} : S^3 \to S^2$ of $\pi_3(S^2)$. $\eta_{\mathrm{top}}$ has a purely algebraic representative, as the quotient map $(\mathbb{C}^2 \setminus \{0\}, \{(0,1)\} \to (\mathbb{CP}^1, \infty)$ identifying the Riemann sphere $\mathbb{CP}^1$ with $\mathbb{C}^2 \setminus \{0\}/\mathbb{C}^\times$. We let $\eta : (\mathbb{A}^2 \setminus \{0\}, (0,1)) \to (\mathbb{P}^1, \infty)$ be the corresponding map in $\mathcal{H}_\bullet(k)$. Noting that $(\mathbb{A}^2 \setminus \{0\}, (0,1)) \cong S^{3,2}$, $(\mathbb{P}^1, \infty) \cong S^{2,1}$, this gives us the stable version $\eta \in \pi^{\mathbb{A}^1}_{1,1}(\mathbb{S}_k)$, the *stable algebraic Hopf map.* $\eta$ is closely related to the automorphism $\tau : \mathbb{S}_k \to \mathbb{S}_k$ induced by the exchange-of-factors symmetry $\tau_{\mathbb{P}^1,\mathbb{P}^1} : \mathbb{P}^1 \wedge \mathbb{P}^1 \to \mathbb{P}^1 \wedge \mathbb{P}^1$, by the identity

$$\tau = \mathrm{Id} + \eta \circ \rho = \mathrm{Id} + \rho \circ \eta,$$

where $\rho : S^0_k \to \mathbb{G}_m$ is the map sending the non-basepoint of $S^0_k$ to $-1$. In addition, we have

$$\eta(1 + \tau) = 0$$

After inverting 2, we can decompose $\mathbb{S}_k$ into the $\tau$ $+1$ and $-1$ eigenspaces, via the idempotents $(1-\tau)/2$, $(1+\tau)/2$. Since $\mathbb{S}_k$ is the unit for the monoidal structure on $\mathrm{SH}(k)$, this decomposes $\mathrm{SH}(k)[1/2]$ as $\mathrm{SH}(k)[1/2] = \mathrm{SH}(k)^+ \times \mathrm{SH}(k)^-$, with

$$\mathrm{SH}(k)^+ := \ker((1-\tau)/2 : \mathrm{SH}(k)[1/2] \to \mathrm{SH}(k)[1/2]);$$

$$\mathrm{SH}(k)^- = \ker((1+\tau)/2 : \mathrm{SH}(k)[1/2] \to \mathrm{SH}(k)[1/2])$$

Thus $\mathrm{SH}(k)^+ \subset \mathrm{SH}(k)[1/2]$ is the $\eta$-torsion subcategory, and $\mathrm{SH}(k)^- \subset \mathrm{SH}(k)[1/2]$ is the $\eta$-local subcategory $\mathrm{SH}(k)[1/2, \eta^{-1}]$.

For $k \subset \mathbb{C}$, we have the topological $\mathbb{C}$-realization

$$\Re_{\mathbb{C}} : \mathrm{SH}(k) \to \mathrm{SH}$$

sending $\Sigma^\infty_{\mathbb{P}^1} X_+$ to $\Sigma^\infty_{S^1} X(\mathbb{C})_+$. Clearly $\Re_\mathbb{C}(\eta)$ is the usual Hopf map in $\pi_1^s(\mathbb{S}) \cong \mathbb{Z}/2$. However, if $k \subset \mathbb{R}$, we have a corresponding real realization

$$\Re_\mathbb{R} : \mathrm{SH}(k) \to \mathrm{SH}$$

sending $\Sigma^\infty_{\mathbb{P}^1} X_+$ to $\Sigma^\infty_{S^1} X(\mathbb{C})_+$, and $\Re_\mathbb{R}(\eta)$ is induced by the map $\times 2 : S^1 \to S^1$. Thus, after inverting 2 everywhere, we see that $\Re_\mathbb{C}$ factors through the projection $\mathrm{SH}(k)[1/2] \to \mathrm{SH}(k)^+$ and $\Re_\mathbb{R}$ factors through the projection $\mathrm{SH}(k)[1/2] \to \mathrm{SH}(k)^-$.

Returning to motivic cohomology, the Hopf map in $\mathrm{DM}(k)$ becomes a morphism $\mathbb{Z}(2)[3] \to \mathbb{Z}(1)[2]$, that is, an element of $H^{-1}(k, \mathbb{Z}(-1)) = 0$. Thus $H_{\mathrm{mot}}\mathbb{Z}[1/2]$ lives in $\mathrm{SH}(k)^+$. This says that the slice tower on $\mathrm{SH}(k)^-$ is the constant tower of identity maps. Another way to see this is that $\Sigma_{\mathbb{P}^1} = \Sigma^{2,1} = \Sigma_{\mathbb{G}_m} \circ \Sigma_{S^1}$. Since $\mathrm{SH}^{\mathrm{eff}}(k)$ is triangulated, we have $\Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k) = \Sigma^n_{\mathbb{G}_m}\mathrm{SH}^{\mathrm{eff}}(k)$, and in $\mathrm{SH}(k)^{-1}$ $\rho : \mathbb{S} \to \Sigma_{\mathbb{G}_m}\mathbb{S}$ and $\eta : \Sigma_{\mathbb{G}_m}\mathbb{S} \to \mathbb{S}$ are inverse isomorphisms. Thus $\Sigma^n_{\mathbb{G}_m}\mathrm{SH}^{\mathrm{eff}}(k)^- = \mathrm{SH}^{\mathrm{eff}}(k)^- = \mathrm{SH}(k)^-$.

3.3. **Six functors and motives over a base.** As we mentioned at the beginning of these lectures, Beilinson envisaged an abelian category of mixed motivic sheaves on each scheme $X$, $\mathrm{Sh}^{\mathrm{Mot}}_X$, with the Grothendieck six operations: the adjoint pair of derived pushforward and pulback functors for each morphism $f : Y \to X$

$$f^* : D(\mathrm{Sh}^{\mathrm{Mot}}_X) \rightleftarrows D(\mathrm{Sh}^{\mathrm{Mot}}_Y) : f_*$$

the adjoint pair of exceptional functors

$$f_! : D(\mathrm{Sh}^{\mathrm{Mot}}_Y) \rightleftarrows D(\mathrm{Sh}^{\mathrm{Mot}}_X) : f^!$$

internal Hom $\mathcal{H}om(-,-)$ and tensor product $-\otimes-$, satisfying the "usual" relations, e.g., smooth and proper base-change isomorphisms, and a natural transformation $f_! \to f_*$ that is an isomorphism for proper $f$.

As we have seen, the lack of the Beilinson-Soulé vanishing conjectures puts this beyond the realm of the current technology, but one could hope for this type of setup for triangulated categories of motives over a base-scheme $X$, $\mathrm{DM}(X)$. There are a number of approaches for this, many of which rely on having the Grothendieck six operations for the motivic stable homotopy categories $X \mapsto \mathrm{SH}(X)$.

Without going into detail, the structure of the Grothenieck six operations for $X \mapsto \mathrm{SH}(X)$, $X \in \mathbf{Sch}_B$, with $B$ a fixed noetherian base-scheme of finite Krull dimension, has been constructed by Ayoub, with a somewhat more streamlined construction (extended to the equivariant case for a "tame" group $G$) by Hoyois. In fact, this has been extended to such a theory on a fairly general subcategory of algebraic stacks by Khan-Ravi, and with a similar theory by Chirantan Choudhury. At present however, the exceptional functors are limited to representable morphisms (I'm told that this should not be a serious restriction, however the natural transformation $f_! \to f_*$ really is restricted to representable $f$).

As we have seen, one can construct $\mathrm{DM}(k)$ as a homotopy category of $H\mathbb{Z}$-modules, so one could ask if there is a reasonable object $H\mathbb{Z}_S \in \mathrm{SH}(S)$ for which the category of $H\mathbb{Z}_S$-modules would be a reasonable choice for $\mathrm{DM}(S)$. This is in fact the case, and there are two such constructions, one by Markus Spitzweck, one by Marc Hoyois.

3.3.1. *Motivic Borel-Moore homology over a base.* We have seen that the higher Chow groups agree with the theory given by the sheaves $z_{\mathrm{equi},r}(X)$ via the inclusion/quasi-isomorphism $C_*^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X))(k) \subset z_r(X,*)$ for $r \geq 0$. Recalling that $C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X) \cong \underline{\mathcal{H}om}(\mathbb{Z}(r)[2r], \mathbb{Z}(X)^c)$, we saw that

$$\begin{aligned}\mathrm{CH}_r(X,n) &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(0)[n], C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X))) \\ &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(r)[2r+n], \mathbb{Z}(X)^c)\end{aligned}$$

suggesting that it would have been better to consider $C^{\mathrm{Sus}}(z_{\mathrm{qfin}}(X))_{\mathrm{Nis}}$ as the Borel-Moore motive of $X$ and define $\mathbb{Z}_{\mathrm{B.M.}}(X) := C^{\mathrm{Sus}}(z_{\mathrm{qfin}}(X))_{\mathrm{Nis}}$ and

$$H_p^{\mathrm{B.M.}}(X, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(p)[q], \mathbb{Z}_{\mathrm{B.M.}}(X))) \cong \mathrm{CH}_q(X, p-2q).$$

In fact, for $X$ of finite type over a Dedekind scheme $B$, one can give a reasonable extension of the definition we gave for a field to yield a cycle complex $z_r(X/B,*)$ and a good definition of motivic Borel-Moore homology

$$H_p^{\mathrm{B.M.}}(X/B, \mathbb{Z}(q)) := \mathbb{H}_{p-2q}(B_{\mathrm{Zar}}, p_{X*}z_q(*))$$

Here $p_{X*}z_q(*)$ is the Zariski sheaf on $B$ associated to the presheaf $U \mapsto z_q(p_X^{-1}(U)/U,*)$.

3.3.2. *Beilinson motivic cohomology.* Homotopy invariant algebraic $K$-theory is represented in $\mathrm{SH}(S)$ by Voevodsky's algebraic $K$-theory spectrum $\mathrm{KGL}_S$. Cisinski-Déglise note that $\mathrm{KGL}_S$ admits Adams operations $\Psi_k$ and $\mathrm{KGL}_{S\mathbb{Q}}$ breaks up into the $k^q$-eigenspectra:

$$\mathrm{KGL}_{S\mathbb{Q}} = \oplus_i \mathrm{KGL}_S^{(i)},$$

with $\mathrm{KGL}_S^{(i)}$ representing the $i$th graded piece of (rational) $K$-theory for the $\gamma$-filtration (assuming $S$ is regular). This gives them a nice commutative monoid object (i.e. commutative ring spectrum) $H_S^{\mathbb{B}} := \mathrm{KGL}_S^{(0)} \in \mathrm{SH}(S)_{\mathbb{Q}}$, whose module category they call the category of *Beilinson motives over* $S$. This construction is cartesian, that is, for $f: Y \to X$ a morphism of schemes, one has a canonical isomorphism $f^*H_X^{\mathbb{B}} \cong \mathbb{H}_Y^{\mathbb{B}}$, which is essentially what one needs to induce a six-functor formalism on $S \mapsto H_S^{\mathbb{B}} - \mathbf{Mod}$ from $\mathrm{SH}(-)$.

3.3.3. *Spitzweck's motivic cohomology.* Spitzweck constructs a motivic cohomology theory over an arbitrary base-scheme. The Bloch cycle complex gives rise to a general version of Bloch's higher Chow groups for finite type schemes over a Dedekind domain, which has nice localization properties but has poor functoriality and lacks a multiplicative structure. On the other hand, using the Bloch-Kato conjectures, established by Voevodsky *et al.*, the $\ell$-completed higher Chow groups are recognized as a truncated $\ell$-adic étale cohomology, for $\ell$ prime to all residue characteristics. The theorem of Geisser-Levine describes the $p$-completed higher Chow groups in characteristic $p > 0$ in terms of logarithmic de Rham-Witt sheaves. Finally, there is the good theory with $\mathbb{Q}$-coefficients given by Beilinson motivic cohomology of Cisinski-Déglise, as described above.

Each of these three theories: $\ell$-adic étale cohomology, the cohomology of the logarithmic de Rham-Witt sheaves, and rational Beilinson motivic cohomology, have good functoriality and multiplicative properties. Gluing the $\ell$-adic, $p$-adic and rational theories together via their respective comparisons with the Bloch cycle complex, Spitzweck constructs a theory with good functoriality and multiplicative properties, and which is described by a presheaf of complexes on smooth schemes over a given Dedekind domain as base-scheme. The corresponding theory agrees with Voevodsky's motivic cohomology for smooth schemes over a perfect field, and is given additively by the hypercohomology of the Bloch complex for smooth schemes over a Dedekind domain (even in mixed characteristic), as described above.

Taking the base-scheme to be $\operatorname{Spec}\mathbb{Z}$, Spitzweck's construction yields a representing object $M\mathbb{Z}_{\mathbb{Z}}$ in $\mathrm{SH}(\mathbb{Z})$ and one can thus define absolute motivic cohomology for smooth schemes over a given base-scheme $X$ by pulling back $M\mathbb{Z}_{\mathbb{Z}}$ to $M\mathbb{Z}_S \in \mathrm{SH}(S)$. The resulting motivic cohomology agrees with Voevodsky's for smooth schemes of finite type over a perfect base-field, and with the hypercohomology of the Bloch cycle complex for smooth finite type schemes over a Dedekind domain. This gives rise to triangulated category of motives $\mathrm{DM}_{\mathrm{Sp}}(S)$ over a base-scheme $X$, defined as the homotopy category of $M\mathbb{Z}_S$-modules, and the functor $S \mapsto \mathrm{DM}_{\mathrm{Sp}}(S)$ inherits a Grothendieck six-functor formalism from that of $S \mapsto \mathrm{SH}(S)$.

3.3.4. *Hoyois' motivic cohomology.* Spitzweck's construction gives a solution to the problem of constructing a triangulated category of motives over an arbitrary base, admitting a six-functor formalism and thus yielding a good theory of motivic cohomology. His construction is a bit indirect and it would be nice to have a direct construction of a representing motivic ring spectrum $H\mathbb{Z}_S \in \mathrm{SH}(S)$ for each base-scheme $S$, still satisfying the cartesian condition.

Hoyois has constructed such a theory of motivic cohomology over an arbitrary base-scheme by using a recent breakthrough in our understanding of the motivic stable homotopy categories $\mathrm{SH}(S)$. This is a new construction of $\mathrm{SH}(S)$ more in line with Voevodsky construction of $\mathrm{DM}(k)$. The basic idea is sketched in notes of Voevodsky, which were realized in a series of works by Ananyevskiy, Garkusha, Panin, Neshitov (authorship in various combinations). Building on these works, Elmanto, Hoyois, Khan, Sosnilo and Yakerson construct an infinity category of framed correspondences, and use the basic program of Voevodsky's construction of $\mathrm{DM}(k)$ to realize $\mathrm{SH}(S)$ as arising from presheaves of spectra *with framed transfers*, just as objects of $\mathrm{DM}(k)$ arise from presheaves of complexes of sheaves with transfers for finite correspondences. It is not our purpose here to give a detailed discussion of this beautiful topic; we content ourselves with sketching some of the basic principles.

An integral closed subscheme $Z \subset X \times Y$ that defines a finite correspondence from $X$ to $Y$ can be thought of a special type of a *span* via the two projections

$$X \xleftarrow{\ p_1\ } Z \xrightarrow{\ p_2\ } Y$$

For $X$ and $Y$ smooth and finite type over a given base-scheme $S$, a framed correspondence from $X$ to $Y$ is also a span,

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y$$

satisfying some conditions, together with some additional data (the framing). For simplicity, assume that $X$ is connected. The morphism $p$ is required to be a finite, flat, local complete intersection (lci) morphism, called a finite *syntomic* morphism (the terminology was introduced by Mazur). The lci condition means that $p$ factors as closed immersion $i : Z \to P$ followed by a smooth morphism $f : P \to X$, and the closed subscheme $i(Z)$ of $P$ is locally defined by exactly $\dim_X P - \dim_X Z$ equations. The morphism $p$ factored in this way has a relative cotangent complex $\mathbb{L}_p$ admitting a simple description, namely

$$\mathbb{L}_p = [\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^*\Omega_{P/X}];$$

the conditions on $i$ and $p$ say that both $\mathcal{I}_Z/\mathcal{I}_Z^2$ and $i^*\Omega_{P/X}$ are locally free coherent sheaves on $Z$ of rank $\dim_X P - \dim_X Z$ and $\dim_X P$, respectively. The perfect complex $\mathbb{L}_p$ defines a point $\{\mathbb{L}_p\}$ in the space $\mathcal{K}(Z)$ defining the $K$-theory of $Z$ of virtual rank $\dim_X Z$; in the case of a finite syntomic morphism the virtual rank is zero.

A *framing* for a syntomic map $p : Z \to X$ is a choice of a path $\gamma : [0,1] \to \mathcal{K}(Z)$ connecting $\{\mathbb{L}_p\}$ with the base-point $0 \in \mathcal{K}(Z)$. For a framing to exist, the class $[\mathbb{L}_p] \in K_0(Z)$ must be zero, but the choice of $\gamma$ is additional data. The morphism $q : Z \to Y$ is arbitrary.

One has the usual notion of a composition of spans:

$$\begin{array}{ccc} & Z' & \\ {\scriptstyle p'}\swarrow & & \searrow{\scriptstyle q'} \\ Y & & W \end{array} \circ \begin{array}{ccc} & Z & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X & & Y \end{array} := \begin{array}{ccc} & Z \times_Y Z' & \\ {\scriptstyle p\circ p_1}\swarrow & & \searrow{\scriptstyle q'\circ p_2} \\ X & & W \end{array}$$

which preserves the finite syntomic condition. However, one needs a higher categorical structure to take care of associativity constraints. The composition of paths is even trickier, since we are dealing here with actual paths, not paths up to homotopy. In the end, this produces an infinity category $\mathbf{Corr}^{fr}(\mathbf{Sm}_S)$ of framed correspondences on smooth $S$-schemes, rather than a category; roughly speaking, the composition is only defined "up to homotopy and all higher homotopies".

Via the infinity category $\mathbf{Corr}^{fr}(\mathbf{Sm}_S)$, we have the infinity category of framed motivic spaces, $\mathbf{H}^{fr}(S)$, this being the infinity category of $\mathbb{A}^1$-invariant, Nisnevich sheaves of spaces on $\mathbf{Corr}^{fr}(\mathbf{Sm}_S)$. There is a stable version, $\mathbf{SH}^{fr}(S)$, an infinite suspension functor $\Sigma^\infty_{fr} : \mathbf{H}^{fr}(S) \to \mathbf{SH}^{fr}(S)$, and an equivalence of infinity categories $\gamma_* : \mathbf{SH}^{fr}(S) \to \mathbf{SH}(S)$, where $\mathbf{SH}(S)$ is the infinity category version of the triangulated category $\mathrm{SH}(S)$, that is, the homotopy category of $\mathbf{SH}(S)$ is $\mathrm{SH}(S)$. The equivalence $\gamma_*$ can be thought of as a version of the construction of infinite loop spaces from Segal's $\Gamma$-spaces, with a framed correspondence $X \leftarrow Z \to Y$ of degree $n$ over $X$ being viewed as a generalization of the map $[n]_+ \to [0]_+$ in $\Gamma^{\mathrm{op}}$.

With this background, we can give a rough idea of Hoyois' construction of the spectrum representing motivic cohomology over $S$. He considers spans $X \xleftarrow{p} Z \xrightarrow{q} Y$, $X, Y \in \mathbf{Sm}_S$, with $p : Z \to X$ a finite morphism such that $p_*\mathcal{O}_Z$ is a locally free $\mathcal{O}_X$-module; note that this condition is satisfied if $p$ is a syntomic morphism, but not conversely. These spans form a category $\mathbf{Corr}^{flf}(\mathbf{Sm}_S)$ under span composition ("flf" stands for "finite, locally free") and forgetting the paths $\gamma$ defines a morphism of (infinity) categories $\pi_{ad} : \mathbf{Corr}^{fr}(\mathbf{Sm}_S) \to \mathbf{Corr}^{flf}(\mathbf{Sm}_S)$.

Given a commutative monoid $A$, the constant Nisnevich sheaf on $\mathbf{Sm}_S$ with value $A$ extends to a functor

$$A_S : (\mathbf{Corr}^{flf})^{\mathrm{op}} \to \mathbf{Ab},$$

where pullback from $Y$ to $X$ by $X \xleftarrow{p} Z \xrightarrow{q} Y$ is given by multiplication by $\mathrm{rank}_{\mathcal{O}_X}\mathcal{O}_Z$, if $X$ and $Y$ are connected; one extends to general smooth $X$ and $Y$ by additivity. This gives us the presheaf (of abelian monoids) with framed transfers $A_S^{fr} := A_S \circ \pi_{ad}^{\mathrm{op}}$, and the machinery of Elmanto, et al., converts this into the motivic spectrum $\gamma_* \Sigma_{fr}^\infty A_S^{fr} \in \mathrm{SH}(S)$. Hoyois shows that this construction produces a cartesian family, and that taking $A = \mathbb{Z}$ recovers Spitzweck's family $S \mapsto M\mathbb{Z}_S$.

This gives us a conceptually simple construction of a motivic Eilenberg-MacLane spectrum, and the corresponding motivic category $\mathrm{DM}_H(S)$, much in the spirit of Voevodsky original construction of $\mathrm{DM}(k)$ and the Röndigs-Østvær theorem identifying $\mathrm{DM}(k)$ with the homotopy category of $\mathrm{EM}(\mathbb{Z}(0))$-modules.