6.3.23

J. Wildeshaus, "Shimura varieties and motives", Section 1: "Weight structures"

1 Basic definitions

Fix a triangulated category $\mathcal{C}$.

Definition 1.1: (Bondarko, Pauksztello.)

A weight structure on $\mathcal{C}$ is a pair $(\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 0})$ of full sub-cat.s of $\mathcal{C}$, st., putting $\mathcal{C}_{w\leq n} := \mathcal{C}_{w\leq 0}[n]$, $\mathcal{C}_{w\geq n} := \mathcal{C}_{w\geq 0}[n]$ $\forall n \in \mathbb{Z}$, we have:

(1) $\mathcal{C}_{w\leq 0}$ and $\mathcal{C}_{w\geq 0}$ are Karoubi-closed, i.e., closed under direct summands in $\mathcal{C}$,

(2) $\mathcal{C}_{w\leq 0} \subset \mathcal{C}_{w\leq 1}$, $\mathcal{C}_{w\geq 0} \supset \mathcal{C}_{w\geq 1}$,

(3) (Orthogonality.) $\forall M \in \mathcal{C}_{w\leq 0}$, $N \in \mathcal{C}_{w\geq 1}$,

$$\mathrm{Hom}_{\mathcal{C}}(M, N) = 0,$$

(4) (Weight filtrations.) $\forall M \in \mathcal{C}$ $\exists$ exact triangle

$$A \longrightarrow M \longrightarrow B \longrightarrow A[1]$$

in $\mathcal{C}$, with $A \in \mathcal{C}_{w\leq 0}$, $B \in \mathcal{C}_{w\geq 1}$.

Remarks: (a) Pauksztello: "co-t-structures".

(b) weight str. $\longleftrightarrow$ t-str.: same axioms, but

$$\mathcal{C}^{t\leq n} = \mathcal{C}^{t\leq 0}[-n], \quad \mathcal{C}^{t\geq n} = \mathcal{C}^{t\geq 0}[-n]$$

(and then (1) is redundant).

Definition 1.2: The heart of a weight structure $w = (\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 0})$ is the full sub-cat. $\mathcal{C}_{w=0} := \mathcal{C}_{w\leq 0} \cap \mathcal{C}_{w\geq 0}$.

Remark: Let $A \in \mathcal{C}_{w=0}$. Then

$$A \longrightarrow 0 \longrightarrow A[1] = A[1]$$

is a wt. filtration of the zero object!

→ wt filtrations are rarely unique!!

Proposition 1.3: (Bondarko.) Let $\mathcal{H} \subset \mathcal{C}$ be full. Assume $\mathcal{H}$ generates $\mathcal{C}$ (as a triang. cat.).

(a) If there is a wt. str. on $\mathcal{C}$ whose heart $\mathcal{C}_{w=0}$ contains $\mathcal{H}$, then it is unique.

In this case, $\mathcal{C}_{w\leq 0}$ is the Karoubi closure of $\mathcal{H}$ in $\mathcal{C}$.

(b) The following are equivalent:

(i) $\exists$ wt. str. on $\mathcal{C}$ whose heart contains $\mathcal{H}$,

(ii) $\mathcal{H}$ is <u>negative</u>, i.e.,

$$\mathrm{Hom}_{\mathcal{C}}(A, B[i]) = 0 \quad \forall A, B \in \mathcal{H},\ i \in \mathbb{Z}_{\geq 1}.$$

<u>Idea of proof:</u> (b) (i) $\Rightarrow$ (ii); obvious by def.

$\mathcal{C}_{w\leq 0} = \{$ success. ext.s of obj.s $A[n]$, $n \leq 0\}$, $\Big\{$ necess.ly, as $\mathcal{C}_{w\leq 0}$

$\mathcal{C}_{w\geq 0} = \{$ ——— " ———, $n \geq 0\}$. $\Big\}$ and $\mathcal{C}_{w\geq 0}$ are stable under ext.s

Bondarko: (ii) $\Rightarrow$ this defines a wt. str. (q.e.d.)

<u>Perspective:</u> (tomorrow...)

$\mathcal{C}$ = Voevodsky mot. $DM_{gm}(k)$, $\mathcal{H}$ = Chow motives $Mot_{CH}(k)$

$\xrightarrow[\text{(Bondarko)}]{}$ the <u>motivic weight structure</u>!

<u>2 weight filtrations avoiding certain weights</u>

$\mathcal{C}$ as before, $w = (\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 0})$ a wt. str. on $\mathcal{C}$.

<u>Definition 2.1:</u> $M \in \mathcal{C}$, $m \leq n$ integers. <u>A weight filtration of M avoiding weights $m, m+1, \dots, n$</u> is an ex. tr.

$$M_{\leq m-1} \longrightarrow M \longrightarrow M_{\geq n+1} \longrightarrow M_{\leq m-1}[1],$$

with $M_{\leq m-1} \in \mathcal{C}_{\leq m-1}$, $M_{\geq n+1} \in \mathcal{C}_{\geq n+1}$.

<u>Exercise 2.2:</u> $m \leq n$ integers,

$$M_{\leq m-1} \longrightarrow M \longrightarrow M_{\geq n+1} \longrightarrow M_{\leq m-1}[1] \quad \text{and}$$

$$N_{\leq m-1} \longrightarrow N \longrightarrow N_{\geq n+1} \longrightarrow N_{\leq m-1}[1]$$

two wt. filtr. avoiding wt.s $m, m+1, \dots, n$. Then any morphism $M \longrightarrow N$ extends uniquely to a morph. of wt. filtr.

<u>Main points:</u> $\mathrm{Hom}_{\mathcal{C}}(M_{\leq m-1}, N_{\geq n+1}) = 0$, $\mathrm{Hom}_{\mathcal{C}}(M_{\leq m-1}, M_{\geq n+1}[-1]) = 0$ (orthogonality).

hence: if a wf filtr. avoiding wt.s $m, m+1, \dots, n$ exists, then it is unique up to unique isomorphism!

Definition 2.3: $M \in \mathcal{C}$ *avoids weights* $m, m+1, \dots, n$, *or is without weights* $m, m+1, \dots, n$ if M admits a wf filtr avoiding wt.s $m, m+1, \dots, n$.

Definition 2.4: (a) $\mathcal{C}_{w\le 0, \neq -1}$ is the full sub-cat of $\mathcal{C}_{w\le 0}$ of objects without wt. -1

(b) $\mathcal{C}_{w\ge 0, \neq +1}$ ——— // ——— $\mathcal{C}_{w\ge 0}$ ——— // ——— $+1$.

3 The heart

$\mathcal{C}$, w as before.

Exercise 3.1: (a) The inclusion $\iota_- : \mathcal{C}_{w=0} \hookrightarrow \mathcal{C}_{w\le 0, \neq -1}$ admits a left adjoint $Gr_0 : \mathcal{C}_{w\le 0, \neq -1} \longrightarrow \mathcal{C}_{w=0}$.

On objects, it is given by sending M to the term $M_{\ge 0}$ of a wf filtr. $M_{\le -2} \longrightarrow M \longrightarrow M_{\ge 0} \longrightarrow M_{\le -2}[1]$ avoiding wt. -1. We have $Gr_0 \circ \iota_- = id_{\mathcal{C}_{w=0}}$.

(b) Similarly: $Gr_0 : \mathcal{C}_{w\ge 0, \neq +1} \longrightarrow \mathcal{C}_{w=0}$ = right adj. to ι_+ ...

hence: if $M \in \mathcal{C}_{w\le 0, \neq -1}$ carries the action of a group / an algebra / whatever (example: the Hecke algebra!), then so does $Gr_0 M$.

FIX: $u : M_- \longrightarrow M_+$ morph in $\mathcal{C}$, with $M_- \in \mathcal{C}_{w\le 0}$, $M_+ \in \mathcal{C}_{w\ge 0}$,

$C \xrightarrow{v_-} M_- \xrightarrow{u} M_+ \xrightarrow{v_+} C[1]$ ex.tr. in $\mathcal{C}$.

ASSUMPTION: C is without wt.s -1 and 0:

$C_{\le -2} \xrightarrow{c_-} C \xrightarrow{c_+} C_{\ge 1} \xrightarrow{\delta_C} C_{\le -2}[1]$ ex.tr., $C_{\le -2} \in \mathcal{C}_{w\le -2}$, $C_{\ge 1} \in \mathcal{C}_{w\ge 1}$

Remarks: $M_- = M(X)^e$, $M_+ = M^c(X)^e$ direct factors of the motive and the motive with compact support of a smooth variety X over a field, $C = \partial M(X)^e$ the corresponding direct factor of the boundary motive of X.

(⇝ The ASSUMPTION puts a restriction on the projector e on $\partial M(X)^e$. ($e = id$ will not do i.g.!))

Theorem 3.2: In the above (general) situation, and under the ASSUMPTION:

(a) $M_- \in \mathcal{C}_{w\le 0,\ne -1}$, $M_+ \in \mathcal{C}_{w\ge 0,\ne +1}$.

(b) The wt. filtr.s according wt -1, resp. wt. $+1$, are

$$C_{\le -2} \xrightarrow{v_- c_-} M_- \xrightarrow{\pi_0} Gr_0 M_- \xrightarrow{\delta_-} C_{\le -2}[1],$$

$$C_{\ge 1} \xrightarrow{\delta_+} Gr_0 M_+ \xrightarrow{i_0} M_+ \xrightarrow{(c_+[1])v_+} C_{\ge 1}[1].$$

(c) There is a canonical isomorphism $Gr_0 M_- \xrightarrow{\sim} Gr_0 M_+$.

As a morphism, it is uniquely determined by the commutativity of

$$\begin{tikzcd} M_- \arrow[r, "u"] \arrow[d, "\pi_0"'] & M_+ \\ Gr_0 M_- \arrow[r] & Gr M_+ \arrow[u, "i_0"'] \end{tikzcd}$$

Perspective: $u: M(X)^e \longrightarrow M^c(X)^e \rightsquigarrow Gr_0 M(X)^e = Gr_0 M^c(X)^e$ = the e-part of the interior motive.

Proof of Theorem 3.2: Choose and fix cones G_- of $v_- c_-$ and G_+ of $c_+(v_+[-1])$:

$$C_{\le -2} \xrightarrow{v_- c_-} M_- \xrightarrow{\pi_0'} G_- \xrightarrow{\delta_-} C_{\le -2}[1],$$

$$C_{\ge 1} \xrightarrow{\delta_+} G_+ \xrightarrow{i_0'} M_+ \xrightarrow{(c_+[1])v_+} C_{\ge 1}[1].$$

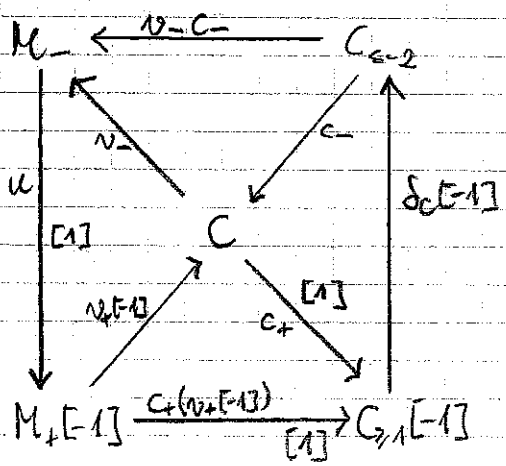
hence: $G_- \in \mathcal{C}_{w\le -0}$, $G_+ \in \mathcal{C}_{w\ge 0}$.

let us show: $\exists\, \alpha: G_- \xrightarrow{\sim} G_+$ making the diagram

$$\begin{tikzcd} M_- \arrow[r, "u"] \arrow[d, "\pi_0'"'] & M_+ \\ G_- \arrow[r, "\alpha"] & G_+ \arrow[u, "i_0'"'] \end{tikzcd}$$

commute ($\Longrightarrow$ (a), (b), (c)):

consider



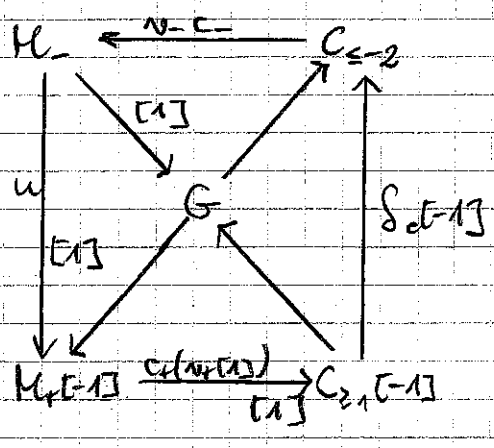
$$(*) \qquad \begin{tikzcd} M_- \arrow[dd, "u"', "{[1]}"] & & C_{\le -2} \arrow[ll, "v_- c_-"'] \arrow[dl, "c_-"] \\ & C \arrow[ul, "v_-"] \arrow[dr, "c_+"', "{[1]}"] & \\ M_+[-1] \arrow[ur, "v_+[-1]"] \arrow[rr, "c_+(v_+[-1])", "{[1]}"'] & & C_{\ge 1}[-1] \arrow[uu, "\delta_c[-1]"'] \end{tikzcd}$$

here:
- the 3 arrows marked [1] link the source to the target, shifted by [1],
- the upper and lower triangles are commutative,
- the left and right triangles are exact.

By definition, this means that (*) is a calotte inférieure.

One of the axioms of triang. cat.s (TR4' in BBD's formulation) implies:

∃ <u>calotte supérieure</u>



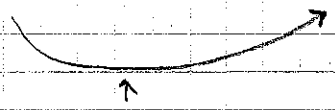
· same meaning of [1],
· left and right: commutative
· upper and lower: exact.

in particular: (i) the same object G[1] can be chosen as cone of ν_C_ and of C_(ν+[1]),
(ii) G[1] provides a factorization of u. q.e.d.

<u>Corollary 3.3:</u> In the setting of Theorem 3.2, let $M_- \to N \to M_+$ be a factorization of u through $N \in \mathcal{C}_{w=0}$. Then $G r_0 M_- = G r_0 M_+$ is canonically identified with a direct factor of N, admitting a canonical direct complement.

<u>Idea of proof:</u> Ex. 3.1 ⇝ $M_- \xrightarrow{\pi_0} G r_0 M_- \longrightarrow N \longrightarrow G r_0 M_+ \xrightarrow{i_0} M_+$

Thm 3.2(c) ⇝



(q.e.d.)

4 Semi-primary categories

F: a coefficient ring ⇝ F-categories, F-functors...

<u>Definition 4.1:</u> (Kelly.)

Or an F-cat. The <u>radical</u> of Or is the ideal $\mathrm{rad}_{\mathcal{O}}$, which associates to each pair of objects $A, B \in \mathcal{O}$

$$\mathrm{rad}_{\mathcal{O}}(A,B) := \{ f \in \mathrm{Hom}_{\mathcal{O}}(A,B) \mid \forall g \in \mathrm{Hom}_{\mathcal{O}}(B,A),\ \mathrm{id}_A - gf \text{ invertible} \} \subset \mathrm{Hom}_{\mathcal{O}}(A,B).$$

(This <u>is</u> a two-sided ideal of Or.)

Definition 4.2: (André-Kahn.)

(a) An F-functor $T: \mathcal{A} \to \mathcal{B}$ between F-cats is <u>radicial</u> if

$$T(\mathrm{rad}_{\mathcal{A}}) \subset \mathrm{rad}_{\mathcal{B}}.$$

(b) An F-cat. $\mathcal{A}$ is <u>semi-primary</u> if

(1) $\forall A \in \mathcal{A}$, $\mathrm{rad}_{\mathcal{A}}(A,A)$ is nilpotent:

$$\exists N \in \mathbb{N},\ \mathrm{rad}_{\mathcal{A}}(A,A)^N = 0 \subset \mathrm{End}_{\mathcal{A}}(A),$$

(2) the quotient cat. $\overline{\mathcal{A}} := \mathcal{A}/\mathrm{rad}_{\mathcal{A}}$ is semi-simple.

Proposition 4.3: (André-Kahn.)

Assume F to be a field. Let $T: \mathcal{A} \to \mathcal{B}$ be an F-functor.

Assume: (i) $\mathcal{A}$ is semi-primary and pseudo-Abelian,

(ii) T is radicial,

(iii) $\forall A \in \mathcal{A}$: $T(A) = 0 \Longrightarrow A = 0$.

Then T is conservative.

<u>Idea of proof:</u> $\overline{\mathcal{A}}$ is Abelian semi-simple, and $\mathcal{B} \to \overline{\mathcal{B}}$ is conservative. (q.e.d.)

<u>Remarks:</u> T = the restriction of the (ℓ-adic or Hodge theoretic) regulator to $\mathcal{A}$ = Chow motives (of a certain type - in order to guarantee 4.3 (i), (ii), (iii)).

J. Wildeshaus, "Shimura varieties and interior motives",

Lecture 2: "Chow motives relating to interior cohomology of Shimura varieties"

5 The motivic weight structure

recall from M. Kerz's talks: $Hot^{eff}_{CH}(k) \hookrightarrow DM^{eff}_{gm}(k)$

$\cup \qquad\qquad \cup$

$Hot_{CH}(k) \hookrightarrow DM_{gm}(k)$

$\left\{ \begin{array}{l} \mathbb{Z}[1/p]\text{-coefficients}, \\ p' = \begin{cases} 1, & \mathrm{char}\, k = 0 \\ \mathrm{char}\, k, & \mathrm{char}\, k \neq 0 \end{cases} \end{array} \right.$

- $M^c(X)$ + localisation

Theorem 5.1: (Bondarko)

(a) There is a unique wt. str. on $DM^{eff}_{gm}(k)$, whose heart equals $Hot^{eff}_{CH}(k)$.

(b) There is a canonical extension of the wt. str. from (a) to one on $DM_{gm}(k)$. It is unique wrt. the requirement that its heart equal $Hot_{CH}(k)$.

Proof: $DM^{eff}_{gm}(k) \supset DM^S$: full subcat. gen.^{d} by $\{M(X) \mid X \in Sm_k\}$,

$\cup\, j_0$: full add. subcat. gen.^{d} by $\{M(X) \mid X \in Sm\,Proj_k\}$.

$\rightsquigarrow DM^{eff}_{gm}(k) = (DM^S)^\natural$,

$Hot^{eff}_{CH}(k) = j_0^\natural$.

(1): j_0 generates DM^S Δ-ly:

- local.^{ton} + ind. on dim: $\langle j_0 \rangle^\Delta \supset \{M^c(U) \mid U \subset X \in Sm\,Proj_k\}$,
- alteration + — " — : $\langle j_0 \rangle^\Delta \supset \{M^c(U) \mid U \in Sm_k\}$.
- locality + — " — : $\langle j_0 \rangle^\Delta = DM^S$.

(2): j_0 is negative: $Hom_{DM_{gm}(k)}(M(X), M(Y)(n)[i]) = 0 \;\forall X, Y \in Sm\,Proj_k, n \geq 1$:

M. Kerz's talks.

(1), (2) $\overset{\text{Prop. 1.3.1}}{\Longrightarrow}$ unique wt. str. on DM^S whose heart contains j_0 (and $DM^S_{wt=0} = j_0^{(\mathbb{Q})}$).

Bondarko: w is compatible with $\natural$! Whence (a).

(b): implied formally by (a). q.e.d.

Definition 5.2: The weight structures from Thm. 5.2 are called the <u>motivic weight structure</u> on $DM^{eff}_{gm}(k)$ and $DM_{gm}(k)$, respectively.

6 The boundary motive

recall from M. Levine's talks: • $NST(k)$ Nisnevich sheaves with transfers,

$$\begin{tikzcd} \mathbb{D}(NST(k)) \arrow[r, "RC^{Sus}"] & DM^{eff}(k) \arrow[l, bend left] \\ & DM^{eff}_{gm}(k) \arrow[u, hook, "i"'] \end{tikzcd}$$

st. $i(M(X)) = RC^{Sus}(\mathbb{Z}^{tr}(X)) \quad \forall X \in Sm_k$,

with $\mathbb{Z}^{tr}(X) := Hom_{Cor(k)}(\cdot, [X])$.

• $M^c(X) := RC^{Sus}(z_{qfin}(X))$, $X \in Sm_k$,

with $z_{qfin}(X) \subset Z_{dim X}(X \times \cdot)$

gen^d by quasi-finite cycles over $\cdot$.

$\mathbb{Z}^{tr}(X) \subset z_{qfin}(X)$ in $NST(k)$.

Definition 6.1: The <u>boundary motive of X</u> is defined as

$$\partial M(X) := RC^{Sus}(z_{qfin}(X)/\mathbb{Z}^{tr}(X))[-1] \in DM^{eff}(k).$$

Proposition 6.2: (a) There is a canonical exact triangle

$$(*) \quad \partial M(X) \to M(X) \xrightarrow{u} M^c(X) \longrightarrow \partial M(X)[1].$$

(b) $\partial M(X) \in DM^{eff}_{gm}(k)$.

Proof: (a): by definition:

(b): M. Levine's talks: $M^c(X) \in DM^{eff}_{gm}(k)$. q.e.d.

The exact Δ $(*)$ is equivariant under

$$\mathbb{Z}^{tr}(X)(X) \cap (\mathbb{Z}^{tr}(X)(X))^t \subset Z_{dim X}(X \times X)$$

gen^d by cycles finite over both copies of X

$\rightsquigarrow$ source of idempotents e on $(*)$.

§7 The (e-part of the) integral motive

$X \in Sm_k$, e an idempotent acting on $(*)$.

Set up the situation from §3:

$u: M(X)^e \longrightarrow M^c(X)^e$. Indeed:

Proposition 7.1: $M(X) \in DM_{gm}^{eff}(k)_{w\le 0}$, $M^c(X) \in DM_{gm}^{eff}(k)_{w\ge 0}$.

Proof: clearly $\Rightarrow$ suffices to show:

ind. on $\dim X$: $X = pt$: $X \in Sm\,Proj_k$, so $M^c(X) = M(X) \in DM_{gm}^{eff}(k)_{w=0}$;

if $X \hookrightarrow \overline{X}$, $\overline{X} \in Sm\,Proj_k$, then

localization: $M^c(\overline{X}) \longrightarrow M^c(X) \longrightarrow M^c(\overline{X}\setminus X)[1] \to M^c(\overline{X})[1]$

$M^c(\overline{X}) = M(\overline{X})$, $w \le 0$; $M^c(\overline{X}\setminus X)[1]$: $w \ge 0$ by ind., $w \ge 1$.

... q.e.d.

ASSUMPTION: $\partial M(X)^e$ is without weights -1 and 0.

$\Rightarrow$ Thm. 3.2 applies. In particular:

Theorem 7.2: Under the ASSUMPTION,

$M(X)^e$ is without weight -1, $M^c(X)^e$ is without weight $+1$, and

$Gr_0 M(X)^e = Gr_0 M^c(X)^e$ ($\in DM_{gm}^{eff}(k)_{w=0} = Mot_{CH}(k)^{eff}$).

Definition 7.3: $Gr_0 M(X)^e$ is *the e-part of the integral motive of X.*

Recall from M. Levine's talks: realizations

- $R_B : DM_{gm}(k)^{op} \longrightarrow D(Vec_{\mathbb{Q}})$ (for $k \hookrightarrow \mathbb{C}$),

  returning to

  $R_{MHS}: DM_{gm}(k)^{op} \longrightarrow D(MHS)$,

- $R_{\acute{e}t,\ell}: DM_{gm}(k)^{op} \longrightarrow D_{\acute{e}t,cts}(k_\ell, k)$ (for $\ell \neq \mathrm{char} k$).

For $X \in Sm_k$, the cohomology objects of $R_?(M(X))$ compute (Betti or ℓ-adic) cohomology of X.

weights on the target of R_{MHS} / $R_{\acute{e}t,\ell}$ (if k is fin. gen. over its prime field): in the sense of Deligne! Meaning: $R(M(X))$ is of weight zero for $X \in Sm\,Proj_k$.

Theorem 7.4: Under the ASSUMPTION:

(a) $\mathcal{H}^n(R_?(Gr_0 M_{gm}(X)^e)) = H^n_!(X, \mathbb{Q}_{(\ell)})^e \quad \forall n \in \mathbb{Z}.$

$\uparrow$ $\mathrm{im}(H^n_c \to H^n)$

(b) For $? =$ MHS or ($? =$ et, ℓ and k fin^{-l} gen^{l} over its prime field), $H^n_!(X, \mathbb{Q}_{(\ell)})^e$ equals the lowest weight filtr. step W_n of $H^n(X, \mathbb{Q}_{(\ell)})^e$, $\forall n \in \mathbb{Z}$.

Proof: Thm. 3.2(b): ex. seq.

$$C_{\leq -2} \longrightarrow M(X)^e \longrightarrow Gr_0 M(X)^e \longrightarrow C_{\leq -2}[1]$$

$\uparrow$ $\in \mathcal{C}_{w \leq -2}$

$\Rightarrow \mathcal{H}^n(R_?(C_{\leq -2}))$ of wt.s $\geq n+2$, $\forall n \in \mathbb{Z}$... q.e.d.

PROBLEM: How to verify the ASSUMPTION?

More generally, given $M \in DM_{gm}(k)$, how to recognize the absence of certain weights in M?

8 Weight conservativity

Definition 8.1: $\bar{k} :=$ alg. closure of k.

(a) $Mot^{Ab}_{CH}(\bar{k}) \subset Mot_{CH}(\bar{k})$ full add. ps.-Abelian sub-cat. gen.^{d} by

- $\mathbb{Z}[m][2m] \quad \forall m \in \mathbb{Z}$,
- $M(A)$, $A/\bar{k}$ Abelian variety.

Chow motives of Abelian type over $\bar{k}$.

(b) $Mot^{Ab}_{CH}(k) \subset Mot_{CH}(k)$ full sub-cat. of obj. M st. $M \times_k \bar{k} \in Mot^{Ab}_{CH}(\bar{k})$.

Chow motives of Abelian type over k.

(c) $DM^{Ab}_{gm}(k) \subset DM_{gm}(k)$ full s sub-cat. gen.^{d} by $Mot^{Ab}_{CH}(k)$.

Voevodsky motives of Abelian type over k.

Proposition 8.2: The motivic weight structure induces a weight structure on $DM^{Ab}_{gm}(k)$.

Proof: apply Prop. 1.3. q.e.d.

Bondarko: as $Mot^{Ab}_{CH}(k)$ is pseudo-Abelian, so is $DM^{Ab}_{gm}(k)$.

from now on: need $\mathbb{Q}$-linear versions of everything:

$$\mathrm{Mot}^{bb}_{CH}(k)_{\mathbb{Q}}\ ,\ \mathrm{DM}^{bb}_{CH}(k)_{\mathbb{Q}}\ ,\ \ldots$$

<u>Proposition 8.3:</u> All objects of $\mathrm{Mot}^{bb}_{CH}(k)_{\mathbb{Q}}$ are <u>finite dimensional</u>, and $\mathrm{Mot}^{bb}_{CH}(k)_{\mathbb{Q}}$ is semi-primary.

<u>Proof:</u> Künnemann: all objects in $\mathrm{Mot}^{bb}_{CH}(\bar{k})_{\mathbb{Q}}$ are f.d.

O'Sullivan: —— " —— $\mathrm{Mot}^{bb}_{CH}(k)_{\mathbb{Q}}$ are f.d.

André-Kahn: all obj. f.d. $\Rightarrow$ the cat. is semi-primary. q.e.d.

<u>Corollary 8.4:</u> (Kimura.)

Let $M \in \mathrm{Mot}^{bb}_{CH}(k)_{\mathbb{Q}}$ be such that $R_{\mathcal{B}}(M)$ or $R_{et,\ell}(M)$ equals 0.

Then $M = 0$.

<u>Proof:</u> This holds actually as soon as M is f.d. (q.e.d.)

Let R be one of $R_{\mathcal{B}}$ or $R_{et,\ell}$. $\mathcal{H}^* R := \bigoplus_{n\in\mathbb{Z}} \mathcal{H}^n \circ R$.

<u>Theorem 8.5:</u> The restriction of $\mathcal{H}^* R$ to $\mathrm{Mot}^{bb}_{CH}(k)_{\mathbb{Q}} \subset \mathrm{DM}^{bb}_{gm}(k)_{\mathbb{Q}} \subset \mathrm{DM}_{gm}(k)_{\mathbb{Q}}$ maps the radical $\mathrm{rad}_{\mathrm{Mot}^{bb}_{CH}(k)_{\mathbb{Q}}}$ to zero. (In particular, it is radicial.)

<u>Corollary 8.6:</u> The restriction of $\mathcal{H}^* R$ to $\mathrm{Mot}^{bb}_{CH}(k)_{\mathbb{Q}}$ is conservative.

<u>Proof:</u> apply Prop. 4.3: (i) = Prop. 8.3,

(iii) = Cor. 8.4,

(ii) = Thm. 8.5. q.e.d.

"<u>Corollary</u>" 8.7: The restriction of $(\mathcal{H}^*)R$ to $\mathrm{DM}^{bb}_{gm}(k)_{\mathbb{Q}}$ is conservative.

(Proof omitted; a few technical efforts are needed to deduce the result from 8.5...)

<u>Corollary 8.8:</u> The restriction of $(\mathcal{H}^*)R$ to $\mathrm{DM}^{bb}_{gm}(k)_{\mathbb{Q}}$ is <u>weight conservative</u>:

$M \in \mathrm{DM}^{bb}_{gm}(k)_{\mathbb{Q}}$ is without weights $\alpha, \alpha+1, \ldots, \beta$ iff

$\mathcal{H}^n R(M)$ is without wt.s $n-\beta, \ldots, n-(\alpha+1), n-\alpha$, $\forall n \in \mathbb{Z}$.

<u>Proof:</u> "only if": ✓.

"if": $\mathrm{Mot}^{bb}_{CH}(k)_{\mathbb{Q}}$ semi-primary $\Longrightarrow$ can find <u>minimal weight filtr.</u>

$$M_{\leq \alpha-1} \longrightarrow M \longrightarrow M_{\geq \alpha} \xrightarrow{\delta_\alpha} M_{\leq \alpha-1}[1]\ ,$$

$$M_{\leq \beta} \longrightarrow M \longrightarrow M_{\geq \beta+1} \xrightarrow{\delta_{\beta+1}} M_{\leq \beta}[1]\ ,\ \text{i.e.,}$$

wh. fibr.s st. δ_α and $\delta_{\beta+1}$ belong to the radical.

Ex. morph. $H_{<\alpha-1} \to H \to H_{\geq\alpha} \xrightarrow{\delta_\alpha} H_{<\alpha-1}[1]$

$\downarrow m \quad \| \quad \downarrow \quad \downarrow m[1]$

$H_{<\beta} \to H \to H_{\geq\beta+1} \xrightarrow{\delta_{\beta+1}} H_{\geq\beta}[1]$

Thm 8.5 $\Rightarrow \mathcal{H}^* R(\delta_\alpha) = 0, \ \mathcal{H}^* R(\delta_{\beta+1}) = 0.$

$\rightsquigarrow$ l.e. seq.

$$\begin{array}{ccccc} 0 & \to & \mathcal{H}^* R(H_{<\alpha-1}) & \to & \mathcal{H}^* R(H) \\ & & \downarrow \mathcal{H}^* R(m) & & \| \\ 0 & \to & \mathcal{H}^* R(H_{<\beta}) & \to & \mathcal{H}^* R(H) \end{array}$$

our hyp. on wh.s: $\mathcal{H}^* R(m)$ is an iso.

$\Rightarrow$ m is an iso. (Cor. 8.7) q.e.d.

Proof of Theorem 8.5: comparison: WLOG $R = R_B$.

O'Sullivan: $X_k \mathbb{C}$ is radicial $\rightsquigarrow$ WLOG: $k = \mathbb{C}$.

Andre-Kahn: $B \in \mathrm{SmProj}_{\mathbb{C}}$ st. $H(B)$ is f.d., then

$\mathrm{rad}_{\mathrm{Hom}_{\mathrm{CH}}(k)}(H(B), H(B)) =$ classes of num. lg trivial cycles on $B \times_{\mathbb{C}} B$.

Lieberman: B Abelian var. $\Rightarrow$ (num lg trivial = hom. lg trivial).

q.e.d.

J. Wilderhaus, "Shimura varieties and motivic motives", Lecture 3: "Verification and applications"

3 The canonical construction

$(G, \mathcal{H})$: (pure) Shimura datum à la Pink:

(among others:) $G/\mathbb{Q}$ connected reductive,

$\mathcal{H}$ homogeneous under $G(\mathbb{R})$,

$h: \mathcal{H} \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, G_{\mathbb{R}})$ with finite fibers,
($\mathbb{S} := \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$; $G(\mathbb{R})$-equiv.)

$\Rightarrow$ $V \in \mathrm{Rep}_{\mathbb{Q}}(G)$ yields: $\forall x \in \mathcal{H}$, $h_x: \mathbb{S} \longrightarrow G_{\mathbb{R}} \longrightarrow GL(V_{\mathbb{R}})$
(V fin. dim. alg. rep.s)
$\triangleq V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}$ s.t. $V^{q,p} = \overline{V^{p,q}}$

$\Rightarrow \mu: \mathrm{Rep}_{\mathbb{Q}}(G) \longrightarrow \mathrm{VHHS}_{\mathcal{H}}(\mathcal{H})$.
(variation of $\mathbb{Q}$-HS)

If V is irreducible, then $\mu(V)$ is pure
$V = V_{\underline{\alpha}}$, $\underline{\alpha}$ dominant weight. } say of wt. $\tau(\underline{\alpha})$

$K \subset G(\mathbb{A}_f)$ open, compact

$\Rightarrow M^K(G,\mathcal{H})(\mathbb{C}) := G(\mathbb{Q}) \backslash (\mathcal{H} \times (G(\mathbb{A}_f)/K))$ $\mathbb{C}$-valued pt.s of $M^K(G,\mathcal{H})$: Shimura variety of level K.

- def.d over a number field,
- normal,
- smooth if K is neat.

$V \in \mathrm{Rep}_{\mathbb{Q}}(G) \rightsquigarrow \mu(V)/\mathcal{H}$ descends to a $\mathrm{VHHS}_{\mathbb{Q}}$ on $M^K(G,\mathcal{H})_{\mathbb{C}}$ if K is neat.

$\rightsquigarrow$ canonical construction $\mu_{K,\mathrm{HHS}}: \mathrm{Rep}_{\mathbb{Q}}(G) \longrightarrow \mathrm{VHHS}_{\mathbb{Q}}(M^K(G,\mathcal{H})_{\mathbb{C}})$ for K neat.
- exact
- $\otimes$-comp.
- $V = V_{\underline{\lambda}} \Rightarrow \mu_{K,\mathrm{HHS}}(V)$ pure of wt. $\tau(\underline{\lambda})$.

A similar construction is possible in the l-adic context:

$$\mu_{K,\ell}: \mathrm{Rep}_{\mathbb{Q}_\ell}(G) \longrightarrow \mathrm{Et}^{\mathrm{lisse}}_{\mathbb{Q}_\ell}(M^u(G,\mathcal{K})) \quad \text{for } K \text{ neat.}$$

Example 3.1: $(G,\mathcal{K}) = (GL_{2,\mathbb{Q}}, \mathcal{K}_f := \hat{\mathbb{Z}}^\times K_\infty)$.

$\underline{\alpha} \in \{(k,\tau) \in \mathbb{N}_{\geq 2} \times \mathbb{Z} \mid k \equiv \tau \bmod 2\} \rightsquigarrow V_{\underline{\alpha}} = \mathrm{Sym}^k V \otimes \det^{-\frac{\tau+k}{2}}$, $\tau(\underline{\alpha}) = \tau$

(V = standard repr. of GL_2)

$\rightsquigarrow \mu_{K,!}(V_{\underline{\alpha}}) \cong \mathrm{Sym}^k\left(R^1\pi_* \mathbb{Q}_\ell(1)\right)^\vee\left(-\frac{\tau+k}{2}\right)$, for

$\pi: \mathcal{E} \longrightarrow M^u(G,\mathcal{K})$ the univ. ell. curve.

(mod. curve!)

In this context, recall: Eichler-Shimura:

$$H^1_!\left(M^u(G,\mathcal{K})(\mathbb{C}), \underbrace{\mathrm{Sym}^k\left(R^1\pi_* \mathbb{Q}\right)^\vee(-k)}_{= \mathrm{Sym}^k(R^1\pi_*\mathbb{Q})}\right) \otimes_{\mathbb{Q}} \mathbb{C} = S_{k+2}(K) \oplus \overline{S_{k+2}(K)}.$$

($S_{k+2}(K)$: cusp forms of level K and "weight $k+2$")

Denote by $j: M^u(G,\mathcal{K}) \hookrightarrow M^u(G,\mathcal{K})^*$ the open immersion into the Baily-Borel compactification, and by

$i: \partial M^u(G,\mathcal{K})^* \hookrightarrow M^u(G,\mathcal{K})^*$ the closed complement.

Conjecture 3.2: Assume $(G,\mathcal{K})$ to be of Hodge type (meaning:

$\exists$ morph. of Shimura data: $(G,\mathcal{K}) \longrightarrow (GSp_{2g}, \mathcal{K}_g)$

for some $g \geq 1$, st. $G \longrightarrow GSp_{2g}$ is a monomorphism).

Let $V_{\underline{\alpha}} \in \mathrm{Rep}_{\mathbb{Q}}(G)$ irred., st. $\underline{\alpha}$ is regular. Then

$$i^* R^j_* \mu_{K,!}(V_{\underline{\alpha}}) \,/\, \partial M^u(G,\mathcal{K})^*$$

carries weights $\tau(\underline{\alpha})$ and $\tau(\underline{\alpha})+1$; $\forall n \in \mathbb{Z}$,

$i^* R^n j_* \mu_{K,!}(V_{\underline{\alpha}})$ is a (Hodge module / l-adic sheaf) on $\partial M^u(G,\mathcal{K})^*_{(\mathbb{C})}$

without weights $n+\tau(\underline{\alpha})$ and $n+\tau(\underline{\alpha})+1$.

Theorem 9.3: The conjecture is true for the following Shimura data:

(1) $(GL_{2,\mathbb{Q}}, \mathcal{H}_1)$: mod. curves (well known),

(2) $(\mathrm{Res}^F_{\mathbb{Q}} GL_{2,F}, \mathcal{H}_1^{[F:\mathbb{Q}]})$, $F/\mathbb{Q}$ tot. real, $F \neq \mathbb{Q}$: Hilbert-Blumenthal var.s (W.),

(3) $(GU(2,1), \mathcal{H}_2)$, for $GU(2,1) \subset \mathrm{Res}^F_{\mathbb{Q}} GL_{3,F}$, F quad. imag.: Picard surfaces (Annual),

(4) $(\mathrm{Res}^{E^+}_{\mathbb{Q}} GU(2,1), \mathcal{H}_2)$, for $GU(2,1) \subset \mathrm{Res}^E_{E^+} GL_{3,E}$, E CM-field, $E^+ \neq \mathbb{Q}$: Picard varieties (Kloosterman),

(5) $(GSp_4, \mathcal{H}_2)$: Siegel threefolds (W.),

(6) $(\mathrm{Res}^F_{\mathbb{Q}} GSp_4, \mathcal{H}_2^{[F:\mathbb{Q}]})$, $F/\mathbb{Q}$ tot. real, $F \neq \mathbb{Q}$: Hilbert-Siegel var.s (Mantovan),

(7) $(GSp_6, \mathcal{H}_3)$: Siegel sixfolds (W.).

Indication for (1): $\underline{k} = (k, r)$, $V_{\underline{k}} = \mathrm{Sym}^k V \otimes \det^{-\frac{r+k}{2}}$.

Then

$$i^* R^n j_* \mu_{N,\ell}(V_{\underline{k}}) = \begin{cases} 0, & n \neq 0, 1 \\ \mathbb{Q}_\ell(k) \otimes \mathbb{Q}_\ell(-\frac{r+k}{2}) = \mathbb{Q}_\ell(-\frac{r-k}{2}), & n = 0 \\ \mathbb{Q}_\ell(-1) \otimes \mathbb{Q}_\ell(-\frac{r+k}{2}) = \mathbb{Q}_\ell(-\frac{r+k+2}{2}), & n = 1 \end{cases}.$$

$\rightsquigarrow$ wt. $r-k$ for $n=0$, wt. $r+k+2$ for $n=1$. $\Big\}$ $< r$, $> r+2$ $\Big\}$ if $k > 0 \Leftrightarrow \underline{k}$ regular.

Remarks: (a) In cases (2), (4) and (6): certain (mildly) regular $\underline{k}$ give rise to $V_{\underline{k}}$ still satisfying the conclusion of Conj. 9.2.

(b) The analogue of Conj. 9.2 is (very) false i.g. when the BB-comp. is replaced by a toroidal comp.

10 The relative structure

Black box 10.1: (Voevodsky-Déglise.)

$DM_{gm}(\cdot)_{\mathbb{Q}}$ can be defined for other general base schemes $\cdot$, and there is a formalism of 6 functors.

Black box 10.2: (Hébert.)

There is a motivic weight structure on $DM_{gm}(S)_{\mathbb{Q}}$, uniquely determined by the following:

$\pi: X \longrightarrow S$ proper, X regular

$\Rightarrow \pi_* \mathbb{Q}_X \in DM_{gm}(S)_{\mathbb{Q},w=0}$.

Weight str.: $f: S \longrightarrow T \rightsquigarrow$

f_* and $f^!$ respect $DM_{gm}(\cdot)_{\mathbb{Q},w\geq 0}$

f^* and $f_!$ respect $DM_{gm}(\cdot)_{\mathbb{Q},w\leq 0}$.

Definition 10.3: $Mot_{CH}(S)_{\mathbb{Q}} := DM_{gm}(S)_{\mathbb{Q},w=0}$.

Black box 10.4: Let $X \in Sm_k$, $j: X \hookrightarrow \overline{X}$ a compactification,

$$X \xrightarrow{j} \overline{X} \xleftarrow{i} \partial\overline{X} := \overline{X} \setminus X$$

$$\downarrow \overline{a}$$

$$\mathrm{Spec}\, k$$

Then the image under $\overline{a}_*$ of the exact triangle

$$j_! \mathbb{Q}_X \longrightarrow j_* \mathbb{Q}_X \longrightarrow i_* i^* j_* \mathbb{Q}_X \longrightarrow j_! \mathbb{Q}_X[1]$$

(in $DM_{gm}(\overline{X})_{\mathbb{Q}}$) is canonically dual to

$$(*) \qquad H^c(X) \leftarrow H(X) \leftarrow \partial H(X) \leftarrow H^c(X)[-1].$$

Observe: $j_! \to j_* \to i_* i^* j_* \to j_![1]$ is exact, whatever coeff. we take

$\rightsquigarrow$ may replace $\mathbb{Q}_X$ by some other object of $Mot_{CH}(X)_{\mathbb{Q}}$

$\rightsquigarrow$ $(*)$ "with coefficients":

$$\partial H(V) \longrightarrow H(V) \longrightarrow H^c(V) \longrightarrow \partial H(V)[1],$$

$V \in Mot_{CH}(X)_{\mathbb{Q}} \Longrightarrow$ same wt. estimates as usual for $H(V)$ and $H^c(V)$.

Theorem 9.4: (Ancona.) K neat.

Assume $(G,\mathcal{H})$ is of PEL-type ($\Rightarrow$ of Hodge type).

There is a $\mathbb{Q}$-linear $\otimes$-functor $\mu_{K,\mathrm{Mot}}: \mathrm{Rep}_{\mathbb{Q}}(G) \longrightarrow \mathrm{Mot}^{S}_{CH}(M^K(G,\mathcal{H}))$.

Smooth Chow motives

It has the following properties:

(a) $\mathcal{H}^* R_? \circ \mu_{K,\mathrm{Mot}} = \mu_{K,?}$, $? \in \{MHS, \ell\}$. More precisely:

$V = V_{\underline{\alpha}}$ irred. $\Rightarrow R_? \mu_{K,\mathrm{Mot}}(V_{\underline{\alpha}})$ conc.^{d} in degree $r(\underline{\alpha})$ and its value equals $\mu_{K,?}(V_{\underline{\alpha}})$.

(b) $\mu_{K,\mathrm{Mot}}$ maps the restr. of the stand. rep. $V \in \mathrm{Rep}_{\mathbb{Q}}(CSp_{2g})$ to $h^1(\mathcal{A}/M^K(G,\mathcal{H}))^\vee$: $\mathcal{A}$ the pull-back of the univ. Ab. scheme over $M^L(CSp_{2g}, \mathcal{H}_g)$.

$\rightarrow$ for $\underline{\alpha}$ arbitrary (dominant), $\mu_{K,\mathrm{Mot}}(V_{\underline{\alpha}})$ is a direct factor of a Lefschetz twist of some $h(\mathcal{A}^{m_{\underline{\alpha}}}/M^K(G,\mathcal{H}))$:

we'll ignore the twist

$$\mu_{K,\mathrm{Mot}}(V_{\underline{\alpha}}) = h(\mathcal{A}^{m_{\underline{\alpha}}}/M^K(G,\mathcal{H}))^{e_{\underline{\alpha}}}$$

($m_{\underline{\alpha}}$ and $e_{\underline{\alpha}}$ depending on $\underline{\alpha}$).

$\rightarrow$ $H^*(M^K(G,\mathcal{H}), \mu_{K,?}(V_{\underline{\alpha}})) = H^{*+r(\underline{\alpha})}(\mathcal{A}^{m_{\underline{\alpha}}}, \mathbb{Q}_{(\ell)})^{e_{\underline{\alpha}}}$,

and likewise for H^*_c, ∂H^* and $H^*_!$.

Theorem 9.5: K neat, $(G,\mathcal{H})$ of PEL-type. Assume that Conj. 9.2 holds for $(G,\mathcal{H})$. Then for any regular $\underline{\alpha}$,

$\partial M(\mathcal{A}^{m_{\underline{\alpha}}})^{e_{\underline{\alpha}}}$ is without weights -1 and 0.

(Hence the $e_{\underline{\alpha}}$-part of the interior motive of $\mathcal{A}^{m_{\underline{\alpha}}}$ is defined.)

Proof: our normalisation: $\mathcal{H}^* R_?(\partial M(\mathcal{A}^{m_{\underline{\alpha}}})^{e_{\underline{\alpha}}}) = \partial H^{*-r(\underline{\alpha})}(M^K(G,\mathcal{H}), \mu_{K,?}(V_{\underline{\alpha}}))$,

which is the image of $i^* R j_* \mu_{K,?}(V_{\underline{\alpha}})[-r_{\underline{\alpha}}]$ under $R^* a_*$, for

$$a: \partial M^K(G,\mathcal{H})^* \longrightarrow *$$

a is proper, hence Ra_* preserves weights.

Our claim would thus follow from Cor. 8.8 if $\partial M(\mathcal{A}^{m_{\underline{\alpha}}})^{e_{\underline{\alpha}}}$ were of Abelian type.

Problem: $\mathcal{IH}(\mathcal{A}^{m\leq})^{e\leq}$ is of Abelian type if $\mathcal{IH}^u(G,\mathcal{H})^*$ is of dimension zero.
If $\dim \mathcal{IH}^u(G,\mathcal{H})^* \geq 1$, then $\mathcal{IH}(\mathcal{A}^{m\leq})$ is i.g. not of Abelian type.

Solution: (1) recall everything for relative motives:

- $DM(\cdot)_{\mathbb{Q}}$: Voisin-Beilinson ,
- wt. st. on $DM(\cdot)_{\mathbb{Q}}$: Hilbert ,
- notion of motives of Abelian type (over a stratified scheme),
- weight conservativity in that relative context,

(2) prove: $V \in \mathrm{Rep}_{\mathbb{Q}}(G)$, then

$$i^* Rj_* \mu_{K,\mathrm{mot}}(V) \in DM(\mathcal{IH}^u(G,\mathcal{H})^*)_{\mathbb{Q}}$$

is of Abelian type. (Thus uses Pink's thesis...)

Hence Conj. 9.2 actually implies: $i^* Rj_* \mu_{u,\mathrm{mot}}(V_{\leq})$ is without wt.s 0 and 1 as soon as $\leq$ is regular (and $\mathcal{IH}(\mathcal{A}^{m\leq})^{e\leq}$ is dual to its image under $R\alpha_*$...) q.e.d.

How to verify Conjecture 9.2?

The boundary $\mathcal{IH}^u(G,\mathcal{H})^*$ is stratified; to control $i^* Rj_* \mu_{K,\ell}(V)$, one is led to try to control $i_{\mathcal{Z}}^* Rj_* \mu_{K,\ell}(V)$ for all strata $\mathcal{Z}$ (and the way these restrictions glue together...). Let $\mathcal{Z}$ be one such.

Pink: up to the free action, $\mathcal{Z} = M^{*_1(K)}(G_1,\mathcal{H}_1)$ is a Shimura variety, for "smaller" Shimura data $(G_1,\mathcal{H}_1)$. G_1 is obtained as follows:

Q : a (certain) parabolic sub-gr of G,

W_1 : its unip. radical,

$Q \xrightarrow{\pi_1} Q/W_1 = G_1 \cdot G_\ell$ almost direct product.

The inertia group of $\mathcal{Z}$ in $M^u(G,\mathcal{H})^*$ is an arithmetic sub-group of $*_1^{-1}(G_\ell)(Q)$.

Theorem: (Pink, proving a conjecture of Harder (1989).)

$$\forall V \in \mathrm{Rep}_{\mathbb{Q}}(G),\quad i_{\mathcal{Z}}^* R^n j_* \mu_{K,\ell}(V) = \bigoplus_{p+q=n} \mu_{*_1(K),\ell}\left(H^p(\pi_1(H_C), H^q(W_1, \mathrm{Res}^G_Q V))\right).$$

(can (in principle) be controlled using Kostant's formula...)